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ON QUINTIC K-3 SURFACES**

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Abstract

If a non-normal quintic surface is birational to a $K3$ surface, then there are three possibilities: either it is singular along a conic; or it is singular along two mutually intersecting lines; or it is singular along a line and has an isolated triple point outside the line. Conversely if a $K3$ surface contains a hyperelliptic curve of genus three with a node or simple cusp, then it is birational to a quintic surface of the first type mentioned above. For the other two cases, the minimal models are also characterized.

A K-3 surface is a regular (complex) surface with trivial canonical bundle. The simplest classical example is a nonsingular quartic surface in $\mathbf{P}^3$. Some singular quintic surfaces aro also $K3$. In [8], the quintic $K3$ surfaces with isolated singularities were discussed. They might have upto three triple points. Their minimal models were characterized by the existence of certain special divisors. The aim of this paper is to find the minimal models of quintic $K3$ surfaces which are singular in codimension one. The main result is that there are essentially three kinds of such quintic surfaces. The first kind of quintic surfaces is singular along a conic. The second kind is singular along two coplanar lines and the third kind is singular along a line and has an isolated triple point away from that line.

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§ 1. Projective Maps of $K3$ Surfaces

In this section we are going to discuss the sufficient conditions for a $K3$ surface to be birational to a quintic surface which is singular in codimension one.

Theorem 1. *Let X be a $K3$ surface. Assume that there is a hyperelliptic curve C of geometric genus 3 which has a node or simple cusp p as its only singularity. Then X is birational to a quintic surface in $\mathbf{P}^3$ which is singular along a conic.*

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Here C being hyperelliptic means that the normalization of C has a g_2^1.

Proof Let X' be the blowing-up of X at the point p. Let E be the exceptional divisor and D be the proper transform of C. Since X is $K3$, E is the canonical divisor of X'. Let p_1 and p_2 be the intersection points of E and D. It is possible that $p_1=p_2$. The adjunction formula implies that $D^2=2$. Take the divisor $A=E+D$. Then $A^2=5$ since $ED=2$. We will show that the complete linear system $|A|$ gives rise to the desired birational map.

First we have the exact sequence

$$0 \to O(E) \to O(D+E) \to O_D(D+E) \to 0. \tag{1}$$

Since X is regular by the definition of $K3$ surfaces, $H^1(O_X)=0$. Thus $H^1(O(E))=0$ by Serre's duality. We have $h^0(O(E))=h^2(O(E))=1$ for E is the canonical divisor. The adjunction formula implies that the restriction of the divisor $D+E$ on D is the canonical divisor of D. Thus $h^0(O_D(D+E))=3$ and $h^1(O_D(D+E))=1$. Serre's Duality implies that $h^2(O(D+E))=h^0(O(-D))=0$. Hence (1) implies that

$$h^0(O(D+E))=4 \tag{2}$$

and $h^1(O(D+E))=0$.

Lemma 1. $h^0(O(D))=2$.

Proof Serre's Duality implies that $h^2(O(D))=h^0(O(E-D))=0$. By the Riemann-Roch theorem, $h^0(O(D))\geqslant 2$. On the other hand, the exact sequence

$$0 \to O \to O(D) \to O_D(D) \to 0$$

implies that $h^0(O(D))\leqslant 3$, for $h^0(O_D(D))\leqslant 2$ by Clifford's theorem ([5, p. 343]). Suppose $h^0(O(D))=3$. Then $h^0(O_D(D))=2$ and the map $H^0(O(D))\to H^0(O_D(D))$ is surjective. The unique g_2^1 is the restriction of $|D|$ on D. Thus the complete linear system $|D|$ has no base points on D. Since the restriction of $D+E$ on D is the canonical divisor of D, which is twice of the g_2^1, the divisor p_1+p_2 also belongs to the g_2^1. So any member of $|D|$ passing through p_1 must pass through p_2 as well. But a generic member of $|D|$ meets E at two points different from p_1 and p_2. Hence $|D|$ has no base points on E. Therefore $|D|$ is base point free. Let Φ_D be the morphism decided by $|D|$. It is a morphism into $\mathbf{P}^2$. Obviously, $\Phi_D(D)$ and $\Phi_D(E)$ are distince lines on $\mathbf{P}^2$. The map from E to its image is a double cover. By the canonical resolution of double coverings (see, e. g., [4]), there is a double covering $\pi: X'\to Y$, where Y is obtained from a sequence of blowing-ups of $\mathbf{P}^2$. Since $\pi: E\to\pi(E)$ is a double cover, $\pi(E)$ is not in the branch locus of π. So E^2 must be an even integer. This contradicts $E^2=-1$.

Therefore $h^0(O(D))=2$.

Next we show that the complete linear system $|A|=|D+E|$ has no base point.

Since $h^0(O(E))<h^0(O(A))$, D is not a fixed component of $|A|$. Since $h^0(O(D))<h^0(O(A))$, E is not a fixed component of $|A|$. Therefore $|A|$ has no fixed components.

Applying Lemma 1 to the exact sequence

$$0\to O(D)\to O(D+E)\to O_E(1)\to 0$$

we see that the map $H^0(O(D+E))\to H^0(O_E(1))$ is surjective. Thus $|A|$ has no base points on E. For the same reason, (1) implies that $|A|$ has no base points on D.

Thus far we have proved that $|A|$ is base point free. Hence the linear system $|A|$ induces a morphism from X' into $\mathbf{P}^3$. (Recall that $h^0(O(A))=4$ by (2).) Denote this morphism by Φ. Since $AE=1$, $\Phi(E)$ is a line. Since Φ induces the canonical map of tne hyperelliptic curve D of genus 3, $\Phi(D)$ is a conic. Since $\Phi(E)$ and $\Phi(D)$ are different curves, $\Phi(X')$ must be a surface. Since $A^2=5$, which is a prime number, Φ is a birational morphism onto a quintic surface. Since $\Phi:D\to\Phi(D)$ is a double cover, the quintic surface is singular along the conic $\Phi(D)$.

Theorem 2. *Let X be a K3 surface with two nonsingular elliptic curves C_1 and C_2 intersecting with each other at three distinct points p_1, p_2 and p_3. Then X is birational to a quintic surface singular along two lines crossing each other.*

Proof Let X' be the blowing-up of X at the point p_3. Let E be the exceptional divisor, D_1 and D_2 be the proper transforms of C_1 and C_2 respectively. Take the divisor $A=E+D_1+D_2$. We show that the complete linear system $|A|$ gives rise to the desired morphism.

Since E is the canonical divisor of X', a simple application of the adjunction formula reveals that $D_i^2=-1$ for $i=1, 2$. Thus $A^2=5$.

Since $D_1^2=-1$, $h^0(O(D_1))=1$. By the Riemann-Roch theorem, we also have $h^1(O(D_1))=0$. Then the exact sequence

$$0\to O(D_1)\to O(D_1+D_2)\to O_{D_2}(D_1+D_2)\to 0$$

implies that $h^0(O(D_1+D_2))=2$ and $h^1(O(D_1+D_2))=0$. Finally the exact sequence

$$0\to O(D_1+D_2)\to O(A)\to O_E(1)\to 0$$

implies that $h^0(O(A))=4$.

Since the map $H^0(O(A))\to H^0(O_E(1))$ is surjective, $|A|$ has no base points on E. The exact sequence

$$0\to O(D_1)\to O(D_1+E)\to O_E\to 0$$

shows that $h^0(O(D_1+E))=2$ and $h^1(O(D_1+E))=0$. And the exact sequence

$$0\to O(E+D_1)\to O(A)\to O_{D_2}(A)\to 0$$

implies that the map $H^0(O(A))\to H^0(O_{D_2}(A))$ is surjective. Thus $|A|$ has no base points on D_2. By symmetry $|A|$ has no base points on D_1 either. Therefore the complete linear system $|A|$ is base point free.

Let $\Phi: X' \to \mathbf{P}^3$ be the morphism defined by $|A|$. Since the subsystem of $|A|$ containing D_1 has codimension 2, the image of D_1 is contained in a line L_1 in $\mathbf{P}^3$. But $D_1 A = 2$. So the map $\Phi: D_1 \to L_1$ is a double cover. For the same reason Φ induces a double cover from D_2 onto a line L_2. Since

$$h^0(O(D_1)) = h^0(O(D_2)) < h^0(O(D_1 + D_2)),$$

L_1 and L_2 are distinct. Thus $\Phi(X')$ must be a surface. Since $A^2 = 5$, Φ must bea birational morphism onto a quintic surface which is singular along $L_1 + L_2$.

Theorem 3. *Let X be a $K3$ surface with two curves C_1 and C_2 on it satisfying the following conditions:*

i) *C_1 is a nonsingular curve of genus 2;*

ii) *C_2 is a nonsingular elliptic curve;*

iii) *C_1 and C_2 intersect at three diftinct points. Then X is birational to a quintic surface which is singular along a line and has an isolated triple point.*

Proof (Sketch) Let X' be the blowing-up of X at the three intersection points of C_1 and C_2. Let E_1, E_2, E_3 be the exceptional divisors and let D_1, D_2 be the proper transforms of C_1 and C_2 respectively. Using the adjunction formula, we have $D_1^2 = -1$ and $D_2^2 = -3$. Take $A = E_1 + E_2 + E_3 + D_1 + D_2$. Then $A^2 = 5$.

Using the similar method as in the proof of Theorem 2, one can easily see that the complete linear system $|A|$ is base point free and it defines a birational morphism Φ from X' onto a quintic surface in $\mathbf{P}^3$.

Since $AE_i = 1$ for $i = 1, 2, 3$, the images of E_1, E_2, E_3, are lines. These three lines are distinct. The map Φ induces a double cover from C_1 onto another lino. Hence $\Phi(X')$ is singular along $\Phi(C_1)$. Since $AD_2 = 0$, $\Phi(D_2)$ is a point p and D_2 is the only curve contracted to p under Φ. Hence p is an isolated triple point on the quintic surface (cf. [7]).

§ 2. Quintic Surfaces Singular in Codimension One

In this section we reverse the direction of the discussion. Given a quintic surface singular in codimension one, we want to find the conditions for the quintic surface to be birational to a $K3$ surface. Here we restrict the discussions to the generic cases. In particular, all $K3$ surfaces mentioned in Section 1 do exist.

Let X_0 bo a quintic surface in $\mathbf{P}^3$. Let C be the union of all curves along which X_0 is singular. Since the case that X_0 is normal is already discussed in [8], here we always assume that C is not empty. As a matter of fact, C is a reduced curve. We call C the singular locus of X_0 of codimension 1. Let C_i be an irreducible component of C. Then the multiplicity of X_0 at the generic point of C_i is called the multiplicity of C_i.

Lemma 2. *If the singular locus of codimension* 1 *of a quintic surface* X_0 *contains a component of multiplicity greater than* 2, *then* X_0 *is birational to a ruled surface.*

Proof Let C' be an irreducible curve of multiplicity $\geqslant 3$ on X_0. First assume that C' is not a line. Take two distinct points in general position on C'. Let L be the line passing through these two points. Then the generic plane passing through L cuts X_0 at an irreducible plane quintic curve with two n-tuple points, where $n\geqslant 3$ is the multiplicity of C'. So n must be 3 and the quintic curve is rational. Hence the projection with center at L gives rise to a rational map from X_0 onto $\mathbf{P}^1$ whose generic fiber is a rational curve. Thus X_0 is a ruled surface.

Next assume that C' is a line. Then the intersection of X_0 and a generic plane passing through C' is the union of C' (of multiplicity 3 or 4) and a rational curve. Hence the projection with center at C' gives rise to a rational map from X_0 onto $\mathbf{P}^1$ whose generic fiber is a rational curve. Therefore X_0 is a ruled surface.

Lemma 3. *If a quintic surface* X_0 *singular in codimension one is birational to a* K3 *surface, then the singular locus of codimension one must be a planar curve.*

Proof Let $C=\sum_{i=1}^{r} C_i$ be the singular locus of codimension one, with irreducible components $C_1, \cdots, C_r$. According to Lemma 2, the multiplicity of each C_i is equal to 2. Let $\pi: T\to\mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ with center at C. Let $E=\pi^{-1}(C)$ be the exceptional divisor. Then the canonical divisor K_T is linearly equivalent to $\pi^*(K_{\mathbf{P}^3})+E$. Let X be the proper transform of X_0 in T. As a divisor in T, X is linearly equivalent to $\pi^*(X_0)-2E$. Let H be a hyperplane in $\mathbf{P}^3$. Then the divisor K_T+X is linearly equivalent to $\pi^*(H)-E$. Let X' be the minimal resolution of X. By the adjunction formula, for $h^0(O(K_{X'}))>0$ it is necessary that $\pi^*(H)-E$ is effective for some hyperplane H in $\mathbf{P}^3$. In other words, C is necessarily a planar curve.

Lemma 4. *Let* S *be a nonsingular surface with an effective divisor* D *which is linearly equivalent to the canonical divisor of* S. *Suppose that every connected component of* D *is a nonsingular rational curve. Then all these components are* (-1)*-curves.*

Proof Let E be a component of D. Then $2E^2=E^2+EK_S=-2$ by the adjunction formula. Hence $E^2=-1$.

Lemma 5. *Every irreducible curve on an Abelian surface has non-negative self-intersection.*

Proof Let C be an irreducible curve on an Abelian surface S. Since $CK_S=0$, the adjunction formula implies that $C^2<0$ if and only if C is a rational curve with $C^2=-2$. Suppose there were one such curve. Then any translation of C would also be a (-2)-curve. This would contradict a well-known fact that a complete surface has only finitely many (-2)-curves.

Definition. *An effective divisor D on a nonsingular surface is called numerically n-connected if $D_1D_2\geqslant n$ for all divisors $D_1>0$, $D_2>0$ with $D_1+D_2=D$.*

Vanishing Theorem ([2, p. 178]) *If an effective divisor D on a surface S is numerically 1-connected with $D^2>0$, then $H^1(S, O_S(-D))=0$.*

We briefly mention some standard notions concerning isolated singularities of surfaces. For details see [1], [6] etc.

Let p be an isolated singularity of a surface V and let $\pi: M\to V$ be the minimal resolution of p. The set $A=\pi^{-1}(p)$ is called the exceptional set of p. Write $A=\bigcup_{i=1}^{n} A_i$, where $A_1, \cdots, A_n$ are the irreducible components of A. A cycle on A is an integral combination of the A_i's. There is a natural partial ordering, denoted by $<$, among all cycles. There is a unique cycle Z, called the fundamental cycle, satisfying

i) $ZA_i<0$ for all i;

ii) $Z\leqslant Y$ for any cycle Y such that $YA_i\leqslant 0$ for all i.

If $\chi(Z)=0$ then p is a rational point. If $\chi(Z)=1$ then p is called a weakly elliptic point. An ordinary isolated triple point is a weakly elliptic point. An isolated singularity is called essential if it is not a rational double point.

Let X_0 be a quintic surface with singular locus C of codimension one. Denote the equation for X_0 by $f(x, y, z)=0$ under the affine coordinates. According to Lemma 2 and Lemma 3, in order that X_0 is $K3$ we only need to discuss the following three cases:

Case 1 C is a conic.

Without loss of generality, we may assume that C is the zeros of the equations $z=0$ and $g(x, y)=0$, where $g(x, y)$ is an irreducible quadratic equation.

The equation $f(x, y, z)=0$ may be written as

$$z^2a(x, y, z)+zg(x, y)b(x, y)+g^2(x, y)c(x, y)=0, \tag{3}$$

where a, b ond c are polynomials of degrees 3, 2 and 1 respectively. Let H be the x, y-plane. Then the canonical divisor of the minimal resolution of X_0 is cut otu by H. So X_0 has no essential isolated singularities outside H. On the other hand, (3) also shows that X_0 has no singularities on $H-C$. Hence X_0-C has no essential isolated singularities. For the simplicity of discussion, we ignore the rational double points. From (3) we also see that $H\cap X_0$ is the union of C and the line L given by $c(x, y)=0$.

Let $\Delta(x, y)=b^2(x, y)-4a(x, y, 0)c(x, y)$. Let $\pi: T\to \mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ with center at C. Let E be the exceptional divisor $\pi^{-1}(C)$ and let X be the proper transform of X_0. Then π induces a map from X to X_0. Let $\tilde{C}$ be the inverse image of C in X. Let H' be the proper transform of H in T. Since H intersects X_0 along C with multiplicity two, $\tilde{C}$ is not contained in H'. Thus $H'\cap X$ is the proper transform L' of L.

We are interested in the case where $a(x, y, z)$, $b(x, y)$, $c(x, y)$ are sufficiently general. So we may assume that $\Delta(x, y)$ is not identically equal to zero. Then the map $\pi:\tilde{C}\to C$ is a double cover. The ramificatoin divisor is decided by the common solutions of equations $\Delta(x, y)=0$ and $g(x, y)=0$. There are 8 distinct solutions due to the generality of a, b, c. Thus the eight ramification points are all distinct, and none of them is on the line L. Hurwitz's formula implies that $\tilde{C}$ is a nonsingular curve of genus 3 and $L'\ \tilde{C}=2$. Since $\pi:\tilde{C}\to C$ is a double cover over a rational curve, $\tilde{C}$ admits a g_2^1. The two intersection points of L' and $\tilde{C}$ are mapped to distinct points. So these two points do not belong to the g_2'. Now X is nonsingular, so L' is a (-1)-curve by Lemma 4. Let $\xi:X\to S$ be the blowing-down of L'. Then $\xi(C')$ is a noded curve of geometric genus 3. The canonical divisor of S is zero. According to the classification theory of surfaces[3], S is either a $K3$ surface or an abelian surface.

Suppose that S were an abelian surface. Let Q_0 be a generic hyperplane section of X_0 and let Q be the proper transform of Q_0 in X. Then Q is a nonsingular curve meeting L' at one point. Thus the genus of Q is 4 for $Q^2+QK_X=5+1=6$. Let $D=\xi(Q)$. Then D is a nonsingular curve of genus 4 on S. We also have

$$h^0(S,\ O_S(D))\geqslant h^0(\mathbf{P}^3,\ O(1))=4.$$

We claim that D is numerically 1-connected. Assume that $D_1>0$, $D_2>0$ and D_1+D_2 is linearly equivalent to D. By Lemma 5 the intersection number of any two effective divisors on S is nonnegative. If $D_1D_2>0$ then we are done. If $D_1D_2=0$, then $D_iD>0$ since every curve in X_0 meets a hyperplane section. So $S_i^2=D_iD>0$ for $i=1$, 2, which contradicts the algebraic index theorem. Thus D is numerically 1-connected.

By the vanishing theorem we have $h^1(S,\ O_S(-D))=0$. By Serre's duality $h^2(S,\ O_S(-D))=h^0(S,\ O_S(D))\geqslant 4$. Hence $\chi(O_S(-D))\geqslant 4$. From the standard sequence $0\to O(-D)\to O\to O_D\to 0$ we have $\chi(O_S)=\chi(O_S(-D))+\chi(O_D)\geqslant 1$. But we know that $h^0(O_S)=h^2(O_S)=1$. Hence $h^1(O_S)\leqslant 1$. This contradicts the assumption that S is abelian.

The above discussion is summarized as the following.

Theorem 4. *In the collection of all quintic surfaces singular along a conic, a general member is birational to a K3 surface satisfying the conditions in Theorem 1.*

Case 2. C is the union of two lines crossing each other.

Without loss of generality, we may assume that C is the zeros of the equations $z=0$ and $xy=0$.

The equation $f(x, y, z)=0$ may be written as

$$z^2a(x, y, z)+zxyb(x, y)+x^2y^2c(x, y)=0, \tag{4}$$

where a, b and c are polynomials of degrees 3, 2 and 1 respectively. Let H be the x, y-plane. Then the canonical divisor of the minimal resolution of X_0 is cut out by H. From (4) we also see that $H\cap X_0$ is the union of C and the line L given by $c(x, y)=0$. Denote the lines $x=0$ and $y=0$ on H by C_1 and C_2 respectively. By the generality of a, b, c the three lines L, C_1 and C_2 have no common points.

Let $\pi: T\to \mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ with center at C. Let E be the exceptional divisor $\pi^{-1}(C)$ and let X be the proper transform of X_0. Then π induces a map from X to X_0. Let $\widetilde{C}_i$ be the inverse image of C_i in X for $i=1$, 2. Let H' be the proper transform of H in T. Since H intersects X_0 along C_i with multiplicity two, $\widetilde{C}_i$ is not contained in H' for $i=1$, 2. Thus $H'\cap X$ is the proper transform L' of L.

We are interested in the case where $a(x, y, z)$, $b(x, y)$, $c(x, y)$ are sufficiently general. So the map $\pi:\widetilde{C}_1\to C_1$ is a double cover. The ramification divisor is decided by the common solutions of equations $\Delta(x, y)=0$ and $x=0$. There are 4 distinct solutions due to the generality of a, b, c. Hurwitz's formula implies that $\widetilde{C}_1$ is a nonsingular elliptic curve and $E'\widetilde{C}_1=1$. For the same reason $\widetilde{C}_2$ is also a nonsingular elliptic curve with $L'\widetilde{C}_2=1$. Furthermore, the inverse image of $C_1\cap C_2$ consists of two points. Hence $\widetilde{C}_1\widetilde{C}_2=2$.

Now X is nonsingular, so L' is a (-1)-curve by Lemma 4. After shrinking L' we get a minimal surface S. The same argument as in Case 1 shows that $h^1(O_S)=0$. So S is a $K3$ surface satisfying the conditions in Theorem 2.

The above discussion is summarized as the following

Theorem 5. *In the collection of all quintic surfaces singular along two coplanar lines, a general member is birational to a $K3$ surface with two elliptic curves intersecting at three points.*

Case 3. C is a line.

Without loss of generality, we may assume that C is the zeros of the equations $z=0$ and $x=0$.

The equation $f(x, y, z)=0$ may be written as

$$z^2a(x, y, z)+zxb(x, y)+x^2c(x, y)=0, \tag{5}$$

where a, b and c are polynomials of degrees 3. If $a(x, y, z)$, $b(x,y)$, $c(x,y)$ are generic, then the surface woull not be $K3$, because all hyperplanes passing through C would cut out an effective divisor linearly equivalent to the canonical divisor of the minimal resolution of the quintic surface. So there should be some conditions on a, b, c for X_0 to be $K3$.

Subcase 3.1. Blowing-up at C does not resolve the codimension one singularity.

Since the multiplicity of C is 2, this case happens when there is a plane H intersecting X_0 at C with multiplicity at least 4. We may assume that H is the x, y-plane. Then equation (5) can be written as

$$z^2a(x, y, z)+zx^2d(x, y)+x^4e(x, y)=0, \qquad (6)$$

where $d(x, y)$, $e(x, y)$ are polynomials of degrees 2 and 1 respectively. Let L be the line $e(x, y)=0$ on H. Then $H\cap X_0=C\cup L$.

Let $\pi:T\to \mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ with center at C. Let E be the exceptional divisor $\pi^{-1}(C)$ and let X be the proper transform of X_0. Let H' be the proper transform of H in T. For general $a(x, y, z)$, the equation $a(0, y, 0)$ has three distinct roots. So $X\cap E$ is the union of four rational curves M, M_1, M_2 and M_3, where M is also contained in H' and $MM_i=1$, $M_iM_j=0$ for $i\neq j$. By (6) X is singular along M. Let $\sigma:T'\to T$ be the blowing-up of T with center at M. Let E' be the exceptional divisor $\sigma^{-1}(M)$ and let X' be the proper transform of X. Let $H^\wedge$ be the proper transform of H' in T'.

Assume that $a(x, y, z)$, $d(x, y)$, $e(x, y)$ are sufficiently general. Then X' is nonsingular. Let D be the curve $X'\cap E'$. Then $\sigma\pi:D\to C$ is a double cover ramified at 4 points, for the ramification is decided by the equation

$$d^2(0, y)-4a(0, y, 0)e(0, y)=0.$$

Hence D is an elliptic curve.

Let F be the proper transform of L in X' and let E_1, E_2, E_3 be the proper transforms of M_1, M_2, M_3 respectively. Then F, E_1, E_2, E_3 are mutually disconnected while each of them meets D at exactly one point.

The canonical divisor of X' is cut out by H', which is the intersection of $H^\wedge$ and X'. By the above construction we see that $H^\wedge\cap X'=F$. By Lemma 4 the curve F is a (-1)-curve. Let $X'\to S$ be the blowing-down of S.

Suppose that S were an abelian surface. Let Q_0 be a generic hyperplane section of X_0 and let Q be the proper transform of Q_0 in X'. Then Q is a nonsingular curve meeting F at one point. Thus the genus of Q is 4 for $Q^2+QK_{X'}=5+1=6$. Let D be the image of Q in S. We also have $h^0(S, O_S(D))\geqslant h^0(\mathbf{P}^3, O(1))=4$.

The same argument as in Case 1 shows that D is numerically 1-connected. By the vanishing theorem we have $h^1(S, O_S(-D))=0$. By Serre's duality

$$h^2(S, O_S(-D))=h^0(S, O_S(D))\geqslant 4.$$

Hence $\chi(O_S(-D))\geqslant 5$. From the standard sequence $0\to O(-D)\to O\to O_D\to 0$ we have $\chi(O_S)=\chi(O_S(-D))+\chi(O_D)\geqslant 2$. But we know that $h^0(O_S)=h^2(O_S)=1$. Hence $h^1(O_S)=0$. This contradicts the assumption that S is abelian.

Therefore S is a $K3$ surface. Let Z, Z_1, Z_2, Z_3 be the images of D, E_1, E_2, E_3 in S respectively. It is easy to see that $h^0(O_S(Z+Z_1+Z_2+Z_3))>1$ and $(Z+Z_1$

$+Z_2+Z_3)^2=0$. Bertini's theorem implies that there is a nonsingular elliptic curve Y linearly equivalent to $Z+Z_1+Z_2+Z_3$. Obviously $ZY=3$. Hence S is a $K3$ surface satisfying the conditions in Theorem 2.

Subcase 2 Blowing-up X_0 at C resolves the singularity of codimension one.

In this case X_0 should have at least one essential singularity in order to be $K3$. All essential singularities together with C should be on a plane H because otherwise there would be no effective canonical divisor on the minimal resolution of X_0. Since $H\cap X_0$ is the union of C and a curve of degree 3, X_0 has at most one isolated singularity on H. Hence we only need to consider the case that X_0 is singular along C and at a point p (not belonging to C), and nonsingular elsewhere.

Lemma 6. *Let X_0 be a quintic surface which is singular along a line L. If X_0 has one essential isolated double point p away from L, then X_0 is not $K3$.*

Proof Let $\phi:T\to\mathbf{P}^3$ be the blowing-up of $\mathbf{P}^3$ at the point p and let E be the exceptional plane. Let X be the proper transform of X_0. The canonical divisor K_T of T is $\phi^*(K_{\mathbf{P}^3})+2E$ and the divisor X in T is linearly equivalent to $\phi^*(X_0)-2E$. Thus K_T+S is linearly equivalent to $\phi^*(H)$ where H is a hyperplane in $\mathbf{P}^3$.

Suppose that X were birational to a $K3$ surface. Then the effective canonical divisor of the minimal resolution X' of X is cut out by the hyperplane H_0 spanned by p and L in $\mathbf{P}^3$. Since X has at most double points or double curves on E, the canonical divisor of X' contains the exceptional set A of the double point p. Since X' is birational to a $K3$ surface, A should be part of the exceptional set of a smooth point on a $K3$ surface, which contradicts the assumption that p is an essential singularity. Therefore X_0 cannot be $K3$.

By this lemma we may assume that p is a triple point of X_0. The intersection of H and X_0 is $2C+L_1+L_2+L_3$ where L_1, L_2 and L_3 are three lines passing through p. In the generic case these three lines are distinct and the triple point p is ordinary. In other words the exceptional set of p is a nonsingular elliptic curve of self intersection -3.

Let X be the minimal resolution of X_0. Let D, M_1, M_2, M_3 be the inverse images of C, L_1, L_2, L_3 in X respectively. Let E be the exceptional curve in X. Using the similar argument as in the previous cases we can easily see that D is a double cover of C ramified at 6 points Thus, D is a nonsingulvr curve of genus 2. The configuration of the five curves are easily decided: $M_iM_j=0$ for $i\neq j$, $M_iE=1$ and $M_iD=1$ for $i=1, 2, 3$. The canonical divisor of X is $M_1+M_2+M_3$. By Lemma 4, M_1, M_2 and M_3 are (-1)-curves. Let $\phi:X\to S$ be the contraction of these three curves. Then $K_S=0$. Using an argument similar to that in Case 1, we can show that $h^1(O_S)<2$. Thus S cannot be an abelian surface, for the first cohomology of an abelian surface has dimension 2. Therefore S is a $K3$ surface satisfying the conditions in Theorem 3.

Therefore we have

Theorem 6. *In the collection of all quintic surface which has an isolated triple point and has a line as its singular locus of codimension one, a general member is birational to a $K3$ surface which has a nonsingular curve of genus* 2 *and a nonsingular elliptic curve intersecting at three points.*

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