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QUASI PERIODIC SOLUTIONS OF NEUTRAL VOLTERRA INTEGRAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY**

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Abstract

This paper deals with the uniqueness, existence and non-existence of quasi periodic solutions of neutral Volterra integral differential equations of the form

$$\frac{d}{dt}(x(t)+\int_{-\infty}^{t}D(s-t)x(s)ds)=f(t)+\int_{-\infty}^{t}C(t, x)x(s)ds \tag{1}$$

and

$$\frac{d}{dt}(x(t)+\int_{-\infty}^{t}D(s-t)x(s)ds)=Bx(t)+\int_{-\infty}^{t}C(s-t)x(s)ds+f(t). \tag{2}$$

Some new unique existence criteria and non-existence criteria of quasi periodic solutions for (1) and (2) are obtained.

This paper deals with the uniqueness and existence of quasi periodic solutions of neutral Volterra integral differential equations of the form

$$\frac{d}{dt}(x(t)+\int_{-\infty}^{t}D(s-t)x(s)ds)=f(t)+\int_{-\infty}^{t}C(t, s)x(s)ds, \tag{1}$$

$$\frac{d}{dt}(x(t)+\int_{-\infty}^{t}D(s-t)x(s)ds)=Bx(t)+\int_{-\infty}^{t}C(s-t)x(s)ds+f(t), \tag{2}$$

where $x\in R^n$, B, C, D are $n\times n$ matrices, C, D are continuous, f is a quasi periodic n-dimentional vector function.

The existence and uniqueness of almost periodic solutions of Volterra integro-differential equations have been studied by many authors[1-4]. Using the technique of [5], we present some new unique existence criteria for (1) and (2).

If $x=(x_1, x_2, \cdots, x_n)\in C^n$, $A=(a_{ij})$ is an $n\times n$ matrix, define

$$|x|=\sum_{i=1}^{n}|x_i|,\ |A|=\sum_{i,j=1}^{n}|a_{ij}|.$$

Definition 1. *A continuous function $F(t)$: $R\to R^n$ is said to be a quasi periodic function if*

$$F(t)=f(\omega_1 t, \omega_2 t, \cdots, \omega_m t)$$

where $f(u_1, u_2, \cdots, u_m)$: $R^m\to R^n$ is a cotinuous and periodic function of $u_1, \cdots, u_m$ with

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period 2π, *and* $\omega_1, \cdots, \omega_m>0$.

Definition 2. *A quasi periodic function*

$$F(t)=f(\omega_1 t, \cdots, \omega_m t)$$

is said to be a strong quasi periodic function if it satisfies the following conditions

1° $f(u_1, \cdots, u_m)\in C^\infty$, *thus* $F(t)$ *can be expanded into Fourier series*

$$F(t)=\sum_{k\neq 0} a_k e^{i(k,\omega)t},\ a_k\in C^n,\ \sum|a_k|<\infty,$$

where $k=(k_1, \cdots, k_m)$ *is the integer vector,* $\omega=(\omega_1, \cdots, \omega_m)$ *and* $(k, \omega)=\sum_{s=1}^{m} k_s\omega_s$;

2° *there is a* $k(\omega)$, *such that*

$$|(k, \omega)|>k(\omega). \tag{3}$$

Let SQP be the set of all strong quasi periodic functions. Define $\|q\|=\sum|a_k|$, for $q\in$ SQP, $q=\sum a_k e^{i(k,\omega)t}$. It is easy to see that (SQP, $\|\cdot\|$) is a normed vector space.

For fixed $\omega=(\omega_1, \cdots, \omega_m)$ satisfying (3), let ω-SQP denote the set of all functions $f(\omega t)=f(\omega_1 t, \cdots, \omega_m t)$ in SQP.

Lemma 1. (ω-SQP, $\|\cdot\|$) *is a Banach space for any given* ω *satisfying* (3).

Proof If $F_j(\omega_1 t, \cdots, \omega_m t)=\sum a_k^{(j)} e^{i(k,\omega)t}$, $j=1, 2, \cdots$, and $\{F_j\}$ is a Cauchy sequece of ($\omega-$SQP, $\|\cdot\|$), then $\{a_k^{(j)}\}$ $j=1, 2, \cdots$ is a Cauchy sequece for every fixed k. Suppose $a_k^{(j)}\to b_k$, $j\to\infty$, and $G(t)=\sum b_k e^{i(k,\omega)t}$. Then $G\in\omega-$SQP and $\|F_j-G\|\to 0$, $j\to\infty$. Therefore, ($\omega-$SQP, $\|\cdot\|$) is a Banach space.

Lemma 2.[6] *If* $F(u_1, \cdots, u_m)\in C^\tau$, *and* $\omega=(\omega_1, \cdots, \omega_m)$ *satisfies*

$$|(k, \omega)|>\tilde{k}(\omega)\left(\sum_{j=1}^{m}|k_j|\right)^{-(m+1)},\ \tilde{k}(\omega)>0,$$

then the definite integral $G(t)=\int_0^t F(\omega_1 s, \cdots, \omega_m s)ds$ *is quasi periodic of* C^s, *where* $\tau=2(m+1)+s$.

Lemma 3. *If the quasi periodic function* $F(\omega s)\in$ SQP, *then the definite integral* $G(t)=\int_0^t F(\omega s)ds$ *is in* SQP.

This Lemma is a corollary of Lemma 2.

Lemma 4. *If* $F(\omega t)\in$ SQP, $G(t)=\int_0^t F(\omega s)ds$, *then*

$$\|G\|\leqslant k^{-1}(\omega)\|F\|.$$

Proof If

$$F(\omega t)=\sum a_k \exp(i(k, \omega)t),$$

then

$$G(t)=\sum \frac{a_k}{i(k, \omega)}\exp(i(k, \omega)t).$$

Thus

$$\|G\|=\sum\frac{|a_k|}{|(k, \omega)|}\leqslant\sum\frac{|a_k|}{k(\omega)}=k^{-1}(\omega)\|F\|.$$

Lemma 5. *If* $C(u)$, $D(u)$ *are continuous and*

$$a=\int_{-\infty}^{0}|C(u)|du<+\infty,\ b=\int_{-\infty}^{0}|D(u)|du<+\infty,\ G\in\omega-\mathrm{SQP},$$

$$(QG)(t)=\int_{-\infty}^{t}C(s-t)G(s)ds,\ (RG)(t)=\int_{-\infty}^{t}D(s-t)G(s)ds,$$

then $QG,\ RG\in\omega-\mathrm{SQP}$ *and* $\|QG\|\leqslant a\|G\|,\ \|RG\|\leqslant b\|G\|$.

Proof Since

$$(QG)(t)=\int_{-\infty}^{t}C(s-t)G(s)ds=\int_{-\infty}^{0}C(u)G(u+t)du,$$

if $G(t)=g(\omega_1 s,\ \cdots,\ \omega_m t)=\sum a_k\exp(i(k,\ \omega)t)$, then

$$\begin{aligned}(QG)(t)&=\int_{-\infty}^{0}C(u)g(\omega_1(u+t),\ \cdots,\ \omega_m(u+t))du\\&=\int_{-\infty}^{0}C(u)\sum a_k\exp(i(k,\ \omega)(u+t))du\\&=\sum a_k\int_{-\infty}^{0}C(u)\exp(i(k,\ \omega)u)du\cdot\exp(i(k,\ \omega)t).\end{aligned}$$

Thus

$$\|QG\|\leqslant\sum|a_k|\int_{-\infty}^{0}|C(u)|du=a\|G\|.$$

Let

$$(qg)(u_1,\ \cdots,\ u_m)=\int_{-\infty}^{0}C(s)g(\omega_1 s+u_1,\ \cdots,\ \omega_m s+u_m)ds.$$

Then $(QG)(t)=(qg)(\omega_1 t,\ \cdots,\ \omega_m t)$, so QG is a quasi periodic function.

Since

$$\sum\left|a_k\int_{-\infty}^{0}C(u)\exp(i(k,\ \omega)u)du\right|\leqslant\int_{-\infty}^{0}|C(u)|du\cdot\sum|a_k|=a\sum|a_k|<\infty,$$

the condition 1° of Definition 2 holds.

Since

$$\frac{\partial^{i_1+\cdots+i_m}}{\partial u_1^{i_1}\cdots\partial u_m^{i_m}}(qg)(u_1,\ \cdots,\ u_m)=\int_{-\infty}^{0}C(s)\frac{\partial^{i_1+\cdots+i_m}}{u_1^{i_1}\cdots u_m^{i_m}}g(\omega_1 s+u_1,\ \cdots,\ \omega_m s+u_m)ds$$

we have $qg\in C^\infty$. The same argument holds for Rg.

Lemma 6. *The maps* $Q,\ R$: $\omega-\mathrm{SQP}\to\omega-\mathrm{SQP}$ *are continuous.*

Proof If $g,\ h\in\omega-\mathrm{SQP}$, then

$$\begin{aligned}\|Qg-Qh\|&\leqslant\int_{-\infty}^{t}|C(s-t)||g(s)-h(s)|ds\leqslant\int_{-\infty}^{0}|C(u)||g(u+t)-h(u+t)|du\\&\leqslant\|g-h\|\int_{-\infty}^{0}|C(u)|du.\end{aligned}$$

Therefore, $\|Qg-Qh\|\to 0$, as $\|g-h\|\to 0$, and Q is continuous. The same argument holds for the map R.

Lemma 7. *If* $G(t)=g(\omega_1 t,\ \cdots,\ \omega_m t)\in\omega-\mathrm{SQP}$, *then* RG *is differentiable and* $(RG)'\in\omega-\mathrm{SQP}$.

Proof If $G\in\omega-\mathrm{SQP}$,

$$(RG)(t)=\int_{-\infty}^{t}D(s-t)G(s)ds=\int_{-\infty}^{0}D(u)G(u+t)du,$$

since $G\in C^\infty$,

$$\frac{d}{dt}(RG)(t)=\int_{-\infty}^{0}D(u)G'(u+t)du=\int_{-\infty}^{0}D(u)\frac{\partial g(\omega_1(u+t),\cdots,\omega_m(u+t))}{\partial(u_1,\cdots,u_m)}\cdot\omega du.$$

Thus, the Lemma is proved.

Theorem 1. *If*

1° $f\in\omega-\mathrm{SQP}$,

2° $a=\int_{-\infty}^{0}|C(u)|du<\infty,\ b=\int_{-\infty}^{0}|D(u)|du<\frac{1}{2},\ a<(1-2b)k(\omega)$,

then equation (1) *has one and only one solution in* $\omega-\mathrm{SQP}$.

Proof If $f\in\omega-\mathrm{SQP}$ is given, define map P: $\omega-\mathrm{SQP}\to\omega-\mathrm{SQP}$ by the following way

$$(Pg)(t)=\int_0^t(f(s)+(Qg)(s)-(Rg)'(s))ds.$$

From Lemmas 2, 4, 5, 6, 7 the map P is a linear continuous map of $\omega-\mathrm{SQP}$ into itself.

If $g\in\omega-\mathrm{SQP}$, then

$$(Pg)(t)=\int_0^t(f(s)+(Qg)(s))ds-(Rg)(t)+(Rg)(0),$$

$$\|Pg\|\leqslant\left\|\int_0^t(f+Qg)(s)ds\right\|+2\|Rg\|\leqslant k^{-1}(\omega)\|f+Qg\|+2\|Rg\|$$

$$\leqslant k^{-1}(\omega)(\|f\|+a\|g\|)+2b\|g\|=k^{-1}(\omega)\|f\|+(ak^{-1}(\omega)+2b)\|g\|.$$

Take $M>0$ so large that $(1-ak^{-1}(\omega)-2b)M>k^{-1}(\omega)$.

If $f\neq 0$, let

$$S=\{g\in\omega-\mathrm{SQP}:\ \|g\|\leqslant M\|f\|\}.$$

Then $\|Pg\|\leqslant M\|f\|$, for $g\in S$, so P is a map of S into itself.

If $g,h\in S$, then

$$\|Pg-Ph\|\leqslant\left\|\int_0^t(f+Qg)(s)ds-\int_0^t(f+Qh)(s)ds\right\|+2\|Rg-Rh\|$$

$$\leqslant\left\|\int_0^t Q(g-h)(s)ds\right\|+2\|R(g-h)\|\leqslant(ak^{-1}(\omega)+2b)\|g-h\|.$$

Therefore, P is a contraction on S. The contraction principle implies there is a unique fixed point g^* of P on S, that is, $Pg^*=g^*$,

$$\frac{d}{dt}(g^*(t)+(Rg^*)(t))=f(t)+(Qg^*)(t).$$

Therefore, $g^*(t)$ is a solution of (1) on S. If (1) has another ω-quasi periodic solution $h^*(t)$, then

$$h^*(t)=\int_0^t(f(s)+(Qg)(s)-(Rg)'(s))ds+h^*(0).$$

Since h^* and $\int_0^t(f(s)+(Qg)(s)-(Rg)'(s))ds$ belong to $\omega-\mathrm{SQP}$, $h^*(0)\in\omega-\mathrm{SQP}$, that is, $h^*(0)=0$. We can take $M>0$ so large that $\|h^*\|\leqslant M\|f\|$. Then h^* is a fixed

Since

$$f(t)+Q(g-\beta)(t)=f(t)+\int_{-\infty}^{t}C(s-t)(g(s)-\beta)ds$$
$$=f_1(t)+\alpha+\int_{-\infty}^{t}C(s-t)g(s)ds-\int_{-\infty}^{t}C(s-s)ds\cdot\beta$$
$$=f_1(t)+\int_{-\infty}^{t}C(s-t)g(s)ds=(f_1+Qg)(t)\in\omega-\text{SQP},$$

we have $Pg\in\omega-\text{SQP}$, and

$$\|Pg\|\leqslant k^{-1}(\omega)\|f_1+Qg\|+2\|R(g-\beta)\|\leqslant k^{-1}(\omega)\|f_1\|+2\|R\beta\|+(ak^{-1}(\omega)+2b)\|g\|.$$

If $f_1\neq 0$, take $M>0$ so large that

$$((1-2b)k(\omega)-a)M>1+2k(\omega)\|R\beta\|/\|f_1\|$$

and let

$$S=\{g\in\omega-\text{SQP}:\ \|g\|\leqslant M\|f_1\|\}.$$

If $f_1=0$, let $S=\{g\in\omega-\text{SQP}:\ \|g\|\leqslant 1\}$.

By the same argument as above, we can prove that P is a contraction on S. Therefore, there is a unique fixed point g^* of P on S, that is,

$$g^*(t)=\int_0^t(f(s)+Q(g^*-\beta)(s))ds-(R(g^*-\beta))(t)+(R(g^*-\beta))(0),$$

$$\frac{d}{dt}((g^*-\beta)(t)+(R(g^*-\beta)(t))=f(t)+Q(g^*-\beta)(t),$$

and $g^*-\beta$ is a solution of (1) in $\omega-\text{SQP}\oplus R^n$.

If there is another solution $h^*(t)-\gamma$ of (1) in $\omega-\text{SQP}\oplus R^n$, then

$$\frac{d}{dt}((h^*(t)-\gamma)+(R(h^*-\gamma)(t))=f(t)+(Q(h^*-\gamma))(t),$$

and

$$h^*(t)-h^*(0)+(Rh^*)(t)-(Rh^*)(0)=\int_0^t(f_1+Qh^*)(s)ds-(A\gamma-\alpha)t.$$

It follows that $A\gamma-\alpha=0$, $\gamma=\beta$ and $h^*(0)=0$, $(Rh^*)(0)=0$. Take $M>0$ so large that $\|h^*\|<M\|f\|$, and h^* is a fixed point of P on S, thus, $h^*=g^*$. This implies the uniqueness of solutions af (1) in $\omega-\text{SQP}\oplus R^n$. The theorem is proved.

Corollary 1. *If A is singular and*

1° $f=f_1+\alpha$, $f_1\in\omega-\text{SQP}$, $\alpha\in AR^n$;

2° $a=\int_{-\infty}^{0}|C(u)|du<\infty$, $b=\int_{-\infty}^{0}|D(u)|du<\frac{1}{2}$, $a<(1-2b)k(\omega)$,

then (1) *has infinite quasi periodic solutions.*

Proof Since A is singular, there are infinite β such that $A\beta=\alpha$. By the same argument as in Theorem 3, we can find a solution $g^*(t)-\beta$ of (1) in $\omega-\text{SQP}\oplus R^n$ for each β. It is obvious that these solutions are different from each other.

Theorem 4. *If A is nonsingular and*

1° $f\in\omega-\text{SQP}\oplus AR^n$;

point of P on S, from the uniqueness of fixed points of P on S, $h^*=g^*$. This implies the uniqueness of $\omega-$SQP periodic solution of (1).

If $f=0$, let $S=\{g\in\omega-\text{SQP}: \|g\|\leqslant M\}$. The remainder of argument proceeds as in case $f\neq 0$.

Lemma 8. *If B is an $n\times n$ constant matrix, then*

$$Bg\in\omega-\text{SQP}, \text{ for } g\in\omega-\text{SQP},$$

and $\|Bg\|\leqslant|B|\,\|g\|$.

Proof Suppose

$$g(t)=\sum a_k\exp(i(k,\omega)t).$$

Then

$$Bg(t)=\sum Ba_k\exp(i(k,\omega)t).$$

Thus, $Bg\in\omega-$SQP, and

$$\|Bg\|\leqslant\sum|Ba_k|\leqslant\sum|B|\,|a_k|=|B|\,\|g\|.$$

Theorem 2. *If A is nonsingular and*

1° $f\in\omega-$SQP,

2° $a=\int_{-\infty}^{0}|C(u)|du<\infty$, $b=\int_{-\infty}^{0}|D(u)|du<\frac{1}{2}$,

$$a+|B|<k(\omega)(1-2b),$$

then equation (2) has one and only one solution in $\omega-$SQP.

Proof Suppose $f\in\omega-$SQP is given. Define map P: $\omega-\text{SQP}\to\omega-\text{SQP}$ by the following way

$$(Pg)(t)=\int_0^t(f(s)+Bg(s)+(Qg)(s))ds-(Rg)(t)+(Rg)(0), \text{ for } g\in\omega-\text{SQP}.$$

We have

$$\|Pg\|=k^{-1}(\omega)\|f+Bg+Qg\|+2\|Rg\|\leqslant k^{-1}(\omega)\|f\|+(k^{-1}(\omega)(|B|+a)+2b)\|g\|.$$

Take $M>0$ so lagre that

$$((1-2b)k(\omega)-|B|-a)M>1.$$

If $f\neq 0$, let $S=\{g\in\omega-\text{SQP}: \|g\|\leqslant M\|f\|\}$; if $f=0$, let $S=\{g\in\omega-\text{SQP}: \|g\|\leqslant M\}$. The remaider of argument proceeds as in case $B=0$.

Let $A=\int_{-\infty}^{0}C(u)du$, $AR^n=\{A\alpha: \alpha\in R^n\}$, and $\omega-\text{SQP}\oplus AR^n=\{f+\alpha: f\in\omega-\text{SQP}, \alpha\in AR^n\}$.

Theorem 3. *If A is nonsingular and*

1° $f\in\omega-\text{SQP}\oplus AR^n$,

2° $a=\int_{-\infty}^{0}|C(u)|du<\infty$, $b=\int_{-\infty}^{0}|D(u)|du<\frac{1}{2}$, $a<(1-2b)k(\omega)$,

then Equation (1) has one and only one solution in $\omega-\text{SQP}\oplus R^n$.

Proof Suppose $f=f_1+\alpha$, $f_1\in\omega-$SQP, $\alpha\in AR^n$. There is a $\beta\in R^n$ such that $A\beta=\alpha$. Define map P: $\omega-\text{SQP}\to\omega-\text{SQP}$ by the following way

$$(Pg)(t)=\int_0^t(f(s)+Q(g-\beta)(s))ds-(R(g-\beta)(t)+(R(g-\beta)(0).$$

2° $a=\int_{-\infty}^{0}|C(u)|du<\infty,\ b=\int_{-\infty}^{0}|D(u)|du<\frac{1}{2},$

$$a+|B|<k(\omega)(1-2b),$$

then (2) *has one and only one solution in* $\omega-\mathrm{SQP}\oplus R^n$.

The proof is similar to Thearem 2, so we omit it.

Corollary 2. *If* A *is singular and*

1° $f\in\omega-\mathrm{SQP}\oplus AR^n$,

2° $a=\int_{-\infty}^{0}|C(u)|du<\infty,\ b=\int_{-\infty}^{0}|D(u)|du<\frac{1}{2},$

$$a+|B|<k(\omega)(1-2b),$$

then (2) *has infinite quasi periodic solutions in* $\omega-\mathrm{SQP}\oplus R^n$.

Theorem 5. *If*

1° $f=f_1+\alpha,\ f_1\in\omega-\mathrm{SQP},\ \alpha\overline{\in}AR^n$,

2° $\int_{-\infty}^{0}|C(u)|du<\infty, \int_{-\infty}^{0}|D(u)|du<\infty,$

then (1) *and* (2) *have no solutions in* $\omega-\mathrm{SQP}\oplus R^n$.

Proof If $g(t)-\beta\in\omega-\mathrm{SQP}\oplus R^n$ is a solution of (1), that is,

$$\frac{d}{dt}((g-\beta)(t)+R(g-\beta)(t))=f(t)+Q(g-\beta)(t),$$

and

$$g(t)-g(0)+(Rg)(t)-(Rg)(0)=\int_0^t(f_1+Qg)(s)ds-(A\beta-\alpha)t,$$

it follows that $A\beta-\alpha=0$, and this is a contradiction. Therefore, (1) has no solution in $\omega-\mathrm{SQP}\oplus R^n$. The same argument holds for equation (2).

Example 1. The equation

$$\frac{d}{dx}(x(t)+\frac{1}{4}\int_{-\infty}^{t}e^{s-t}x(s)ds)=\sum_{n=1}^{m}\mathrm{con}\sqrt{n}\,t+\frac{1}{4}\int_{-\infty}^{t}(s-t-1)^{-2}x(s)ds$$

has a quasi periodic solution which has the form

$$x(t)=\sum_{n=1}^{m}(a_n\mathrm{con}\sqrt{n}\,t+b_n\sin\sqrt{n}\,t).$$

References

[1] Corduneanu, C., Integrodifferential equations with almost periodic solutions, in Volterra and functional differential equation (K. B. Hannsgen, T. L. Herdman, H. W. Stech, & R. L. Wheeler, Eds), 233—243, New York/Basel, 1982.

[2] Fink, A. M., Almost periodic differential equations, Lecture Notes in Math. Vol. 377, Springer-Verlag, Berlin, 1974.

[3] Seiffet, G., Almost periodic solutions for delay-differential equations with infinite delays, *J. Differential Equations*, 5.4(1981), 416—425.

[4] Langenhop, C. E., Periodic and almost periodic solutions of Volterra integrra diffrential equations with infinite memory, *J. Differential Equations*, 58(1985), 391—403.

[5] Hale, J. K., Ordinary differential equations, Wiley-Interscience, New York, 1969.

[6] Lin Zhensheng, The Floquet theory for quasi periodic linear system, *Appl. Math. and Mech.*, 3: 3(1982), 327—344.