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ON CERTAIN CRITERIA FOR THE PRIMALITY OF MEROMORPHIC FUNCTIONS

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Abstract

The authors obtain, among other things, the following result: Let f be a transcendental meromorphic function, then there exists an integer n, such that the set

$$\{a\in\mathbf{C};\ f(z)z^n(z-a)\text{ is not prime}\}$$

is an at most countable set.

§ 1. Introduction

A meromorphic function $F(z)=f(g(z))$ is said to have f and g as left and right factors respectively, provided f is meromorphic and g entire (g may be meromorphic when f is rational). F is said to be prime (pseudo-prime, left-prime, right-prime), if every factorization of F of the above form into factors implies that either f or g is linear (either f is rational or g is a polynomial, f is linear whenever g is transcendental, g is linear whenever f is transcendental). For an entire function, when factors are restricted to entire functions, we define the corresponding notions such as E-prime, E-pseudo-prime, E-left-prime, etc. Two factorizations $F=f(g)$ and $F=f_1(g_1)$ are said to be equivalent, which is denoted by $f(g)\sim f_1(g_1)$, if there is a linear transformation λ such that

$$f_1=f(\lambda)\quad\text{and}\quad g_1=\lambda^{-1}(g).$$

In what follows we shall employ the notations and theorems of Nevanlinna theory of meromorphic functions[3].

Ozawa[5] gave several criteria for E-left-primality of entire functions. Then Noda[4] improved Ozawa's results and obtained

Theorem A. *Let F be a transcendental entire function with at least one simple zeros satisfying*

$$\Delta=N\left(r,\frac{1}{F'}\right)-\left[N\left(r,\frac{1}{F}\right)-\overline{N}\left(r,\frac{1}{F}\right)\right]>kT\left(r,\frac{F'}{F}\right),\ n.\ e. \tag{1}$$

for some $k>0$, where "n. e." means that the inequality holds as $r\to\infty$ except for a set of r of finite measure. Assume that the simultaneous equations

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$$\begin{cases} F(z) = C, \\ F'(z) = 0 \end{cases} \tag{2}$$

have only finitely many solutions for any non-zero constant C. Then F is E-left-prime.

We shall extend the above result by proving

Theorem 1. *Suppose that F satisfies all the hypotheses of Theorem A. Then F is left-prime.*

Furthermore, we shall give a similar criterion for the left-primality of meromorphic function, that is

Theorem 2. *Let F be a transcendental meromorphic function with at least one simple zero or simple pole satisfying* (1) *for some constant $k > 1/2$. Assume that the simultaneous equations* (2) *have only finitely many solutions for any non-zero constant C. Then F is left-prime.*

Concerning the primality of entire and meromorphic function, we have

Theorem 3. *Suppose that in addition to the hypotheses of Theorem* 1, *the simultaneous equations* (2) *have at most one solution for any non-zero constant C. Then F is prime, unless*

$$F(z) = f(P(z)), \tag{3}$$

where f is entire, and

$$P(z) = w_0 + a(z - z_0)^p \tag{4}$$

with w_0, a, z_0 being constants and $p \geq 2$ an integer.

Theorm 4. *Suppose that in addition to the hypotheses of Theorem* 2, *the simultaneous equations* (2) *have at most one solution for any non-zero constant C. Then F is prime, unless* (3) *holds with f being meromorphic and P being of the form* (4).

Finally, we shall give an application of our criteria in section 4.

§ 2. Preliminary lemmas

Lemma 1[1]. *Let f be meromorphic and g entire, both transcendental. Then*

$$\lim_{r\to\infty} \frac{T(r, f(g))}{T(r, g)} = \infty.$$

Lemma 2. *Let $f(w) = H_1(w)\exp(H_2(w))$, where H_1 is a non-constant rational function with at least one zero, and H_2 a non-constant polynomial. Then there exists a complex w_0 such that $f'(w_0) = 0$ and $f(w_0) \neq 0$.*

Proof Suppose the contrary, i. e. every zero of f' is one of f. Write

$$H_1(w) = A \prod_{j=1}^{M} (w - a_j)^{m_j} \Big/ \prod_{j=1}^{N} (w - b_j)^{n_j}$$

where $A \neq 0$, $a_i \neq b_j$, $M \geq 1$, $N \geq 0$. Put

$$H_3(w)=\frac{\prod_{j=1}^{N}(w-b_j)^{n_j+1}}{\prod_{j=1}^{M}(w-a_j)^{m_j-1}}[H_1'(w)+H_1(w)H_2'(w)]. \tag{5}$$

Obviously

$$H_3(a_j)\neq 0,\ \infty,\quad H_3(b_j)\neq 0,\ \infty, \tag{6}$$

so that H_3 is a polynomial. If x is a zero of H_3, then by (5) and (6), x is a zero of $f'=(H_1'+H_1H_2')\exp(H_2)$. Hence. it is also a zero of H_1. Therefore, there exists an $a_j(1\leqslant j\leqslant M)$ such that $x=a_j$, which contradicts (6). Thus H_3 must have no zeros, i. e. H_3 is a constant, $B(\neq 0)$ say. We have

$$\left[\prod_{j=1}^{N}(w-(b_j)^{n_j+1}\right](H_1'+H_1H_2')=B\prod_{j=1}^{M}(w-a_j)^{m_j-1}. \tag{7}$$

But

$$\text{degree of left hand side in (7)}\geqslant\sum_{j=1}^{M}m_j+N$$

$$>\sum_{j=1}^{M}(m_j-1)=\text{degree of right hand side in (7)},$$

which is a contradiction. And the lemma follows.

Lemma 3. *Let F be a transcendental meromorphic function with at least one zero satisfying* (1) *for some $k>0$. Assume that* (2) *have only finitely many solutions for any non-zero constant C. Then F is pseudo-prime.*

Proof Let $F=f(g)$ with f being meromorphic and g entire, both transcendental. We claim first that there exists a complex w_0 such that $f'(w_0)=0$ and $f(w_0)\neq 0$. Suppose the contrary, i. e. we assume that f' has no zeros or every zero of f' is one of f. Then we have

$$N\left(r,\frac{1}{F'}\right)\leqslant N\left(r,\frac{1}{F}\right)-\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{g'}\right),$$

which implies

$$\varDelta=N\left(r,\frac{1}{F'}\right)-\left[N\left(r,\frac{1}{F}\right)-\overline{N}\left(r,\frac{1}{F}\right)\right]\leqslant N\left(r,\frac{1}{g'}\right).$$

From (1) we deduce

$$\overline{N}\left(r,\frac{1}{F}\right)\leqslant\overline{N}\left(r,\frac{F'}{F}\right)\leqslant T\left(r,\frac{F'}{F}\right)$$

$$\leqslant\frac{\varDelta}{k}\leqslant\frac{1}{k}N\left(r,\frac{1}{g'}\right)\leqslant\frac{1+o(1)}{k}T(r,g),\ n.e. \tag{8}'$$

If f has infinitely many zeros, $\{w_n\}$ say, then by the second fundamental theorem of Nevanlinna's, for $q\geqslant 3$

$$(q-2-o(1))T(r,g)<\sum_{j=1}^{q}\overline{N}\left(r,\frac{1}{g-w_j}\right)\leqslant\overline{N}\left(r,\frac{1}{F}\right)\leqslant\frac{1+o(1)}{k}T(r,g),\quad n.e.$$

which is impossible when q is large. Therefore, f must have only finitely many zeros, and we may write

$$f(w)=H_1(w)\exp(H_2(w)) \tag{9}$$

where H_1 is a meromorphic function having at least one but finitely many zeros, and H_2 is a non-constant entire function. If H_1 has infinitely many poles, so does $H_1'/H_1+H_2'=H$ (say). Hence H is transcendental. But since

$$\frac{F'(z)}{F(z)}=g'(z)H(g(z)), \tag{10}$$

by Lemma 1 we have

$$\frac{T(r, F'/F)}{T(r, g)}\to\infty\ (r\to\infty),\ n.e. \tag{11}$$

which contradicts (8). Consequently, H_1 has at most finitely many poles. Thus we may assume that H_1 in the form (9) is a rational function. On the other hand, if H_2 is transcendental, we may deduce (10) with H transcendental and (11), again a contradiction. Therefore, H_2 must be a non-constant polymonial.

Now we make use of Lemma 2 and see that it is impossible that f' has no zeros or every zero of f' is one of f. Therefore, there is a complex w_0 such that $f'(w_0)=0$ and $f(w_0)\neq 0$. We assert that $g(z)=w_0$ has only finitely many roots, for otherwise the simultaneous equations

$$\begin{cases}F(z)=f(w_0),\\ F'(z)=0\end{cases}$$

would have infinitely many solutions, which violates the assumption. Thus we may write

$$g(z)=w_0+Q(z)\exp(M(z)), \tag{12}$$

where Q is a polynomial, and M entire.

Further, if $x\neq w_0$ is a zero of f', then $f(x)=0$. Otherwise, by the same reasoning just stated, x would be an other Picard's exceptional value of g, which is impossible. Therefore, we have

$$N\left(r,\frac{1}{F'}\right)\leqslant N\left(r,\frac{1}{f'(g)}\right)+N\left(r,\frac{1}{g'}\right)$$
$$\leqslant N\left(r,\frac{1}{F}\right)-\bar{N}\left(r,\frac{1}{F}\right)+O(\log r)+N\left(r,\frac{1}{g'}\right)$$

or

$$\Delta\leqslant N\left(r,\frac{1}{g'}\right)+O(\log r).$$

From (12), we can esaily derive

$$N\left(r,\frac{1}{g'}\right)+O(\log r)=o\{T(r,g)\},\ n.e.$$

Thus we obtain

$$\bar{N}\left(r,\frac{1}{F}\right)\leqslant\bar{N}\left(r,\frac{F'}{F}\right)\leqslant\frac{\Delta}{k}=o\{T(r,g)\},\ n.e.$$

Therefore, if w^* is a zero of f other than w_0 (f must have a zero by the assumption), then by the second fundamental theorem, we have

$$(1-o(1))T(r,\ g)<\bar{N}\left(r,\frac{1}{g-w_0}\right)+\bar{N}\left(r,\frac{1}{g-w^*}\right)$$
$$\leqslant O(\log r)+\bar{N}\left(r,\frac{1}{F}\right)=o\{T(r,\ g)\},\ n.\ e.$$

This contradiction completes the proof of our lemma.

Lemma 4. *Suppose that in addition to the hypotheses of Lemma 3, the simultaneous equations (2) have at most one solution for any non-zero constant C. Assume that F has a factorization of the form (3) with f being transcendental meromorphic and P entire. Then P must be a polynomial of the form (4).*

Proof By Lemma 3, $F=f(P)$ implies that P must be a polynomial. Suppose $P=\deg P\geqslant 2$. We discuss two cases separately.

Case a). f' has infinitely many zeros $\{w_n\}$. We claim that for all large n, w_n must be zeros of f. Otherwise, there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that $f(w_n)\neq 0$ for large n and $P(z)=w_n$ has p ($\geqslant 2$) distinct roots which are the solutions of the simultaneous equations

$$\begin{cases}F(z)=f(w_n),\\ F'(z)=0.\end{cases}$$

This contradicts the assumption. Therefore, $F=f(P)$ has infinitely many zeros. On the other hand, we have

$$\bar{N}\left(r,\frac{1}{F}\right)\leqslant\bar{N}\left(r,\frac{F'}{F}\right)$$
$$\leqslant\frac{1}{k}\left[N\left(r,\frac{1}{f'(P)}\right)+N\left(r,\frac{1}{P'}\right)-N\left(r,\frac{1}{F}\right)+\bar{N}\left(r,\frac{1}{F}\right)\right]$$
$$\leqslant\frac{1}{k}\left[N\left(r,\frac{1}{F}\right)-\bar{N}\left(r,\frac{1}{F}\right)+O(\log r)-N\left(r,\frac{1}{F}\right)+\bar{N}\left(r,\frac{1}{F}\right)\right]$$
$$=O(\log r),$$

which is a contradiction.

Case b). f' has finitely many zeros. Then

$$\bar{N}\left(r,\frac{1}{F}\right)\leqslant\bar{N}\left(r,\frac{F'}{F}\right)\leqslant\frac{1}{k}N\left(r,\frac{1}{F'}\right)$$
$$=\frac{1}{k}\left[N\left(r,\frac{1}{f'(P)}\right)+N\left(r,\frac{1}{P'}\right)\right]=O(\log r).$$

Hence, F has finitely many zeros, so does f. And we may write

$$f(w)=H_1(w)\exp(H_2(w)),$$

where H_1 is a non-constant meromorphic function having finitely many zeros, and H_2 is entire. If H_1 has infinitely many poles, then H_1'/H_1+H_2' is transcendental, so is

$$\frac{F'}{F}=P'\left[\left(\frac{H_1'}{H_1}+H_2'\right)\circ P\right].$$

But

$$T\left(r, \frac{F'}{F}\right) \leq \frac{1}{k} N\left(r, \frac{1}{F'}\right) = \frac{1}{k}\left[N\left(r, \frac{1}{f'(P)}\right) + N\left(r, \frac{1}{P'}\right)\right] = O(\log r),$$

a contradiction. Thus we may assume that H_1 is a rational function. Further, if H_2 is transcendental, then so are $H_1'/H_1 + H_2'$ and F'/F, and we shall get a contradiction again. Therefore, H_2 must be a polynomial.

By applying Lemma 2, we see that there exists a complex w_0 such that $f'(w_2) = 0$ and $f(w_0) \neq 0$. In this case, if $P(z) = w_0$ has at least two distinct roots, then the simultaneous equations

$$\begin{cases} F(z) = f(w_0), \\ F'(z) = 0 \end{cases}$$

would have at least two solutions, which violates the assumption. Hence, $P(z) = w_0$ has only one root with multiplicity p, i. e. P is of the form (4). And the lemma follows.

Lemma 5. *Let $Q(w)$ be a rational function having at least one simple zero. Then there is a complex w_0 such that $Q'(w_0) = 0$ and $Q(w_0) \neq 0$, unless Q is a linear polynomial.*

The proof of Lemma 5 is an elementary work, which should be omitted.

§ 3. Proofs of Theorems

Proof of Theorem 2. Without loss of generality, we may assume that F has a simple zero; otherwise we discuss $\tilde{F} = F^{-1}$.

By Lemma 3, F is pseudo-prime. Let $F = Q(g)$, where g is transcendental meromorphic, and Q rational. Obviously, Q has at least one simple zero, w_1 say. By Lemma 5, there is a complex w_0 such that $Q'(w_0) = 0$ and $Q(w_0) \neq 0$, and $g(z) = w_0$ has at most finitely many roots by the same reasoning as before. We first assume

$$Q(\infty) = \infty. \tag{13}$$

Then we have

$$\begin{aligned} N\left(r, \frac{1}{g'}\right) &\leq N\left(r, \frac{g - w_0}{g'}\right) + N\left(r, \frac{1}{g - w_0}\right) \\ &\leq m\left(r, \frac{g'}{g - w_0}\right) + N\left(r, \frac{g'}{g - w_0}\right) + O(\log r) \\ &\leq o\{T(r, g)\} + \overline{N}(r, g) + \overline{N}\left(r, \frac{1}{g - w_0}\right) + O(\log r) \\ &\leq \overline{N}(r, g) + o\{T(r, g)\}, n.e. \end{aligned} \tag{14}$$

Also,

$$\begin{aligned} N\left(r, \frac{1}{F'}\right) &= N\left(r, \frac{1}{Q'(g)}\right) + N\left(r, \frac{1}{g'}\right) \\ &\leq N\left(r, \frac{1}{F}\right) - \overline{N}\left(r, \frac{1}{F}\right) + O(\log r) + N\left(r, \frac{1}{g'}\right). \end{aligned} \tag{15}$$

(14) and (15) give

$$\Delta \leqslant \bar{N}(r, g) + o\{T(r, g)\}, \quad n. e.$$

Noticing (13), we have

$$\bar{N}(r, g) + \bar{N}\left(r, \frac{1}{F}\right) \leqslant \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F}\right) = \bar{N}\left(r, \frac{F'}{F}\right)$$

$$\leqslant \frac{\Delta}{k} \leqslant \frac{1}{k} \bar{N}(r, g) + o\{T(r, g)\}, \quad n. e.$$

Thus we obtain

$$\bar{N}\left(r, \frac{1}{F}\right) \leqslant \left(\frac{1}{k} - 1\right) \bar{N}(r, g) + o\{T(r, g)\}, \ n. e. \tag{16}$$

We claim that Q has no zeros other than w_1. Otherwise, let $w_2 (\neq w_1)$ be a zero of Q. Then by the second fundamental theorem and (16), we have

$$T(r, g) < \bar{N}\left(r, \frac{1}{g-w_0}\right) + \bar{N}\left(r, \frac{1}{g-w_1}\right) + \bar{N}\left(r, \frac{1}{g-w_2}\right) + o\{T(r, g)\}$$

$$\leqslant O(\log r) + \bar{N}\left(r, \frac{1}{F}\right) + o\{T(r, g)\}$$

$$\leqslant \left(\frac{1}{k} - 1\right) \bar{N}(r, g) + o\{T(r, g)\}, \quad n. e. \tag{17}$$

Since $k > 1/2$, from (17) we obviously get a contradiction.

Thus Q has only one simple zero w_1. Also, due to (13), we conclude that Q is a linear polynomial.

Now we consider the case when $Q(\infty) \neq \infty$. Then Q has a finite pole, w^* say. Let $\zeta = \frac{1}{w - w^*}$, and denote

$$Q^*(\zeta) = Q\left(w^* + \frac{1}{\zeta}\right), \ g^*(z) = \frac{1}{g(z) - w^*}.$$

Then $F = Q(g) = Q^*(g^*)$ and $Q(g) \sim Q^*(g^*)$. Clearly, Q^* satisfies $Q^*(\infty) = \infty$, so that $Q^*(\zeta) = c_1\zeta + c_2$ with c_1, c_2 being constants. Therefore

$$Q(w) = Q^*(\zeta) = c_1 \frac{1}{w - w^*} + c_2,$$

i. e. Q is a fractional linear function. The proof is completed.

Proof of Theorim 1 Entirely similar to the proof of Theorem 2. It suffices to notice that when F is entire and (13) is assumed, g must be entire, and $\bar{N}(r, g) = 0$. Therefore, (17) holds for any $k > 0$, which causes a contradiction.

Remark 1. Under the hypotheses of Theorem 1 (or Theorem 2), we can not obtain the primality of F. For example, let

$$F(z) = (z^p - 1)\exp(z^p) \quad (p \geqslant 2).$$

Then f satisfies the hypotheses of Theorem 1, but F is clearly not prime.

Proof of Theorem 3 It follows from Theorem 1 and Lemma 4.

Proof of Theorem 4 It follows from Theorem 2 and Lemma 4.

Remark 2. The function F in the preceding example shows that in Theorem

3 (or Theorem 4) the factorization (3) with (4) may actually occur. Also, if the simultaneous equations (2) in Theorem 3 have at least two solutions, then the conclusion may not be valid. For example, let

$$F(z)=(z^3+z)\exp(z^3+z).$$

Then the simultaneous equations

$$\begin{cases} F(z)=-e^{-1}, \\ F'(z)=0 \end{cases}$$

have three solutions, while F is neither prime nor of the form (3) with (4).

Remark 3. It is easily seen that the condition (1) in all four theorems may be replaced by

$$kT\left(r, \frac{F'}{F}\right) < \Delta + O(\log r), \quad n.\ e. \tag{1'}$$

We shall use this fact in the next section.

§ 4. Application

We shall solve the following

Problem A. Given any transcendental meromorphic function f, does there exist a meromorphic function g such that $f \cdot g$ is prime?

This problem is originally due to Gross-Osgood-Yang[2], who posed it for entire function f. Noda[4] gave the answer to problem A with f entire by proving

Theorem B. *Let f be a transcendental entire function. Then the set*

$$\{a \in \mathbf{C};\ f(z) \cdot (z-a) \text{ is not prime}\}$$

is at most a countable set.

The answer to problem A in general case is apparently contained in the following theorem, which we are about to prove.

Theorem 5. *Let f be a transcendental meromorphic function. Then there exists an integer n such that the set*

$$\{a \in \mathbf{C};\ f(z) z^n (z-a) \text{ is not prime}\}$$

is at most a countable set.

In proving Theorem 5, we shall need two more lemmas.

Lemma 6. *Let F be a meromorphic function. Then there exists a countable set E such that, for any non-zero constant c and $a \in \mathbf{C} - E$, the simultaneous equations*

$$\begin{cases} F(z) \cdot (z-a) = c, \\ [F(z) \cdot (z-a)]' = 0 \end{cases}$$

have at most one solution.

Lemma 6 is an extension of Lemma 2 in [4], and the proof is similar, which should be omitted.

Lemma 7. *Let f be a meromorphic function. Then there exists an integer n and a countable set E_1 such that for any $a\in \mathbf{C}-E_1$, the factorization*

$$(z-a)z^n f(z)=g(P(z))$$

with g being transcendental and P a polynomial implies that P is linear.

Proof We choose an integer n such that $z=0$ is a simple zero of $z^n f(z)$. Denote

$$E_1=\{0\}\cup\{z;\ f(z)=0 \text{ or } \infty\}.$$

Suppose that there are uncountably many complex numbers a in $\mathbf{C}-E_1$ such that

$$F_a(z)=(z-a)z^n f(z)=g(P(z)),$$

where g is transcendental, and P a polynomial of degree $p\geqslant 2$. We may assume $P(o)=0$. Since $g(P(o))=g(o)=0$, each zero of P is a zero of F_a. Also, since 0 and a are simple zeros of F_a, we may write

$$P(z)=z(z-a)^\beta\prod_{j=1}^m(z-z_j),$$

where $\beta=0$ or 1, and z_j, $j=1, \cdots, m$, are zeros of f, $z_j\neq 0$. Obviously, all possible $\{z_j\}_1^m$ forms an at most countable set. Therefore, there exist $a_1, a_2\in\mathbf{C}-E_1$ and a polynomial denoted still by

$$\prod_{j=1}^m(z-z_j)=q(z)\quad\text{(say)},\tag{4.1}$$

such that

$$z^n(z-a_i)f(z)=g_i(z(z-a_i)^\beta q(z)),\quad i=1,\ 2,\ (a_1\neq a_2),$$

where g_1 and g_2 are meromorphic. We deal with two cases separately.

Case a). $\beta=0$. Then

$$\frac{z-a_1}{z-a_2}=\frac{g_1(zq(z))}{g_2(zq(z))}=g^*(zq(z))\quad\text{(say)}.\tag{4.2}$$

Since $\deg(zq(z))\geqslant 2$, $g^*(zq(z))=a_1$ has either no roots or at least two roots, which is impossible by (4.2).

Case b). $\beta=1$. Then there are uncountably many $a\in\mathbf{C}-E_1$ such that

$$z^n(z-a)f(z)=g_a(z(z-a)q(z)),$$

where g_a is meromorphic, and $q(z)$ is of the form (4.1). Since $g_a(o)=0$, we may write $g_a(w)=w\,h_a(w)$ with $h_a(o)\neq\infty$. Hence

$$z^n(z-a)f(z)=z(z-a)q(z)h_a(z(z-a)q(z)).$$

Put

$$H(z)=h_a(z(z-a)q(z))=z^{n-1}f(z)/q(z)$$

and

$$G_a(z)=z(z-a)q(z).\tag{4.3}$$

We see that H does not depend on a, and $H'=h_a'(G_a)G_a'$. Therefore, if $G_a'(x)=0$, then $H'(x)=0$ or x is a pole of f. And from (4.3),

$$a[q(x)+xq'(x)]=2xq(x)+x^2q'(x). \tag{4.4}$$

If $q(x)+xq'(x)\neq 0$, we have

$$a=\frac{2xq(x)+x^2q'(x)}{q(x)+xq'(x)}.$$

This indicates that there are uncountably many x which are either poles of f or zeros of H'. Thus H must be a constant, which implies that f is rational, a contradiction. Therefore, any zero x of G'_a satisfies

$$q(x)+xq'(x)=0.$$

And by (4.4),

$$2xq(x)+x^2q'(x)=0.$$

These two equations imply $xq(x)=0$. If $x\neq 0$, then $q(x)=0$. And by (4.4) $q(x)=0$ for $x=0$. That is to say that each zero of G'_a is a zero of $q(z)$. However, it is easy to verify that there must be a zero of $G'_a(z)=[z(z-a)q(z)]'$ which is not a zero of $q(z)$. This contradiction completes the proof.

Proof of Theorem 5 By Lemma 7, there exists an integer n and a countable set E_1, such that for any $a\in \mathbf{C}-E_1$ the function $F_a(z)=(z-a)z^nf(z)$, which has a simple zero a, can not be factorized into $F_a=g(P)$ with g being transcendental and P a polynomial of degree at least two. Put

$$H(z)=z+\frac{z^nf(z)}{(z^nf(z))'}.$$

It is easy to verify

$$T\left(r,\frac{F'_a}{F_a}\right)\leqslant T(r,H)+O(\log r)$$

and

$$\overline{N}\left(r,\frac{1}{H-a}\right)=\overline{N}\left(r,\frac{(z^nf)'}{F'_a}\right)\leqslant N\left(r,\frac{1}{F'_a}\right)-\left[N\left(r,\frac{1}{F_a}\right)-\overline{N}\left(r,\frac{1}{F_a}\right)\right].$$

Also, using the second fundamental theorem, we know that for $a\neq a_1$, a_2 and a_3 (say),

$$\frac{3}{4}T(r,H)<\overline{N}\left(r,\frac{1}{H-a}\right),\quad n.\ e.$$

Put $E_2=\{H(x);\ H'(x)=0\}$. Then we derive from the proceding three inequalities that for $a\notin E_1\cup E_2\cup\{a_1,a_2,a_3\}=E_0$

$$\frac{3}{4}T\left(r,\frac{F'_a}{F_a}\right)\leqslant N\left(r,\frac{1}{F'_a}\right)-\left\{N\left(r,\frac{1}{F_a}\right)-\overline{N}\left(r,\frac{1}{F_a}\right)\right\}+O(\log r),\quad n.\ e.$$

By Lemma 6, there exists a countable set E_3 such that for any $a\notin E_3$,

$$\begin{cases}F_a(z)=c,\\ F'_a(z)=0\end{cases}$$

have at most one solution for any non-zero constant c. Let $E=E_0\cup E_3$. To $F_a(z)=(z-a)z^nf(z)$ applying Theorem 4 and Lemma 7, we conclude that for any $a\in \mathbf{C}-E_1$. F_a is prime. The proof is completed.

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