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ON THE OPERATOR EQUATION $f(A)=A$ (II)

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Abstract

This paper discusses the existence and uniqueness of the solution to the operator eqnation $f(A)=A$ and generalizes the main results obtained in the author's paper: On the operator equation $f(A)=A$ (Acta Math. Sinica, N. S. 1 (1985), 327—334).

§1.

We follow [1] for notation. For the convenience of the reader, we recall that H denotes a complex Hilbert space and $\mathscr{B}(H)$ is the algebra of all (bounded linear) operators on H. By $\mathscr{A}_H(\Delta)$ or simply $\mathscr{A}(\Delta)$ we mean the set of all operator-valued functions f from the open unit disc Δ in the complex plane $\mathbf{C}$, i. e., $\Delta=\{z\in\mathbf{C}: |z|<1\}$, into $\mathscr{B}(H)$ such that f takes the form

$$f(z)=\sum_{n=0}^{\infty}B_nz^n \text{ for } z\in\Delta, \tag{1}$$

where $\{B_n\}\subset\mathscr{B}(H)$ and the series is convergent in the uniform topology for every z in Δ; in other words, f is an operator function analytic on Δ (cf. [1] or [2]). For $f\in\mathscr{A}(\Delta)$ and $T\in\mathscr{B}(H)$ with $\sigma(T)\subset\Delta$, it is easy to see that

$$f(T)\overset{\text{def.}}{=\!=}\sum_{n=0}^{\infty}B_nT^n$$

is justified.

The purpose of this paper is to take further account of the following problems as in [3]: under what conditions does there exist an operator T in $X_f'=\{A\in\mathscr{B}(H):$ A commutes with f and $\sigma(A)\subset\Delta\}$ such that $f(T)=T$? When is such T unique? And we generalize, among others, most of the results obtained in [3].

§2.

We begin with the following

Proposition 1. *Let $f\in\mathscr{A}(\Delta)$ takes the form* (1) *and satisfy the following conditions:*

(i) $\|f(z)\|<1$ *for every z in $\Delta=\{z: |z|<1\}$;*

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(ii) *B_0 is normal and commutes with f (i. e., $B_0B_n=B_nB_0$ for all $n\geqslant 0$).*

Then $B_1^*B_1<(I-B_0^*B_0)^2$.

Remark. For two selfajoint operators A and B, by $A\geqslant B$ we mean that $A-B$ is positive, i. e., $\langle Ax, x\rangle\geqslant\langle Bx, x\rangle$ for all x in H. And $A>B$ means that $A-B$ is both positive and invertible.

Proof By (i), $\|B_0\|=\|f(0)\|<1$. Define

$$F(z)=(f(z)-B_0)(I-B_0^*f(z))^{-1} \text{ for } z\in\Delta.$$

Since B_0 is normal, by Fuglede's theorem, we have $B_0^*B_n=B_nB_0^*(\forall n\geqslant 0)$. An application of Lemma 5.1 in [1] gives $\|F(z)\|<1(z\in\Delta)$. Clearly, $F(0)=0$, and hence F is of the form $F(z)=zh(z)$ where h is analytic on Δ and $\|h(z)\|<1(z\in\Delta)$ by the maximum modulus principle for analytic vector functions ([2], p. 100). Observing that

$$h(0)=F'(0)=B_1(I-B_0^*B_0)^{-1}$$

one can show, by using Lemma 5.1 in [1] again, that $\|B_1(I-B^*B)^{-1}\|=\|h(0)\|\leqslant 1$ if and only if $B_1^*B_1\leqslant(I-B_0^*B_0)^2$, which completes the proof.

Remark. Here it is not necessary that B_n and B_m commute for all n, m.

Corollary 1.1. *Suppose f is a scalar function analytic on Δ with $|f(z)|<1$ ($z\in\Delta$). If $|f'(0)|=1$, then $f(z)=e^{i\theta}z$ where θ is a real constant.*

Proof Clear.

Corollary 1.2. *Suppose $f\in\mathscr{A}(\Delta)$ is of the form (1) such that $\{B_n\}$ is a sequence of operators commuting pairwise. If $\|f(z)\|<1$ for all z in Δ and $\sigma(B_1)\subset\{z: |z|=1\}$, then each $B_n (n\neq 1)$ is nilponent.*

Proof Let $\mathfrak{A}$ be the commutative Banach subalgebra of $\mathscr{B}(H)$ generated by $\{I, B_n, n\geqslant 0\}$ and let $\mathfrak{M}$ denote the maximal ideal space of $\mathfrak{A}$. Take any m in $\mathfrak{M}$ and define

$$f_m(z)=\sum_{n=0}^{\infty}B_n(m)z^n \quad (z\in\Delta).$$

By the Gelfand representation theorem for commutative Banach algebras, we see that

$$|f_m(z)|=|f(z)(m)|\leqslant\|f(z)\|<1 \quad (z\in\Delta).$$

From Corollary 1.1 above, it follows that $f_m(z)=e^{i\theta}z$ and hence $B_n(m)=0$ for $n\neq 1$. The proof is complete.

Proposition 2. *Suppose $\mathfrak{A}$ is a commutative Banach subalgebra with the identity operator of $\mathscr{B}(H)$ and suppose $f\in\mathscr{A}(\Delta)$ is of the form (1) satisfying the following conditions:*

(i) *$B_n\in\mathfrak{A}$ for all $n\geqslant 0$;*

(ii) *$\|f(z)\|<1$ for all z in Δ;*

(iii) *$1\in\sigma(B_1)$;*

(iv) *there exists an operator T in $X_f=\{A\in\mathfrak{A}: \sigma(A)\subset\Delta\}$ such that $f(T)=T$ and T is normal, that is, T is a normal fixed point for f in X_f.*

Then T is the unique fixed point for f in X_f.

Proof Observe, first, that by the Gelfand representation theorem we have $f(X_f)\subset X_f$ and hence the problem of whether f has fixed points in X_f makes sense. Since T is normal and hence the technique employed in the proof of Theorem 5 in [3] still works, we leave it for the reader to show that T is the unique fixed point for f in X_f. The proof is complete.

Remark. $\mathfrak{A}$ in Proposition 2 above is not necessarily generated by $\{I, B_0, B_1, \cdots\}$ and may be much bigger.

Next theorem is the main result of this note.

Theorem 3. *Suppose $\mathfrak{A}$ is a commutative W^*-algebra and suppose $f\in\mathscr{A}(\Delta)$ is of the form (1) satisfying the following conditions:*

(i) *$B_n\in\mathfrak{A}$ for all $n=0, 1, 2, \cdots$;*

(ii) *$\|f(z)\|<1$ for all z in $\Delta=\{z: |z|<1\}$.*

Then that f has a fixed point in $X'_f=\{A\in\mathfrak{A}': \sigma(A)\subset\Delta\}$ ($\mathfrak{A}'$ denotes the commutant of $\mathfrak{A}$, i. e., $\mathfrak{A}'=\{A\in\mathscr{B}(H): A$ commutes with every element of $\mathfrak{A}\}$) implies that f has a fixed point in $X=\{A\in\mathfrak{A}: \sigma(A)\subset\Delta\}$; and has a unique fixed point T in X'_f if and only if $1\bar{\in}\sigma_p(B_1)$ (the point spectrum of B_1) and $T\in X_f$.

Proof Step I. Assume that $1\bar{\in}\sigma(B_1)$ and T is a fixed point for f in X'_f. We claim that $T\in\mathfrak{A}$ and T is the unique fixed point for f in X'_f.

Let $\mathscr{B}$ be the commntative C^*-algebra generated by $\{I, B_n, \forall n\geqslant 0\}$ and $\mathscr{B}_1$ the commutative Banach algebra generated by T and $\mathscr{B}$, and let $\mathfrak{M}$, $\mathfrak{M}_1$ represent the maximal ideal spaces of $\mathscr{B}$, $\mathscr{B}_1$ respectively. For $m\in\mathfrak{M}$, write $f_m(z)=\sum_{n=0}^{\infty}B_n(m)z^n$ for z in Δ. We show that there exists one and only one point $\lambda_m\in\Delta$ such that $f_m(\lambda_m)=\lambda_m$. Clearly, $\{AB: A\in\mathscr{B}_1, B\in m\}$ is a proper ideal of $\mathscr{B}_1$, containing m. Hence there exists an $m_1\in\mathfrak{M}_1$ with $\{AB: A\in\mathscr{B}_1, B\in m\}\subset m_1$. It is readily seen that $m=m_1\cap\mathscr{B}$ since the latter is a proper ideal of $\mathscr{B}$ with m being included in it. Observing that $\{B_n\}_{n=0}^{\infty}$ is a sequence of normal operators commuting pairwise, we see that

$$B_n(m)=B_n(m_1) \quad \text{for all } n\geqslant 0.$$

Put $\lambda_m=T(m_1)$. Then

$$\lambda_m=T(m_1)=\sum_{n=0}^{\infty}B_n(m_1)T^n(m_1)=\sum_{n=0}^{\infty}B_n(m)\lambda_m^n=f_m(\lambda_m)$$

by the hypothesis that $T=\sum_{n=0}^{\infty}B_nT^n$. Since $1\bar{\in}\sigma(B_1)$, it follows from Theorem 7′ in [3] that λ_m is the unique fixed point for $f_m(z)$ in Δ which is what we want to show. Now define a scalar function ξ on Δ by $\xi(m)=\lambda_m$ where λ_m is the unique fixed point for $f_m(z)$ in Δ. We shall prove that $\xi(m)\in C(\mathfrak{M})$, i. e., $\xi(m)$ is a continuous function on $\mathfrak{M}$. Suppose $\{m_\alpha\}$ is a net in $\mathfrak{M}$ and converges to m in $\mathfrak{M}$. By the discussion above, we have $\{\lambda_{m_\alpha}\}\subset\sigma(T)$. Since $\sigma(T)$ is compact, we may assume that $\lambda_{m_\alpha}\to\lambda$ with no loss of generality. Observe that $\|f(z)\|<1$ $(z\in\Delta)$ implies that $\|B_n\|\leqslant 1$ $(\forall n\geqslant 0)$, and that $r=\lim\|T^n\|^{1/n}<1$. Then

$$\left|\lambda_{m_\alpha}-\sum_{n=0}^{\infty}B_n(m)\lambda^n\right|=\left|\sum_{n=0}^{\infty}B_n(m_\alpha)\lambda_{m_\alpha}-\sum_{n=0}^{\infty}B_n(m)\lambda^n\right|$$
$$<\left|\sum_{n=0}^{N}B_n(m_\alpha)\lambda_{m_\alpha}^n-\sum_{n=0}^{N}B_n(m)\lambda^n\right|+2\sum_{n<N}^{\infty}r^n,$$

which shows that $\lambda=\sum_{n=0}^{\infty}B_n(m)\lambda^n$. Now that $B_1(m)\neq 1$, we see $\lambda=\lambda_m$. Hence $\xi\in C(\mathfrak{M})$. So by appealing to Gelfand's representation theorem, we have $B\in\mathscr{B}$ such that $B(m)=\xi(m)$, or $B(m)=\lambda_m$ for every m in $\mathfrak{M}$, and hence $B=f(B)$. Furthermore, by the trick employed in the proof of Theorem 5 in [3], one can easily show that B is the unique fixed point for f in X'_f and therefore $B=T$, which substantiates our claim at the beginning of this Step. In fact, this is a result obtained in [6].

Step II. Assume that $1\overline{\in}\sigma_p(B_1)$ and T is a fixed point for f in X'_f. As above, we shall show that $T\in\mathfrak{A}$ and T is the unique fixed point for f in X'_f as well.

Let E be the spectral measure of B_1. Write $K=\sigma(B_1)$ and $K_n=\left\{z: z\in K, |1-z|\geqslant\frac{1}{n+2}\right\}$ for $n=0, 1, 2, \cdots$. Set $P_n=E(K_n)$ $(n\geqslant 0)$. From the spectral theorem ([4], p. 67), we have $P_nT=TP_n$ and $P_nB_k=B_kP_n$ for $n, k=0, 1, 2, \cdots$. If we put $f_n(z)=f(z)P_n$ $(\forall n\geqslant 0)$, then

$$f_n(TP_n)=f(T)P_n=TP_n.$$

Note that $1\overline{\in}K_n$ and $\sigma(f'_n(0))=\sigma(B_1P_n)\subset K_n$. By Step I, TP_n is contained in the commutative C^*-algebra $\mathfrak{A}_n$ generated by I and $\mathfrak{A}P_n$, and moreover, TP_n is the unique fixed point for f_n in $X'_n=\{A\in\mathfrak{A}'_n: \sigma(A)\subset\Delta\}$. Since $K-\{1\}=\bigcup_{n=0}^{\infty}(K_n-K_{n-1})$, where $K_{-1}=\varnothing$ and $E(\{1\})=0$ by hypothesis, it follows that

$$I=\lim_{n\to\infty}E(K_n)\ (\mathrm{SOT}),$$
$$T=\lim_{n\to\infty}TP_n(\mathrm{SOT}).$$

Therefore $T\in\mathfrak{A}$. Now assume that there exists another operator S in X'_f such that $f(S)=S$. Then we have $f_n(SP_n)=SP_n$ for all $n\geqslant 1$ as well, and hence $SP_n=TP_n$ since both SP_n and TP_n belong to X'_n and f_n has a unique fixed point in X'_n. Thus

$$S=\lim SP_n=\lim TP_n=T,$$

which substantiates our assertion set forth at the beginning of Step II.

Step III. Suppose $1\in\sigma_p(B_1)$. We shall show that if f has a fixed point T in

X'_f, f has infinitely many fixed points in X'_f.

Let $H_1=\ker(I-B_1)$ and P be the projection of H onto H_1. We claim that for any $\lambda\in\Delta$, $A_\lambda=\lambda P+T(I-P)$ is a fixed point for f in X'_f and $A_\lambda\in\mathfrak{A}$. In fact, for x in H_1 we have

$$B_nx=B_nB_1x=B_1B_nx,$$
$$B_n^*x=B_n^*Bx+BB_n^*x$$

and hence

$$PB_n=B_nP \quad \text{for } n\geqslant 0.$$

Write $g(z)=f(z)|H_1$ $(z\in\Delta)$ the restriction of f on H_1. Since $g'(0)=B_1|H_1=I_{H_1}$, it follows from Corollary 1.2 above that $g(z)=z$, namely, $B_1P=P$, $B_nP=0$ for $n\neq 1$. On the other hand, it is easy to prove in a similar way that H_1 is a reducing subspace of T as well. Then $T(I-P)=f(T)(I-P)=f(T(I-P))$. Since $1\overline{\in}\sigma(f'(0)(I-P))$, we derive from Step I that $T(I-P)$ is an element of the commutative C^*-subalgebra generated by $\{I, B_1, B_2, B_3, \cdots\}$ and hence $T(I-P)\in\mathfrak{A}$. Now it is a simple verification that for any $\lambda\in\Delta$, $A_\lambda=\lambda P+T(I-P)$ is a fixed point for f in X'_f and $A_\lambda\in\mathfrak{A}$ (note that $\mathfrak{A}$ is a W^*-algebra), which is the claim. In fact, the proof is complete.

Let $\prod$ denote the open right half-plane, i. e., $\prod=\{z: \mathrm{Re}z>0\}$. In the following corollary we are concerned with operator functions f analytic on $\prod$ and $f(T)$ where $T\in\mathscr{B}(H)$ and $\sigma(T)\subset\prod$. For the definitions and related results the reader is referred to [1].

Corollary 3.1. *Suppose $\mathfrak{A}$ is a commutative W^*-algebra and suppose $f\in\mathscr{A}(\prod)$ satisfies the following conditions:*

(i) *$f(z)\in\mathfrak{A}$ for every z in $\prod$;*

(ii) $\mathrm{Re}f(z)=(f(z)+f^*(z))/2>0(z\in\prod)$.

Then f has fixed points in $X'_f=\{A\in\mathfrak{A}': \sigma(A)\subset\prod\}$ if and only if f has fixed points in $X_f=\{A\in\mathfrak{A}: \sigma(A)\subset\prod\}$ and f has a unique fixed point in X'_f if and only if f has a fixed point in X_f and $1\overline{\in}\sigma_p(4f'(1)-f(1))$.

Proof By Theorem 3 above, one can employ the technique in the proof of Corollary 5.1 in [3] to show the desired results.

Generally speaking, a function f under the above consideration may have no fixed points in question. In what follows we give a very general condition on f under which a fixed point exists.

Theorem 4. *Let $\mathfrak{A}$, f, and X be as in Theorem 3 above. If there exists a positive number $r<1$ so that*

$$\mathrm{Max}\{\|f(z)\|: |z|=r\}\leqslant r$$

holds, then f has a fixed point T in X_f.

Proof Case *I*: $1\overline{\in}\sigma(B_1)$.

Let $\mathscr{B}$ be the commutative C^*-algebra generated by $\{I, B_n, n\geqslant 0\}$ and $\mathfrak{M}$ the maximal idael space of $\mathscr{B}$. For this case, we shall show, indeed, that there exists an operator T in $\mathscr{B}$ such that $\sigma(T)\subset\Delta$ and $f(T)=T$. For any m in $\mathfrak{M}$, put

$$f_m(z)\overset{\text{def.}}{=\!=}\sum_{n=0}^{\infty}B_n(m)z^n \text{ for } z\in\Delta.$$

Then f_m is a scalar analytic function on Δ such that

$$\max\{|f_m(z)|: |z|=r\}\leqslant r<1.$$

Since f_m is continuous on $\{z: |z|\leqslant r\}$ and $|f_m(z)|\leqslant r$ for $|z|\leqslant r$, it follows from the well-known Brouwer fixed point theorem ([5], p. 468) that f_m has a fixed point λ_m in $\{z: |z|\leqslant r\}$. However, that $|f_m(z)|<1$ $(z\in\Delta)$ and $f'(0)=B_1(m)\neq 1$ implies that λ_m is the unique fixed point for f_m in Δ ([3], Theorem 7′). Define

$$g(m)=\lambda_m \text{ for } m\in\mathfrak{M}.$$

We claim that g is a continuous function on $\mathfrak{M}$. Indeed, this is a simple verification, and one can do it in the same way as in Step I of the proof of Theorem 3 above. Hence, by the Gelfand representation theorem, there exists an operator T in $\mathscr{B}$ such that $T(m)=\lambda_m$ for every m in $\mathfrak{M}$. Thus

$$T(m)=\sum_{n=0}^{\infty}B_n(m)T^n(m)\ (m\in\mathfrak{M}),$$

or

$$T=f(T).$$

We consider, next, Case II: $1\overline{\in}\sigma_p(B_1)$ and then Case III: $1\in\sigma_p(B_1)$ as in the proof of Theorem 3 above. So we leave it for the reader. The proof is complete.

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