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A NOTE ON WEAKLY PRIMITIVE RINGS

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Abstract

It is well known that, for a subring of a full linear ring over a vector spaec, 2-fold transitive implies k-fold transitive for every natual integer k, and a primitive ring with minimal oneside ideal is a two side nonsingular ring and every isomorphism can be induced by a semi-linear one to one transformation. This paper generailzes these results to weakly primitive rings.

Throughout this paper, unless specifically indicated otherwise, rings need not possess an identity element. By a module we will mean a right module, and an effort will be made to consistently write module homomorphisms on the side oppssite to that of the scalars. A partial endomorphism of a module M is a homomorphism from a submodule of M into M. A nonzero R-module M is called compressible if it can be embedded in each of its nonzero submodules; it will called critically compressible if it is compressible, and, additionally, cannot be embedded in any of its proper factor modules.

Lemma 1. *The following conditions are equivalent for a compressible module M:*

(i) *M is critically compressible;*

(ii) *Every nonzero partial endomorphism of M is a monomorphism.*

Proof Refer to [6, Proposition 2.1].

A module which satisfies condition (ii) of the above lemma is called a monoform module.

Lemma 2. (i) *If M_R is monoform then elements of $D=\text{End}(M_R)$ have unique extensions to elements of $\Delta=\text{End}(\overline{M}_R)$ and Δ is a division ring, where $\overline{M}_R$ is the quasi-injective hull of M_R.*

(ii) *If M_R is critically compressible then D is a right Ore domain with right quotient division ring Δ.*

Proof Refer to [6, Proposition 1.2].

We call a triple $(\Delta, {}_{\Delta}V_R, M_R)$ an R-lattice if V is a Δ-R-bimodule with Δ being a division ring, $\Delta M=V$, and R acts faithfully on M. And we say that R acts

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on R-lattice $(\Delta, {}_\Delta V_R, M_R)$ k-fold transitive if for given $v_1, v_2, \cdots, v_k \in V$ linearly independent over Δ, there exists $0 \neq a \in \Delta$ such that for any elements $n_1, n_2, \cdots, n_k \in M$ one can find $r \in R$ with $an_i = v_i r$ for each $i = 1, \cdots, k$. A ring R is called a weakly primitive ring if it has a faithful critically compressible module. For weakly primitive ring Zelmanowitz proved the following density theorem.

Theorem (The Density Theorem). *The following conditions are equivalent for a ring R:*

(i) *R is weakly primitive.*

(ii) R acts on R-lattice $(\Delta, {}^\Delta V_R, M_R)$ *k-sold transitive for every integer k.*

Remark. In the (ii) of above theorem, the R-lattice $(\Delta, {}_\Delta V_R, M_R)$ satisfies following conditions: (a) M_R is a critically compressible module, (b) V_R is quasi-injective hull of M, and (c) $\Delta = \mathrm{End}(E(\overline{M}_R))$ where $E(\overline{M}_R)$ is quasi-injective hull of M_R.

Theorem 3. *If R acts on R-lattice $(\Delta, {}_\Delta V_R, M_R)$ 2-fold transitive, then R acts on R-lattice $(\Delta, {}_\Delta V_R, M_R)$ k-fold transitive for every integer k.*

Proof For any $v \in V$ there is $0 \neq a \in \Delta$ such that for any $0 \neq u \in M$ there exists $r \in R$ with $vr = au \in M$, that is, V_R is an essential extension of M.

Let N_R be a submodule of M_R. We take an $0 \neq n \in N$. Then there exists $0 \neq a \in \Delta$ such that for any $m \in M_R$ one can find some $r \in R$ with $nr = am \in N$, i. e., $a \in \mathrm{Hom}_R(M, N)$ and obviously that a is a monomorphism, that is, M_R is a compressible module.

For any $\tau \in \mathrm{End}_\Delta V$ and $v \in V$ and $m \in M$ one can find $r, s \in R$ with $(\tau r - s)|_{\Delta v} = 0$ and $r|_{\Delta m}$ being an automorphism. Indeed, let $u = v\tau$. If u and m are linear independent over Δ, then there exists $0 \neq a \in \Delta$ such that for m, 0 one can find $r \in R$ with $mr = am$, $ur = 0$. For v one can find $s \in R$ with $vs = 0$. Hence $v(\tau r - s) = v\tau r - vs = ur = 0$, i. e., $(\tau r - s)|_{\Delta v} = 0$ and $r|_{\Delta m} = Is$. If u and m are linear dependent over Δ, that is, $u = dm$, for v there exists $0 \neq a \in \Delta$ such that for any $m \in M$ one can find $t \in R$ with $vt = am$. And for $a^{-1}dm$ one can find $0 \neq b \in \Delta$ and $r \in R$ with $a^{-1}dmr = bm \in M$ and also one can find $s \in R$ with $vs = abm$. Then $v\tau r = ur = dmr = a(a^{-1}dmr) = abm = vs$. Thus $(\tau r - s)|_{\Delta v} = 0$ and $r|_{\Delta m}$ is an automorphism.

Secondly, we show that M_R is a critically compreasible module; in fact, we only show that M_R is a monoform module by Lemma 1. Let N_R be a submodule of M_R and let $0 \neq f \in \mathrm{Hom}_R(N, M)$ be given; say $f(m) \neq 0$ for some $m \in N$. Given an arbitrary element $0 \neq n \in N$, we choose $\tau \in \mathrm{End}\ {}_\Delta V$ with $n\tau = m$ and take $r, s \in R$ with $\tau r = s$ on Δn, and with $r|_{\Delta(m)}$ being an automorphism. Then $f(n)s = f(ns) = f(n\tau r) = f(mr) = f(m)r \neq 0$, so $f(n) \neq 0$, and it follows that f is a monomorphism.

Finlly, we must show that $\Delta' = \mathrm{End}(\overline{M}_R) = \Delta$. By Lemma 2 and the fact that V_R is an essential extension of M_R we have $\Delta \subseteq \mathrm{End}(E(M_R)) = \mathrm{End}(\overline{M}_R)$. For any

$\sigma\in \mathrm{End}\,(M_R)$ and for any $m\in M$, $\sigma(m)$ and m must be linear dependent over Δ; if not, then there exists $0\neq a\in\Delta$ such that there exists $r\in R$ with $mr=0$ and $\sigma(m)r=am\neq 0$, but then $0\neq\sigma(m)r=\sigma(mr)=0$ which is a contradiction. Hence $\sigma(m)=dm$ for some $0\neq d\in\Delta$. Now let n be an arbitrary element of M_R. There exists $0\neq a\in\Delta$ such that there exists $r\in R$ with $mr=an\in M$, that is, a is an element of $\mathrm{End}(M_R)$ and $\sigma a(n)=\sigma(an)=\sigma(mr)=(\sigma(m))r=(dm)r=d(mr)=dan$. Hence $\sigma a=da$ or $\sigma=d$. But by lemma $\mathrm{End}(M_R)$ is an Ore domain and $\Delta'=\mathrm{End}(\overline{M}_R)$ is a right quotient ring of $\mathrm{End}(M_N)$, so $\Delta=\mathrm{End}(\overline{M}_R)$. By the density theorem of Zelmanowitz, we know that R acts on R-lattice $(\Delta, {}_\Delta V_R, M_R)$ densely. Thus we compelete the proof.

Lemma 4. *Let M_R and N_R be two R-modules. If there exist monomorphisms $M_R\to N_R$ and $N_R\to M_R$, then $\overline{N}_R\cong\overline{M}_R$ where $\overline{N}_R$ and $\overline{M}_R$ are quasi-injective hulls of N_R aad M_R respectively.*

Paoof Since there exist monomorphisms $N_R\xrightarrow{f}M_R$ and $M_R\xrightarrow{g}N_R$, and we extend two monomorphisms $E(N_R)\to E(M_R)$ and $E(M_R)\to E(N_R)$, by Bumby Theorem ([3, Proposition 3.60]) we know that $E(N_R)$ is isomorphic to $E(M_R)$. Without loss of generality we can assume that $E(M_R)=E(N_R)=E$ and M_R, N_R are two essential submodules of E_R. $S=\mathrm{End}(E_R)$, then $\overline{M}_R=SM_R$ and $\overline{N}_R=SN_R$. f and g can extend two monomorphisms of E, say $\hat{f}$ and $\hat{g}$. Then $\hat{f}(\overline{N}_R)=\hat{f}S(N_R)\subseteq\hat{f}S\hat{f}^{-1}(M_R)\subseteq SM_R=\overline{M}_R$. Similarly, $\hat{g}$: $\overline{M}_R\hookrightarrow\overline{N}_R$. Since $\overline{M}_R\hookrightarrow\overline{N}_R$ and $\overline{N}_R$ is quasi-injective, $\overline{N}_R$ is $\overline{M}_R$-injective by [2, Proposition 16.13]. And $\overline{N}_R\hookrightarrow\overline{M}_R$, we have $\overline{M}_R\cong\overline{N}_R\oplus L_R$ for some submodule of E; but this contradicts the assumption that N_R is essential in E_R. Thus $\overline{M}_R\cong\overline{N}_R$..

Theorem 5. *Let $R_i\,(i=1, 2)$ be two rings which act on R_i-lattice (Δ_i, V_i, M_i) 2-fold ransitive and contain a linear transformation with finite rank. If σ is an isomorphism from ring R_1 to rina R_2, then there exists a semi-linear one to one transformation τ from V_1 to V_2 such that $r^\sigma=\tau^{-1}r\tau$ for every $r\in R_1$.*

Proof We consider rings R_1 and R_2 as the same ring R under isomorphism σ. Then the R_i-lattices (Δ_i, V_i, M_i) are R-lattice, and R acts on (Δ_i, V_i, M_i) 2-fold transitive and contains a linear transformation with finite rank on $V_i\,(i=1, 2)$.

By Theorem 3 we know that R acting on R-lattices (Δ_i, V_i, M_i) is dense. Let r be a linear transformation with finite rank. Then we may write $Vr\subseteq\sum_{i=1}^{t}\Delta m_i$ with $m_1, \cdots, m_t\in M$, linear independent over Δ. And we can choose some $r'\in R$ such that $m_1r'\neq 0$ and $m_ir'=0$ for $2\leqslant i\leqslant t$. Thus we know that $rr'\in R$ with rank 1.

Let us now assume that $r\in R$ is a linear transformation of rank 1 on V. Then $V=\ker r\oplus\Delta m$ and for every nonzero element $r'\in rR$, $\ker r'=\ker r$. So $r'\neq 0$ iff $mr'\neq 0$, that is, $rR\to M$ via: $r'\mapsto mr'$ is a monomorphism. By Theorem 3 we know that

M is a compressible module, so there also exists a monomorphism $M \to rR$.

From above discussion we know that for each R-lattice (Δ_i, V_i, M_i) there exists $r_i \in R$ such that $r_iR \hookrightarrow M_i$ and $M_i \hookrightarrow r_iR$. And by [4, Theorem 4.1] we know that R is a right nonsingular prime, and r_1R is a uniform right ideal of R. So there are two monomorphisms $r_1R \to r_2R$ and $r_2R \to r_1R$. Thus we have two monomorphisms $M_1 \to M_2$ and $M_2 \to M_1$. By Theorem 3, $V_1 = \overline{M}_1$ and $V_2 = \overline{M}_2$. Thus $V_1 \cong V_2$ by Lemma 4, and we write it as τ. We restore τ to an isomorphism from R_1-module V_1 to R_2-module V_1. Then we have

$$(r_1v_1)\tau = (v_1)\tau(r_1)^\sigma, \ v_1 \in V_1 \text{ and } r_1 \in R_1.$$

We think r_1 as an endomorphism of V_1 and $(r_1)\sigma$ as an endomorphism of V_2. Then $r_1\tau = \tau(r_1^\sigma)$, that is,

$$r_1^\sigma = \tau^{-1}r_1\tau, \text{ for every } r_1 \in R_1.$$

It remains to show that τ is a semi-linear transformation from vector space V_1 over Δ_1 to vector space V_2 over Δ_2. Since τ is an isomorphism from abelian group V_1 to abelian group V_2, the correspondence

$$\theta: \operatorname{End}(V_1) \to \operatorname{End}(V_2)$$
$$r_1 \mapsto \tau^{-1}r_1\tau$$

is an isomorphism from ring $\operatorname{End}(V_1)$ to ring $\operatorname{End}(V_2)$ and $\theta(R_1) = R_2$. By Theorem 3 we know that the centralizer of R_i in $\operatorname{End}(V_i)$ is Δ_i. Hence $\theta(\Delta_1) = \Delta_2$. Thus

$$(d_1v_1)\tau = v_1L_{d_1}\tau = v_1\tau\tau^{-1}L_{d_1}\tau = (v_1\tau)L_{d_2}$$
$$= d_2(v_1\tau) = (d_1\varphi)(v, \tau),$$

that is, (τ, φ) is a semi-linear one to one transformation from vector space V_1 to vector space V_2.

Corollary 6. *If R is a right order of $M_{n_i}(D_i)$ $(i=1, 2)$ where D_i is a division ring, then $D_1 \cong D_2$ and $n_1 = n_2$.*

Proof It is obvious by using above theorem, we omit the detail.

Theorem 7 *Let R be a ring with a faithful critically compressible right ideal. Then*

(i) *R is a left nonsingular ring.*

(ii) *If R has a uniform left ideal, then either R is a two side order in a matrix ring Δ_t for some division ring Δ, in case R contains a subring isomorphic to D_t for some two side order D of Δ; or else for each positive integer t there exists a two side order D of Δ and a subring of R which maps homomorphically onto D_t.*

Proof (i) By the theorem of Zelmanowitz ([6, Theorem 4.1]) we know that R is a right nonsingulsr, prime ring with a uniform right ideal. Let I_R be a uniform right ideal of R and $S = \operatorname{End}(I_R)$ and ${}_RM_s = \operatorname{Hom}_R(I, R)$. By Lemma 2 we know that S is a right Ore domain, so $Z(s) = 0$. Let $x \in I$ and $f \in S$ with $f(x) = 0$.

Then $I/\ker f \hookrightarrow I_R$, and since R is right nonsingular and I_R is uniform, we have $f=0$, that is, ${}_SI$ is a faithful module.

Now we take $r' \notin Z_l(R)$. Then there is a large left ideal L of R with $Lr'=0$. Take any $x \in I$ and put $J=\{s \in S: sx \in IL\}$. This is a large left ideal of S; indeed $sx \neq 0$ if $x \neq 0$ and $s \neq 0$. In this case there must exist some $g \in \mathrm{Hom}_R(I, R)$ with $gs(x) \neq 0$. In fact, otherwise we would have $I/\ker g \hookrightarrow R$, but $I/\ker g$ is a singular module and R is nonsingular. Since L is a left large ideal of R, there exists some rR such that $rgsx \in L$ and $rgsx \neq 0$, and then there exists some $x' \in I$ with $x'rgsx \neq 0$ by the prime of R. It is easy to verify that $x'rgs \in S$ and $x'rgsx \in IL$. Hence $x'rgs \in J$, that is, J is a large left ideal of S, and $Jxr' \subseteq ILr'=0$. Then it must be $xr'=0$ by the above discussion and x is arbitrary, so $Ir'=0$. Thus $r'=0$, i. e., $Z_l(R)=0$.

(ii) In fact, we have proved that there exists a Morita context $(R, {}_RM_S, {}_SI_R, S)$ where I_R is a uniform right ideal of R, $S=\mathrm{End}\, I_R$ and $M=\mathrm{Hom}_R(I, R)$, and this Morita context is nondegenerate. Let ${}_RJ$ be a left uniform ideal of R. We assert that IJ as an S-module is a uniform module; if not, let J_1 and J_2 be two nonzero S-submodules of IJ with $J_1 \oplus J_2$ being a direct sum as S-modules in IJ, then $(M, J_1) \oplus (M, J_2)$ is a direct sum of left ideals of R. Indeed if $r_1+r_2=0$, $r_i \in (M, J_i)$ then $Ir_1+Ir_2=0$, but $Ir_i \subseteq I(M, J_i)=[I, M]J_i \subseteq J_i$, hence $Ir_i=0$. Thus $r_i=0$ since R is prime, but $(M, J_i) \subseteq (M, IJ)=(M, I)J \subseteq J$, which contradicts the fact that J is uniform.

We take $0 \neq x \in IJ$. Since IJ is a uniform module as S-module for every s_1, $s_2 \neq 0$ which are two elements of S, $0 \neq s_1x \in IJ$ and $0 \neq s_2x \in IJ$, we can choose s_3, s_4 which satisfy $s_3s_1x=s_4s_2x$. Then by Lemma 2 we know that it must be $s_3s_1=s_4s_2$, that is, S is a left Ore domain.

Applying the dense theorem we know that R acts on R-lattice (Δ, V, M) densily where $M=I_R$, $V=\bar{I}_R$ and $\Delta=\mathrm{End}(\bar{I}_R)$.

Suppose that $\dim \Delta V \geqslant t$ and choose $m_1, \cdots, m_t \in M$ linear indendent over Δ. For each $i=1, \cdots, t$, set $A_i=\bigcap_{j\neq i}(0: M_j)$ by [6, Lemma 2.1], $A_i \not\subseteq (0: m_i)$ for each i, and so $N=\sum_{i=1}^{t} m_iA_i$ is a nonzero submodule of M. Put $D=\{a \in \Delta \mid aM \subseteq N\}$; an easy calculation proves that D is a two side order in Δ. For given $0 \neq \lambda \in \Delta$, $\lambda^{-1}(N) \cap N \neq 0$; so choosing $0 \neq a \in D$ such that $aM \subseteq \lambda^{-1}(N) \cap N$ yields $0 \neq \lambda a \in D$. And since S is a left order of Δ, there also exists $0 \neq b \in S$ such that $b\lambda \in S$. Then taking $0 \neq c \in D$, we would have $0 \in cb \in D$ and $cb\lambda \in D$.

Next we set $W=\sum_{i=1}^{t} Dm_i$, $W'=\sum_{i=1}^{t} D^1m_i$. Observe that $\mathrm{Hom}_D(W', W) \cong D_t$. Now given $f \in \mathrm{Hom}_D(W'W)$, f is completely determined by the values $m_if=\sum_{j=1}^{t} d_{ij}m_j$, $d_{ij} \in D$, $i=2, \cdots, t$. Since each $d_{ij}m_j \subseteq N$, we may write each $d_{ij}m_j=m_ir_{ij}$ for some r_{ij}

$\in A_i$. Thus $m_i f = m_i r_i$ where $r_i \in A_i$. Setting $r = \sum_{i=1}^{t} r_i$ yields $m_i f = m_i r$ for each i. Thus by letting $S' = \{r \in R \mid W'r \subseteq W\}$, the assignment $r \mapsto$ right multiplication by r on W' yields a homomorphism of S' onto $\mathrm{Hom}_D(W', W) \cong D_t$ whose kernel

$$K = \{r \in R \mid W'r = 0\}.$$

If, in fact, $\dim {}_\Delta V = t$, then $m_1, \cdots, m_t$ forms a basis for V and $K = 0$ since M_R is faithful. In this case then, R is a two side order in End V, and S' is a subring of R isomorphic to S_t.

Then we obtain the Theorem 1 of [4] again by using verious methods.

Corollary 8. *Let R be a right prime Goldie. Then R is a left Goldie ring iff R has a uniform left ideal.*

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