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STRONG LIMIT THEOREMS FOR BLOCKWISE *m*-DEPENDENT RANDOM VARIABLES AND A GENERALIZATION OF THE CONJECTURES OF MÓRICZ**

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Abstract

The authors introduce the definition of block-wise *m*-dependent with respect to some positive real sequence for random variables taking values in a separable Banach space, prove some general results on the strong convergence of blockwise *m*-dependent random variables, and in particular, give positive answers to Móricz conjectures ([5]) by a sharper theorem.

§ 1. Introduction

The purpose of this paper is to prove some results on strong limit theorems for blockwise *m*-dependent seqences of *B*-valued random variables which appear to be new even for real-valued random variables. In particular, our results will settle two conjectures raised by Móricz ([5]) when the random variables take values in *R*. Before stating our results precisely we provide some background and motivation for these matters.

Definition [5]. *If $\{X_n;\ n\geqslant 1\}$ is a sequence of random variables (in abbreviation: r. v. 's) and m is a nonnegative integer, $\{X_n;\ n\geqslant 1\}$ is termed blockwise m-dependent if for each p large enough, the two sets*

$$\{X_{2^{p-1}+1}, \cdots, X_k\} \text{ and } \{X_l, X_{l+1}, \cdots, X_{2^p}\}$$

of r. v. 's are independent provided $2^{p-1}<k<l\leqslant 2^p$ and $l-k>m$.

Difinition 2 [5] *If $\{X_n;\ n\geqslant 1\}$ is a sequence of r. v. 's satisfying $EX_n=0$ and $EX_n^2<+\infty$ $(n\geqslant 1)$, $\{X_n;\ n\geqslant 1\}$ is termed blockwise quasiorthogonal if for each $p\geqslant 1$ there exists a nonrandom sequence $\{f_p(j);\ j=0, 1, \cdots, 2^{p-1}-1\}$ such that*

$$|E(X_kX_l)|\leqslant f_p(|k-l|)\sqrt{EX_k^2}\sqrt{EX_l^2} \quad (2^{p-1}<k,\ l\leqslant 2^p) \tag{1.1}$$

and

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$$\sum_{j=0}^{2^{p-1}} f_p(j) \leqslant C, \tag{1.2}$$

where C denotes a positive absolute constant.

In [5] Móricz proved the following results.

Theorem A. *If $\{X_n;\ n\geqslant 1\}$ is blockwise m-dependent with $EX_n=0$, $EX_n^2<+\infty (n\geqslant 1)$, then*

$$\sum_{n=1}^{\infty} \frac{EX_n^2}{n^2} < +\infty \tag{1.3}$$

implies

$$\lim_{n\to+\infty} \sum_{i=1}^{\infty} X_i/n = 0 \ a.\ s. \tag{1.4}$$

Theorem B. *If $\{X_n;\ n\geqslant 1\}$ is blockwise quasiorthogonal, then*

$$\sum_{n=1}^{\infty} (EX_n^2)(\log(n+1))^2 < +\infty \tag{1.5}$$

implies

$$\sum_{n=1}^{\infty} X_n \quad \text{convergence} \quad a.\ s. \tag{1.6}$$

In [5] Móricz also raised the following conjectures.

Conjecture 1. For every $\alpha>1$, there exists a sequence $\{X_n;\ n\geqslant 1\}$ of r. v. 's such that $EX_n=0$, $EX_n^2<+\infty$ $(n\geqslant 1)$ and (1.3) is satisfied, $\{X_n;\ p^\alpha<n\leqslant(p+1)^\alpha\}$ is independent for each $p=1, 2, \cdots$, but

$$\limsup_{n\to+\infty} \frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| = +\infty \quad \text{a. s.} \tag{1.7}$$

Conjecture 2. For every $\alpha>1$, there exists a sequence $\{X_n;\ n\geqslant 1\}$ of r. v. 's such that $EX_n=0$, $EX_n^2<+\infty$ $(n\geqslant 1)$ and

$$\sum_{n=1}^{\infty} \frac{EX_n^2}{n^2}\cdot(\log(n+1))^2 < +\infty \tag{1.8}$$

is satisfied and

$$E(X_k X_l) = 0\ (k\neq 1;\ p^\alpha<k,\ l\leqslant(p+1)^\alpha;\ p\geqslant 1), \tag{1.9}$$

but

$$\limsup_{n\to+\infty} \frac{1}{n}\left|\sum_{i=1}^{n} X_i\right| = +\infty \quad \text{a. s.} \tag{1.10}$$

Let B denote a real separable Banach space with topological dual B^* and norm $\|\cdot\|$. For convenience, we introduce the following definition.

Definition 3. *If $\{X_n;\ n\geqslant 1\}$ is a sequence of B-valued r. v. 's, m is a nonnegative integer and $\{a_n;\ n\geqslant 1\}$ is a sequence of positive real numbers with $a_1\geqslant 1$, $a_{n+1}-a_n\geqslant 1$ $(n\geqslant 1)$ and $(a_{n+1}-a_n)\uparrow+\infty$. $\{X_n;\ n\geqslant 1\}$ is termed blockwise m-dependent with respect to $\{a_n;\ n\geqslant 1\}$ if for each n large enough, the two sets*

$$\{X_{[a_n]+1}, \cdots, X_k\} \text{ and } \{X_l, X_{l+1}, \cdots, X_{[a_{n+1}]}\}$$

of B-valued r. v. 's are independent provided $a_n<k$, $l\leqslant a_{n+1}$ and $1-k>m$.

Obviously, in the real-valued case the Móricz blockwise m-dependence is equivalent to the bolckwise m-dependence with respect to $\{2^n;\ n\geqslant 1\}$.

In section 2 we present a general result on strong limit theorems for blockwise m-dependent sequences of B-valued r. v. 's and some corollaries. From one of the corollaries Móricz's Theorem A follows. It might be worthwhile to mention that the methods of this paper are capable of regarding Mórios blockwise m-dependeut r. v. 's, in some sense, as independent r. v. 's in dealing with the strong law of large numbers (SLLN) and the law of the iterated logarithm (LIL). In section 3 we give positive answers to the conjectures raised by Móricz.

§ 2. A General Result and Some Corollaries

The following Theorem 2.1 is of basic importance in our work.

Theorem 2. 1. *Let $\{X_n;\ n\geqslant 1\}$ be blockwise m-dependent B-valued r. v. 's with respect to $\{2^n;\ n\geqslant 1\}$, $\{Y_n;\ n\geqslant 1\}$ independent B-valued r. v. 's such that for each $n\geqslant 1$ X_n and Y_n are identically distributed, $S_n=X_1+\cdots+X_n$, $S(n)=Y_1+\cdots+Y_n$ $(n\geqslant 1)$, $S_A=\sum_{i\in A}X_i$ and $S(A)=\sum_{i\in A}Y_i$, where $A\subset\{1,\ 2,\ \cdots\}$. Let q be a semi-norm on B. Further, assume $\{b_n;\ n\geqslant 1\}$ is a positive sequence satisfying*

$$b_n\uparrow+\infty,\quad \sum_{k=0}^{n}b_{2^k}\leqslant C\cdot b_{2^n}\ \text{and}\ b_{2^n}\leqslant C\cdot b_{2^{n-1}}\quad (n\geqslant 1), \tag{2.1}$$

where C is a absolute constant. If there are two constants $0\leqslant A_1,\ A_2<+\infty$ such that

$$\varlimsup_{n\to+\infty}\ \sup_{A\subset\{2^{n-1}+1,\cdots,2^n\}} P\left(q\left(\frac{S(A)}{b_{2^n}}\geqslant A_1\right)=a<1\right.$$

and

$$\varlimsup_{n\to+\infty} q\left(\frac{S(n)}{b_n}\right)\leqslant A_2\quad a.\ s., \tag{2.3}$$

then

$$\varlimsup_{n\to+\infty} q\left(\frac{S_n}{b_n}\right)\leqslant 8C(m+1)(A_1+A_2)\quad a.\ s. \tag{2.4}$$

Theorem 2.1 easily implies the following corollary.

Corollary 2. 2. *Let $\{X_n;\ n\geqslant 1\}$ be blockwise m-dependent B-valued r. v. 's with respect to $\{2^n;\ n\geqslant 1\}$, $\{Y_n;\ n\geqslant 1\}$ independent B-valued r. v. 's such that for each $n\geqslant 1$ X_n and Y_n are identically distributed. If*

$$\lim_{n\to+\infty}\frac{S(n)}{n^{1/p}}=0\quad a.\ s. \tag{2.5}$$

and for all $\varepsilon>0$,

$$\lim_{n\to+\infty}\ \sup_{A\subset\{2^{n-1}+1,\cdots,2^n\}} P\left(\frac{\|S(A)\|}{2^{n/p}}\geqslant\varepsilon\right)=0, \tag{2.6}$$

then

$$\lim_{n\to+\infty}\frac{S_n}{n^{1/p}}=0\quad a.\ s., \tag{2.7}$$

where $p>0$, *and* S_n, $S(n)$, S_A *and* $S(A)$ *are as in Theorem* 2.1.

In regard to SLLN, the following corollary extends Móricz's Theorem A to the B-valued case.

Corollary 2. 3. *Let* $\{X_n;\ n\geq 1\}$ *be blockwise m-dependent B-valued r. v. 's with respect to* $\{2^n;\ n\geq 1\}$, $\{Y_n;\ n\geq 1\}$ *independent B-valued r. v. 's such that for each* $n\geq 1$ X_n *and* Y_n *are identically distributed. If* $EY_n=0$, $E\|Y_n\|^2<+\infty$ $(n\geq 1)$,

$$\sum_{n=1}^{\infty}\frac{E\|Y_n\|^2}{n^2}<+\infty \tag{2.8}$$

and

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{Y_i}{n}=0 \quad \text{a. s.,} \tag{2.9}$$

then

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{X_i}{n}=0 \quad \text{a. s.} \tag{2.10}$$

In Theorem 2.4 it is proved that in regard to SLLN and LIL identically distributed and blockwise m-dependent sequence $\{X_n;\ n\geq 1\}$ and the companion sequence $\{Y_n:\ n\geq 1\}$ have, in some sense, the similar limit behaviour.

Theorem 2. 4. *Let* $\{X_n;\ n\geq 1\}$ *be identically distributed and block-wise m-dependent B-valued with respect to* $\{2^n;\ n\geq 1\}$ *and* $\{Y_n;\ n\geq 1\}$ *independent identically distributed B-valued r. v. 's such that* X_1 *and* Y_1 *have the same distribution. We have*

(a) *If* $0<p<2$ *and*

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{Y_i}{n^{1/p}}=0 \quad \text{a. s.,} \tag{2.11}$$

then

$$\lim_{n\to+\infty}\sum_{i=1}^{n}\frac{X_i}{n^{1/p}}=0 \quad \text{a. s.} \tag{2.12}$$

(b) *If*

$$\overline{\lim_{n\to+\infty}}\frac{\|Y_1+\cdots+Y_n\|}{\sqrt{2n\log\log n}}<+\infty \quad \text{a. s.,} \tag{2.13}$$

then

$$\overline{\lim_{n\to+\infty}}\frac{\|X_1+\cdots+X_n\|}{\sqrt{2n\log\log n}}<+\infty \quad \text{a. s.} \tag{2.14}$$

(c) *If*

$$P\left(\left\{\frac{Y_1+\cdots+Y_n}{\sqrt{2n\log\log n}};\ n\geq 16\right\} \text{ is conditionally compact in } B\right)=1, \tag{2.15}$$

then

$$P\left(\left\{\frac{X_1+\cdots+X_n}{\sqrt{2n\log\log n}};\ n\geq 16\right\} \text{ is conditionally compact in } B\right)=1. \tag{2.16}$$

The proofs of Theorem 2.1 and Theorem 2.4 depend on the following lemmas. The first one provides the foundation for key idea in this paper, the second one comes from [1].

Lemma 2. 1. *Let* $\{X_n;\ p\leq n\leq q\ (q-p>m)\}$ *be blockwise m-dependent B-valued r. a. 's, teat is, the two sets*

$$\{X_p, \cdots, X_k\} \text{ and } \{X_1, \cdots, X_q\}$$

of r. v. 's are independent provided $p \leqslant k < 1 \leqslant q$ *and* $l-k>m$. *Then there is a partition* $A_1, \cdots, A_{m+1}$ *of* $\{p, p+1, \cdots, q\}$ *such that* $\{p, p+1, \cdots, q\} = \bigcup_{i=1}^{m+1} A_i$, $A_iA_j=\phi$ $(i \neq j$, $i, j=1, \cdots, m+1)$ *and for each* $1 \leqslant i \leqslant m+1$ $\{X_n;\ n \in A_i\}$ *are independent r. v. 's when* A_i *contains, at least, two elements.*

Proof Define

$$A_i = \{p+jm+i-1;\ j=0, 1, \cdots, \text{ and } p \leqslant p+jm+i-1 \leqslant q\}$$

for $i=1, \cdots, m+1$. Then all that remains is easy to prove.

Lemma 2.2. *Let* $\{Z_n;\ n \geqslant 1\}$ *be B-valued r. v. 's. If*

$$P\left(\overline{\lim_{n\to+\infty}} \|Z_n\| < +\infty\right) = 1 \tag{2.17}$$

and for every $\varepsilon > 0$, *there exists a finite dimensional subspace F of B such that*

$$P\left(\overline{\lim_{n\to+\infty}}\, q_F(Z_n) \leqslant \varepsilon\right) = 1, \tag{2.18}$$

here and later on q_F *denotes the semi-norm given by* $q_F(x) = \inf_{y \in F} \|x-y\|$ $(x \in B)$, *then*

$$P(\{Z_n;\ n \geqslant 1\} \text{ is conditionally compact in } B) = 1. \tag{2.19}$$

Proof of Theorem 2.1 From (2.3) and Borel-Cantelli Lemma we have for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left(q\left(\frac{S(2^n)-S(2^{n-1})}{b_{2^n}}\right) \geqslant 2\Lambda_2 + \varepsilon\right) < +\infty. \tag{2.20}$$

The next step of our proof is to show

$$\overline{\lim_{n\to+\infty}}\, q\left(\frac{S_{2^n}-S_{2^{n-1}}}{b_{2^n}}\right) \leqslant 4(m+1)(\Lambda_1+\Lambda_2) \quad \text{a. s.} \tag{2.21}$$

From Lemma 2.1 there exists a partition $A_1^{(n)}, \cdots, A_{m+1}^{(n)}$ of $\{2^{n-1}+1, \cdots, 2^n\}$ for each n large enough such that for each $1 \leqslant i \leqslant m+1$ $\{X_k;\ k \in A_i^{(n)}\}$ are independent r. v. 's when A_i contains, at least, two elements. For each n large enough, usin (2.2) and the Ottaviani inequality we have for every $\varepsilon > 0$

$$\begin{aligned} &P\left(q\left(\frac{S_{2^n}-S_{2^{n-1}}}{b_{2^n}}\right) \geqslant (m+1)(4(\Lambda_1+\Lambda_2)+2\varepsilon)\right) \\ &\leqslant \sum_{i=1}^{m+1} P\left(q\left(\frac{S(A_i^{(n)})}{b_{2^n}}\right) \geqslant \frac{4(m+1)(\Lambda_1+\Lambda_2)+2(m+1)\varepsilon}{(m+1)}\right) \\ &= \sum_{i=1}^{m+1} P\left(q\left(\frac{S(A_i^{(n)})}{b_{2^n}}\right) \geqslant 4(\Lambda_1+\Lambda_2)+2\varepsilon\right) \\ &\leqslant \frac{m+1}{1-a-\delta} P\left(q\left(\frac{S(2^n)-S(2^{n-1})}{b_{2^n}}\right) \leqslant 2\Lambda_2+\varepsilon\right), \end{aligned} \tag{2.22}$$

where $\delta > 0$ with $a+\delta < 1$. Combining (2.22) with (2.20), we easily see that (2.21) holds. Moreover, using the similar argument we have

$$\overline{\lim_{n\to+\infty}} \max_{2^{n-1}<k\leqslant 2^n} q\left(\frac{S_k-S_{2^{n-1}}}{b_{2^n}}\right) \leqslant 4(m+1)(\Lambda_1+\Lambda_2) \quad \text{a. s.} \tag{2.23}$$

Recalling (2.1) we have

$$\overline{\lim_{n\to+\infty}} q\left(\frac{S_{2^n}}{b_{2^n}}\right)\leqslant \overline{\lim_{n\to+\infty}} \sum_{k=1}^{n}\frac{b_{2^k}}{b_{2^n}} q\left(\frac{S_{2^k}-S_{2^{k-1}}}{b_{2^k}}\right)$$
$$\leqslant 4C(m+1)(A_1+A_2) \quad \text{a. s.} \tag{2.24}$$

from which and (2.23), (2.4) follows. Thus Theorem 2.1 is proved.

Proof of Corollary 2.2 Note that $\sum_{i=0}^{n} 2^{i/p}\leqslant \frac{2^{1/p}}{2^{1/p}-1}\cdot 2^{n/p}$ and $2^{n/p}=2^{1/p}\cdot 2^{n-1/p}$, the Corollary 2.2 follows immediately from Theorem 2.1.

Proof of Corollary 2.3 It is enough to prove that

$$\sup_{A\subset\{1,\cdots,n\}} E\frac{\|S(A)\|}{n}\to 0 \quad (n\to+\infty). \tag{2.25}$$

Recalling $E\|S(A)\|\leqslant E\|S(n)\|$ for all $A\subset\{1, \cdots, n\}$, we have

$$\lim_{n\to+\infty}\sum_{i=1}^{n} E\frac{\|Y_i\|^2}{n^2}=0, \tag{2.26}$$

thus

$$E\frac{\left\|\sum_{i=1}^{n} Y_i I_{\{\|Y_i\|>n\}}\right\|}{n}\leqslant\sum_{i=1}^{n} E\frac{\|Y_i\|^2}{n^2}\to 0, \tag{2.27}$$

by which and (2.9)

$$\sum_{i=1}^{n} Y_i\frac{I_{\{\|Y_i\|\leqslant n\}}}{n}\xrightarrow{P} 0. \tag{2.28}$$

From here on we follow the proof of Lemma 3.1 in [2] to obtain

$$\lim_{n\to+\infty} E\frac{\left\|\sum_{i=1}^{n} Y_i I_{\{\|Y_i\|\leqslant n\}}\right\|}{n}=0. \tag{2.29}$$

Hence (2.25) holds.

Proof of Theorem 2.4 The implications (2.11)⇒(2.12) and (2.13)⇒(2.14) both are immediate corollaries of Theorm 2.1. Now, we prove (2.15)⇒(2.16). First, using Theorem 2.1, we have

$$\overline{\lim_{n\to+\infty}}\frac{\|X_1+\cdots+X_n\|}{\sqrt{2n\log\log n}}<+\infty \text{ a. s.} \tag{2.30}$$

from (2.15). Second, by the results in [3], [4] and [6],

$$\frac{Y_1+\cdots+Y_n}{\sqrt{2n\log\log n}}\xrightarrow{P} 0. \tag{2.31}$$

Moreover there exists a nonrandom compact subset K uniquely determined by the distribution of Y_1 such that

$$P\left(C\left\{\frac{Y_1+\cdots+Y_n}{\sqrt{2n\log\log n}};\ n\geqslant 16\right\}=K\right)=1, \tag{2.32}$$

where $C\{x_n;\ n\geqslant 1\}$ is the cluster set of $\{x_n;\ n\geqslant 1\}$ in B. Applying (2.31), for every $\varepsilon>0$ we have

$$\sup_{A\subset\{1,\cdots,n\}} P\left(\frac{\|\sum_{i\in A} Y_i\|}{\sqrt{2n\log\log n}}\geqslant \varepsilon\right)=\sup_{k\leqslant n} P\left(\frac{\|Y_1+\cdots+Y_k\|}{\sqrt{2n\log\log n}}\geqslant\varepsilon\right)$$
$$\to 0 \ (n\to+\infty). \tag{2.33}$$

For every $\varepsilon>0$, take a finite dimensional subspace F of B such that

$$K\subset\{x;\ q_F(x)\leqslant \varepsilon/8(m+1)\}. \tag{2.34}$$

Thus (2.32) implies the following

$$\overline{\lim_{n\to+\infty}}\, q_F\left(\frac{Y_1+\cdots+Y_n}{\sqrt{2n\log\log n}}\right)\leqslant\frac{\varepsilon}{8(m+1)} \quad \text{a. s.} \tag{2.35}$$

Finally, by Theorem 2.1 we have

$$\overline{\lim_{n\to+\infty}}\, q_F\left(\frac{X_1+\cdots+X_n}{\sqrt{2n\log\log n}}\right)\leqslant C\cdot\varepsilon \quad \text{a. s.,} \tag{2.36}$$

where C satisfies $\sum_{k=4}^{n}\sqrt{2^k\log\log 2^k}\leqslant C\sqrt{2^n\log\log 2^n}$ and

$$\sqrt{2^n\log\log 2^n}\leqslant C\sqrt{2^{n-1}\log\log 2^{n-1}} \quad (n\geqslant 4).$$

In fact, we can take $C=4$. Then (2.15)$\Rightarrow$(2.16) follows from Lemma 2.2.

The following example shows that $\{b_n;\ n\geqslant 1\}$ does not satify (2.1). Then Theorem 2.1 may fail, even if we deal with real-valued r. v. 's.

Example Let $\{Z_n;\ n\geqslant 1\}$ be independent identically distributed real-valued r. v. 's with $Z_1\sim N(0,1)$. Let

$$Y_n=\begin{cases} Z_{2^k}, & \text{if } n=2^k,\ k=0,1,2,\cdots,\\ 0, & \text{if } n\neq 2^k,\ k=0,1,2,\cdots,\end{cases}$$

and

$$X_n=\begin{cases} Z_1, & \text{if } n\neq 2^k,\ k=0,1,2,\cdots,\\ 0, & \text{if } n=2^k,\ k=0,1,2,\cdots.\end{cases}$$

If is evident that for each $n\geqslant 1$ $\{X_k;\ 2^{n-1}<k\leqslant 2^n\}$ are independent r. v. 's, $\{Y_n;\ n\geqslant 1\}$ independent r. v. 's. Moreover, X_n and Y_n are identically distributed for each $n\geqslant 1$. Since

$$\overline{\lim_{n\to+\infty}}\left|\sum_{k=0}^{n} Z_{2^k}\right|\Big/\sqrt{2n\log\log n}=1 \quad \text{a. s.,} \tag{2.37}$$

we have

$$\overline{\lim_{n\to+\infty}}\frac{|Y_1+\cdots+Y_n|}{\sqrt{2\log n\log\log\log n}}=\frac{1}{\sqrt{\log 2}} \quad \text{a. s.} \tag{2.38}$$

But

$$\lim_{n\to+\infty}\frac{(\log 2)\cdot(X_1+\cdots+X_n)}{\log n}\overset{\text{a.s.}}{=\!=}Z_1\sim N(0,1). \tag{2.39}$$

It leads to

$$\frac{Y_1+\cdots+Y_n}{\log n}\to 0 \quad \text{a. s.} \tag{2.40}$$

even though from (2.38), in this case, we have

$$\frac{X_1+\cdots+X_n}{\log n}\not\to 0 \quad \text{a. s.} \tag{2.41}$$

Of course, $\{\log(\max(n, e));\ n\geqslant 1\}$ does not satisfy the assumption (2.1).

§ 3. A Generalization of Miórcz Conjectures

In this section we prove that Móricz conjectures hold, that is, the following sharper theorem is true.

Theorem 3. 1. *For each $\alpha>1$, there exists a sequence $\{X_n;\ n\geqslant 1\}$ of real-valued r. v.'s such that $EX_n=0$, $EX_n^2<+\infty$ $(n\geqslant 1)$ and*

$$\sum_{n=1}^{\infty}\frac{EX_n^2}{n^2}(\log(n+1))^2<+\infty. \tag{3.1}$$

Moreover, for each $p\geqslant 1$ r. v.'s $\{X_n;\ p^\alpha<n\leqslant(p+1)^\alpha\}$ are independent. But for any $0\leqslant\delta<1/2\alpha$,

$$\lim_{n\to+\infty}\frac{|X_1+\cdots+X_n|}{n^{1+\delta}}=+\infty \quad \text{a. s.} \tag{3.2}$$

and

$$\frac{X_1+\cdots+X_n}{n^{1+\frac{1}{2\alpha}}(\log(n+1))^{-2}}\overset{P}{\not\to}0. \tag{3.3}$$

Proof Let X be a real-valued random variable with the distribution $N(0,1)$. Putting

$$X_n=\begin{cases}0, & \text{if } n\neq[k^\alpha]\ k=1,2,\cdots,\\ k^{\alpha-\frac{1}{2}}(\log(k+1))^{-2}X, & \text{if } n=[k^\alpha],\ k=1,2,\cdots,\end{cases} \tag{3.4}$$

we have $EX_n=0$, $EX_n^2<+\infty$ $(n\geqslant 1)$ and

$$\sum_{n=1}^{\infty}\frac{EX_n^2}{n^2}\cdot(\log(n+1))^2=\sum_{n=1}^{\infty}\frac{k^{2\alpha-1}}{[k^\alpha]^2}\cdot(\log(k+1))^{-4}\cdot(\log([k^\alpha]+1))^2 <+\infty. \tag{3.5}$$

Moreover $\{X_n;\ p^\alpha<n\leqslant(p+1)^\alpha\}$ is independent for each $p=1,2,\cdots$. It is easy to prove that

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{k^{\alpha-\frac{1}{2}}(\log(k+1))^{-2}}{n^{\alpha+\frac{1}{2}}(\log(n+1))^{-2}}=\alpha+\frac{1}{2}. \tag{3.6}$$

So we have

$$\lim_{n\to+\infty}\frac{X_1+\cdots+X_n}{n^{1+\frac{1}{2\alpha}}(\log(n+1))^{-2}}=\left(\alpha+\frac{1}{2}\right)X \quad \text{a. s.}, \tag{3.7}$$

from which (3.3) follows and (3.2) holds.

Moreover we have the following interesting result.

Theorem 3. 2. *Let $\{X_n;\ n\geqslant 1\}$ be real-valued r. v.'s such that $EX_n=0$, $EX_n^2<+\infty(n\geqslant 1)$ and for some $\alpha>1$, $\{X_n;\ n\geqslant 1\}$ is blockwise m-dependent with respect to $\{n^\alpha;\ n\geqslant 1\}$. If*

$$\sum_{n=1}^{\infty}\frac{EX_n^2}{n^{2(t-\frac{1}{\alpha})}}<+\infty \tag{3.8}$$

for some $t>0$, then

$$\frac{X_1+\cdots+X_n}{n^t}\to 0 \quad a.\ s. \tag{3.9}$$

Proof By Kroneker Lemma we only need to prove that

$$\lim_{n\to+\infty}\sup_{k<2^{n-1}}\frac{|X_{2^{n-1}+1}+\cdots+X_{2^{n-1}+k}|}{2^{nt}}=0 \text{ a. s.} \tag{3.10}$$

Note that the number of the positive integers k with $2^{n-1}+1\leqslant k^\alpha\leqslant 2^n$ are, at most, $[2^{n/\alpha}]$, say $k_1(n)<k_2(n)<\cdots<k_{p(n)}(n)$, where $p(n)\leqslant 2^{n/\alpha}$. So for each $\varepsilon>0$

$$\begin{aligned}
&P\left(\sup_{k<2^{n-1}}\frac{|X_{2^{n-1}+1}+\cdots+X_{2^{n-1}+k}|}{2^{nt}}\geqslant\varepsilon\right)\\
&\leqslant P\left(\sup_{2^{n-1}+1<k\leqslant(k_1(n))^\alpha}\frac{|X_{2^{n-1}+1}+\cdots+X_k|}{2^{nt}}\geqslant\frac{\varepsilon}{2^{n/\alpha}}\right)\\
&\quad+P\left(\sup_{(k_1(n))^\alpha\vee k\leqslant(k_2(n))^\alpha}\frac{|X_{[k_1^\alpha(n)]+1}+\cdots+X_k|}{2^{nt}}\geqslant\frac{\varepsilon}{2^{n/\alpha}}\right)\\
&\quad+\cdots+P\left(\sup_{(k_{p(n)}(n))^\alpha<k\leqslant 2^n}\frac{|X_{[(k_{p(n)}(n))^\alpha]+1}+\cdots+X_k|}{2^{nt}}\geqslant\frac{\varepsilon}{2^{n/\alpha}}\right).
\end{aligned} \tag{3.11}$$

Recalling Lemma 2.1 and Kolmogorov inequality, we have for $i=0, 1, \cdots, p(n)$

$$\begin{aligned}
&P\left(\sup_{(k_i(n))^\alpha<k\leqslant(k_{i+1}(n))^\alpha}\frac{|X_{[(k_i(n))^\alpha]+1}+\cdots+X_k|}{2^{nt}}\geqslant\frac{\varepsilon}{2^{n/\alpha}}\right)\\
&\leqslant\frac{(m+1)^2}{\varepsilon^2}\sum_{(k_i(n))^\alpha<k\leqslant(k_{i+1}(n))^\alpha}\frac{EX_k^2}{2^{2n\left(t-\frac{1}{\alpha}\right)}},
\end{aligned} \tag{3.12}$$

where $k_0(n)=2^{n-1}+1$, $k_{p(n)+1}(n)=2^n$. Thus for each $\varepsilon>0$

$$\begin{aligned}
&P\left(\sup_{k\leqslant 2^{n-1}}\frac{|X_{2^{n-1}+1}+\cdots+X_{2^{n-1}+k}|}{2^{nt}}\geqslant\varepsilon\right)\\
&\leqslant\frac{(m+1)^2}{\varepsilon^2}\cdot\sum_{k=2^{n-1}+1}^{2^n}\frac{EX_k^2}{k^{2\left(t-\frac{1}{\alpha}\right)}}
\end{aligned} \tag{3.13}$$

which together with (3.8) implies (3.10).

Remrak. Let $\{X_n;\ n\geqslant1\}$ be real-valued r. v. 's such that $EX_n=0$, $EX_n^2<+\infty$ $(n\geqslant1)$ and for some $\alpha>1$, $\{X_n;\ n\geqslant1\}$ is blockwise m-dependent with respect to $\{n^\alpha;\ n\geqslant1\}$. Theorem 3.1 shows that the condition $\sum_{n=1}^\infty\frac{EX_n^2}{n^2}<+\infty$ is not sufficient for $\frac{S_n}{n^{1+\delta}}\to0$ a. s. $\left(0\leqslant\delta<\frac{1}{2\alpha}\right)$, but Theorem 3.2 asserts that the condition $\sum_{n=1}^\infty E\frac{X_n^2}{n^2}<+\infty$ is sufficient for $\frac{S_n}{n^{1+\frac{1}{\alpha}}}\to0$ a. s. $\left(\text{taking } t=1+\frac{1}{\alpha}\right)$. In addition, Theorem 3.2 shows that if $\sum_{n=1}^\infty\frac{EX_n^2}{n^{2\left(1-\frac{1}{\alpha}\right)}}<+\infty$ then $S_n/n\to0$ a. s. (taking $t=1$).

Clearly Theorem 3.2 implies the following result which can be generalized to the case where $\{X_n;\ n\leqslant1\}$ take values in a separable B-convex space.

Corollary 3. 3. *Let $\{X_n;\ n\geqslant1\}$ be real-valued r. v. 's such that $EX_n=0$ and $EX_n^2\leqslant C\,(n\geqslant1)$, where C is a absolute positive constant, and for some $\alpha>2$, $\{X_n;\ n\geqslant1\}$*

be blockwise m-dependent with respect to $\{n^\alpha;\ n\geqslant 1\}$. *Then*

$$S_n/n\to 0 \quad \text{a. s.} \tag{3.14}$$

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