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# THE MORSE CRITICAL GROUPS OF THE MINMAX THEOREM

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**Abstract**

The author gives a new min-max theorem and studies the local homology properties of critical points that is got by min-max theorem. In particular, this result includes many min-max theorem of link type, which can be applicable to nonlinear analysis.

## § 0. Introduction

Since the pioneer work by Ambrosetti and Rabinowitz[1] various min-max theorems of link type, including the Mountain Pass Lemma, have been developed[2,3,4]. Many authors are interested in finding the local topological properties of the critical point which is obtained by the min-max theorem of link type. Only some special cases have been studied, and there are some good results in[5,7,8]. In general, we obtain some results in [9] by use of the methods of deformation.

In this paper we show some results in general case. In order to state our results, we denote by $H$ a Hilbert space, $Q$ a finite dimensional triangulable submanifold in $H$ with or without boundary. We identify the manifold $Q$ with its triangulation. Denote by $\partial$ a chain boundary map, $S$ a connected $C^1$-Hilbert submanifold in $H$ without boundary[17,12]. For each $\tau\in S_q(H)$ and $\tau\neq 0$, then $\tau=\sum\limits_{\sigma} v_\sigma \sigma$, in which $\sigma$ are some $q$-simlexes and $v_\sigma\neq 0$. So one can define

$$|\tau|=\bigcup_{\sigma}\mathrm{Im}(\sigma).$$

The notations of others are taken in [5].

The main theorem of our paper is

**Theorem.** *Let* $\dim Q = co-\dim S = k$, *and there exists a triangulation of* $Q$ *such that* $[\partial Q]$ *is a non-zero element in* $H_{k-1}(H\backslash S)$. *Let* $f\in C^1(H, \mathrm{R})$ *satisfy* $(P, S)$ *condition and*

$$\sup_{x\in Q} f(x) < +\infty;\ \sup_{x\in|\partial Q|} f(x)\leqslant a;\ f(x)>a,\ x\in S \tag{0.1}$$

*for some* $a\in R$. *Then* $f$ *has a critical value* $c\geqslant a$, *where* $c$ *is given by* (2.1).

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*Moreover, if there exists* $\varepsilon>0$ *such that* $c$ *is the unique criticalvalue in* $(c-\varepsilon, c+2\varepsilon)$, *and any connected component of* $K_c$ *only has single point, then we have*

$$H_k(f_c, f_c\backslash K_c)\neq 0. \tag{0.2}$$

*In particular, if* $f\in C^2(H, R)$, *and all of the critical points in* $K_c$ *are non-degenerated, then there exists* $p\in K_c$ *such that* $index(f, p)=k$, *and* $c>a$.

Some illustrations for Theorem are necessary. In section 1 we want to prove an interesting topological lemma (see Lemma 1) by use of Thom isomorphisms. As a corollary we get $\dim H_{k-1}(H\backslash S)=1$ $(k>1)$ and $\dim H_{k-1}(H\backslash S)=2(k=1)$. So $H_{k-1}(H\backslash S)$ has a generator. If $[\partial Q]$ is a nonzero element of $H_{k-1}(H\backslash S)$, the $|\partial Q|$ and $S$ must link in the sense of

$$\begin{gathered} S\cap|\partial Q|=\varnothing \\ \text{if } \psi\in C(Q, H),\ \psi|_{|\partial Q|}=\mathrm{id}_{|\partial Q|}, \text{ then } \psi(Q)\cap S\neq\varnothing. \end{gathered} \tag{0.3}$$

So we find a class of interesting links, which can be called homology link. Homologic link includes most of the links which can be applicable for nonlinear analysis (compare with [1, 2, 3, 4, 11]). In all these applications, $S$ is a $C^1$-Hilbert manifold, $\dim Q=\operatorname{codim} S<+\infty$, $Q$ is triangulable, and $|\partial Q|$ is natural boundary of the manifold $Q$. In this time homologic link is the 'link' that is defined in [6]. Our Theorem includes all of the homologic links. When the link is taken as tho type of Mountain Pats Lemma, the results of Theorem were given by [5] with different conditions. When the link is taken as the type of [4] (i. e. $H=X_1\oplus X_2$, $\dim X_1=k$, $S=X_2$, $Q=X_1\cap B_\rho$), the results of Theorem were given by [7] and [8]. But the Theorem also includes the link that is given in the [2, 3, 11] (i. e. $S=X_2\cap\partial B_\rho$, $\partial Q=(B_R\cap X_1)\cup\{x+te:(x, t)\in X_1\times R^1_+,\ \|x\|^2+t^2=R^2\}$, where $H=X_1\oplus X_2$, $\dim X_1=k-1$, $e\in X_2$, $\|e\|=1$, and $0<\rho<R$). Hence the Theorem is a generalization to [5, 7, 8]. And our proof is uniform. The idea of proof bases on [5, 6, 8]. The existence of critical value in Theorem is new.

Since the space is restricted, we have to omit some applications of Theorem.

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## § 1. Some Topological Lemmas

In this section we want to build the following interesting lemma.

**Lemma 1.** *Let* $M$ *be a* $C^3$-*Hilbert manifold and admit the partitions of unity. Let* $S$ *be a* $C^1$ *connected Hilbert submanifold of* $M$ *with* $\operatorname{codim} S=k$. *Assume*

$$H_q(M)=\begin{cases} Z, & q=0, \\ 0, & q>0. \end{cases}$$

*Then*

$$H_q(M\setminus S)=\begin{cases} Z\oplus Z, & q=0,\\ H_q(S), & q>0,\end{cases}\quad k=1;$$

$$H_q(M\setminus S)=\begin{cases} Z, & q=0,\\ 0, & 0<q<k-1,\\ Z, & q=k-1,\\ H_{q-k+1}(S), & q>k-1,\end{cases}\quad k>1. \tag{1.1}$$

*Proof* From the theorem of the existence of tubular neighborhood for $S$ in $M$ (see [12]), we know that there exists a vector bundle $(E, p, S)$ with the fibre $p^{-1}(s)=R^k$, and a $C^1$-embedding $f$: $E\to M$ such that $f=id$ on $S$. Let $E_0=E\setminus(S\times\{0\})$. Then we have from the Thom isomorphisms (see [13])

$$H_q(E, E_0)=\begin{cases} 0, & q<k,\\ Z, & q=k,\\ H_{q-k}(S), & q>k.\end{cases}$$

Exploiting the excision theorem we have

$$\begin{aligned} H_q(M, M\setminus S) &= H_q(f(E), f(E)\setminus S)\\ &= H_q(E, E_0)\\ &= \begin{cases} 0, & q<k,\\ Z, & q=k,\\ H_{q-k}(S), & q>k.\end{cases}\end{aligned} \tag{1.2}$$

By the portion of the long exact sequence

$$H_{q+1}(M)\to H_{q+1}(M, M\setminus S)\to H_q(M\setminus S)\to H_q(M)\to H_q(M, M\setminus S), \tag{1.3}$$

we have

$$H_q(M\setminus S)=H_{q+1}(M, M\setminus S) \text{ for all } q>0. \tag{1.4}$$

When $q=0$, we have the exact sequence from (1.3)

$$H_1(M, M\setminus S)\to H_0(M\setminus S)\to H_0(M)\to H_0(M, M\setminus S).$$

If $k>1$, then $H_0(M, M\setminus S)=H_0(M, M\setminus S)=0$ (see (1.2)). So we have

$$H_0(M\setminus S)=H_0(M)=Z,\ k>1. \tag{1.5}$$

If $k=1$, then we get the short exact sequence from (1.2) and (1.3)

$$0\to Z\to H_0(M\setminus S)\to Z\to 0.$$

Since $H_0(M\setminus S)$ is a free $Z$-module on $M\setminus S$ with generators as many as path components of $M\setminus S$, we have

$$H_0(M\setminus S)=Z\oplus Z,\ k=1. \tag{1.6}$$

Combing (1.4) with (1.2), (1.5) and (1.6), we complete the proof.

**Corollary 1.** *Let $S$ be a $C^1$ connected Hilbert submanifold in $H$ with* codim $S=k$. *Then*

$$H_q(H\setminus S)=\begin{cases} Z\oplus Z, & q=0,\\ H_q(S), & q>0,\end{cases}\quad k=1;$$

$$H_q(H\backslash S)=\begin{cases} Z, & q=0, \\ 0, & 0<q<k-1, \\ Z, & q=k-1, \\ H_{q-k+1}(S), & q>k-1, \end{cases} \quad k>1.$$

**Remark 1.** This corollary tell us that the conditions of Theorem can be verified by the Thom isomorphisms.

**Corollary 2.** *Let $M$ be a unit sphere in Hilbert space $H$ with* $\dim H=+\infty$. *Let $S$ be a $C^1$ connected Hilbert submanifold in $M$ with* codim $S=k$. *Then* (1.1) *holds.*

*Proof* Since $\dim H=+\infty$, $M$ is contractible (see [14]). Hence

$$H_q(M)=\begin{cases} Z, & q=0, \\ 0, & q>0. \end{cases}$$

By Lemma 1, we have (1.1).

**Remark 2.** In fact, Corollary 2 is the Jordan-Brouwer Seperation Theorem of infinite dimension sphere. For finite dimension sphere this is a wellknown fact (see [16]). We know that many interesting properties of finite dimension sphere are missing in infinite dimension sphere. So our result is meaningful.

**Lemma 2.** *Suppose $[\partial Q]$ is a non-zero element of $H_{k-1}(H\backslash S)$. Let $\tau\in S_k(H)$ such that $\partial\tau=\partial Q$. Then*

$$|\tau|\cap S\neq\varnothing.$$

*Proof* If $|\tau|\cap=S=\varnothing$, then $\tau\in S_k(H\backslash S)$, which implies $[\partial Q]=0$, a contradiction.

## 2. The Proof of Theorem

In this section, we prove the Theorem exploiting the lemmas of above section.

*Proof of Theorem* Let

$$\Gamma=\{|\tau|:\tau\in S_k(H),\ \partial\tau=\partial Q\}.$$

Then $\Gamma$ satisfies the following properties:

1) $\Gamma\neq\varnothing$, in fact $Q\in\Gamma$ in the sence of given triangulation.

2) $\Gamma$ is invariant under the action of the negative gradient flow.

Set

$$c=\inf_{A\in\Gamma}\sup_{x\in A}f(x). \tag{2.1}$$

Then $c\geqslant a$, from which follows Lemma 2 and (0.1). When $c>a$, we can prove that $c$ is a critical value by the standard argument. So we can assume that $c=a$. If $c$ is not a critical value, then there exists $\varepsilon_0>0$ such that there is not any critical value in $(c-\varepsilon_0, c+\varepsilon_0)$ by (P. S) condition. So $f_c$ is deformation retract of $f_{c+\varepsilon_0}$ from the second deformation Lemma (see [5, 15]). Denote by the deformation as $\eta$. By the definition of $c$, there exists $|\tau|\in\Gamma$ such that $|\tau|\subset f_{c+\varepsilon_0}$. Hence $|\eta\circ\tau|$

$\in \Gamma$. But $|\eta\circ\tau| \subset f_c = f_a$. This is in contradiction with Lemma 2 and (0.1). Therefore $c$ is a critical value.

Using above methods, we know that $c > a$ when $c$ is the unique critical value in $(c-\varepsilon, c+2\varepsilon)$. Now consider the following boundary homorphism

$$H_k(f_c, f_c \backslash K_c) \xrightarrow{\partial_*} H_{k-1}(f_c \backslash K_c).$$

By the definition of $c$, for any $\varepsilon > 0$ there is a chain $\tau \in S_k(H)$ such that $\partial\tau = \partial Q$ and $|\tau| \subset f_{c+\varepsilon}$. Since $f$ has no critical value in $(c, c+\varepsilon]$, $f_c$ is a deformation retract of $f_{c+\varepsilon}$ from the second deformation Lemma. Let $\eta$ be this deformation, and $\xi = \eta\circ\tau$. Then $\partial\xi = \partial\tau = \partial Q$, and $|\xi| \subset f_c$. It follows from $c > a$ that

$$[\partial Q] \in \operatorname{Im}(\partial_*). \tag{2.2}$$

Now we show that $[\partial Q]$ cannot be the trivial element of $H_{k-1}(f_c \backslash K_c)$. For if not, there is a $k$-chain $\sigma \in S_k(f_c \backslash K_c)$, $\partial\sigma = \partial Q$. But for $\varepsilon > 0$ small enough, $f_{c-\varepsilon}$ is a deformation retract of $f_c \backslash K_c$ from the first deformation Lemma (see [5]). Using this deformation we get a $k$-chain $\delta$ such that $|\delta| \subset f_{c-\varepsilon} \subset f_c \backslash K_c$, and still have $\partial\delta = \partial Q$. This implies $\sup_{x\in|\delta|} f(x) \leqslant c-\varepsilon$. Naturally $\delta \in S_k(H)$, which contradicts the definition of $c$.

From (2.2), we have $[\partial Q] \in \operatorname{Im}(\partial_*)$. Hence $H_k(f_c, f_c \backslash K_c) \neq 0$. The rest of the Theorem is standard (see [5]).

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