Chin. Ann. of Math.
12B: 2 (1991), 213—218.

THE NONEXISTENCE OF EXPANSIVE FLOW ON A COMPACT 2-MANIFOLD

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Abstract

This paper makes some investigation to the question whether there exists an expansive flow on some compact metric space. The main result is that there exists no expansive flow on a compact 2-manifold.

Many papers have been published for the dynamical properties of expansive homeomorphisms since W. R. Utz introduced the notion of expansiveness for homeomorphisms. At the same time, people paid attention to the existence of such homeomorphisms on a general manifold, for example, [5] and [6] proved the nonexistence of any expansive homeo-morphism on the closed interval, the closed disk and the circle; [7] and [8] showed that there exists a certain expansive homeomorphism on each compact orientable surface with positive genus and on each n-dimensional open ball.

In [1], R. Bowen introduced a notion of expansiveness for continuous flows and gave some corresponding results to homeomorphisms, for example, expansiveness is a conjugacy invariant. Later a series of papers, for example [2—4], investigated the properties of expansive flows. In tnis paper we make some investigation to the question whether there exists an expansive flow on some compact metric space. We shall prove the following results:

Theorem 1 *Let K be a connected n-dimensional finite complex ($n\geqslant 1$). If its Euler characteristic $\chi(K)$ does note equat with zero, then there exists no expansive flow on the polyhedra $|K|$.*

Theorem 2. *There exists no expansive flow on a compact 2-manifold.*

Let (X, d) be a compact metric space, and $\Phi: X\times R\to X$ a continuous flow on X. For every $t\in R$, let Φ_t denote a homeomorphism of X onto X defined by $\Phi_t=\Phi(\cdot, t)$. The orbit (positive semi-orbit) of Φ through x is denoted by $\gamma(x)$ ($\gamma^+(x)$). The notation $\omega(x)$ ($\alpha(x)$) indicates the ω—limit set (α—limit set) of $\gamma(x)$.

Manuscript received October 18, 1988.

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Definition 1. *Φ is expansive if $\forall\varepsilon>0$, $\exists\delta>0$ with the property that if $d(\Phi_t(x), \Phi_{s(t)}(y))<\delta$, $\forall t\in R$ for a pair of points $x, y\in X$ and a continuous map $s:R\to R$, $s(0)=0$ then $y=\Phi_t(x)$, where $|t|<\varepsilon$.*

Let (Y, ρ) be acompact metric space, $f:Y\to Y$ a homeo-morphism, and $\psi:Y\to(0, +\infty)$ a continuous map.

Definition 2. *The suspension of f under ψ is the flow Φ on the space*

$$Y_\psi=\bigcup_{0<t<\psi(y)}\{(y, t)\,|\,(y, \psi(y))\sim(f(y), 0)\}$$

defined for small non-negative time by $\Phi_t(y, s)=(y, s+t)$, $0\leqslant t+s<\psi(y)$.

Lemma 1[1]. *If Φ is an expansive flow on X, then each fixed point of Φ is an isolated point of X.*

Lemma 2[1]. *Let f be a homeomorphism of (Y, ρ), and $\psi:Y\to(0, +\infty)$ be a continuous map. The suspension of f under ψ is expansive if and only if f is expansive.*

Remark. By Lemma 2, we may conclude that any simple clo-sed curve L admits an expansive flow. In fact, let $Y=\{y\}$ be a single-point space, $\psi(y)=1$, then Y_f and a unit circle S^1 are homeomorphic, and so are Y_f and L. f is expansive, so the suspension Φ of f under ψ is also expansive. It follows that there exists an expansive flow on L since expansiveness is a conjugacy invariant.

Lemma 3. *Let K be an m-dimension finite complex and $\chi(K)\neq 0$. Then every continuous self-map on the polyhedra $|K|$ which is homotopic to identity map has a fixed point.*

Proof Because the Lefschetz number of identity map equates with the Euler characteristic $\chi(K)$ of K, and two homotopic maps have the same Lefschetz number, the Lefschetz number $L(f)$ of f does not equate with zero. By Lefschetz fixed point theorem, f has a fixed point.

Proof of Theorem 1 Suppose to the contrary that there exists an expansive flow Φ: $|K|\times R\to|K|$. The homeomorphism $\Phi_{1/m}$ and identity map i_d are homotopic ($\forall m\in\mathbf{Z}^+$). A homotopy between $\Phi_{1/m}$ and i_d is F: $|K|\times[0, 1]\to|K|$ difined by $F(x, s)=\Phi(x, s/m)$, $\forall(x, s)\in|K|\times[0, 1]$. By Lemma 3, $\Phi_{1/m}$ has a fixed point x_m. The assumption $n\geqslant 1$ and Lemma 1 show that the point x_m is not a fixed point of Φ. Hence x_m is a periodic point whose period $0<\tau_m\leqslant 1/m$, so the orbit $\gamma(x)$ is a periodic orbit. Since $|K|$ is compact, without loss of generality, we can assume that x_m converges to y. We claim that y is a fixed point of Φ, i. e. $\Phi(y, t)=y$ for all $t\in R$. In fact, for any $m\in\mathbf{Z}^+$, there exists a $K_m\in\mathbf{Z}$ such that $t=K_m/m+t_m$, where $0\leqslant t_m<1/m$. Therefore, $\lim_{m\to\infty}t_m=0$, $\Phi(x_m, t)=\Phi(x_m, t_m)$, and $\Phi(y, t)=\lim_{m\to\infty}\Phi(x_m, t)=\lim_{m\to\infty}\Phi(x_m, t_m)=y$. So, y is a fixed point of Φ. Lemma 1 shows that y is an isolated point of $|K|$, and this contradicts the assumption that K is connective. The proof of Theorem 1 is completed.

Proof of Theorem 2. Assume this is not so. Let Φ be an expansive flow on M^2. We may assume that M^2 is connective because a flow is also expansive when it is restricted to any of its closed invariant sets. Now the genus, the number of boundary circles and the Euler characteristic of M^2 are denoted by g, m, $\chi(M^2)$, respectively. We have the following relation:

$$\chi(M^2)=\begin{cases}2-2g-m, & \text{when } M^2 \text{ is orientable,}\\ 2-g-m, & \text{when } M^2 \text{ is nonorientable.}\end{cases}$$

It is enough to prove the theorem only for $\chi(M^2)=0$. When $\chi(M^2)=0$, M^2 is one of the following four cases:

(1) The closed circular ring.

(2) The torus.

(3) The Möbius trip.

(4) The Klein bottle.

We shall first show that Φ possesses at most finite periodic orbits any of which is non-homotopic to zero.

Suppose γ is a periodic orbit of Φ. If γ is zero-homotopic, then γ bounds a closed disk D on M^2 which is invariant for Φ. Thus $\Phi|_D$ (Φ restricted to D) is expansive. By Theorem 1, this is impossible, because $\chi(D)=1\neq 0$. Thus γ is non-homotopic to zero.

We now suppose that Φ possesses infinite periodic orbits, any of which is non-homotopic to zero. Take $u_n\in M^2$ such that u_n belongs to a periodic orbit and $u_m\notin\gamma(u_n)$ when $m\neq n$. By the compactness of M^2, $\{u_n\}$ has a convergent subsequence. Without loss of generality, we may assume that $u_n\to u$. Then u is a nonwandering point of Φ. By Lemma 1 and the assumption that M^2 is connected, neither $\omega(u)$ nor $\alpha(u)$ contains fixed point. Hence, the orbit $\gamma(u)$ is a Possion stable orbit. Note that Φ does not have nontrival Possion stable orbit, so $\gamma(u)$ is a periodic orbit.

Assume that $\gamma(u)$ is a two-side periodic orbit. For any $\delta>0$, let $N_{\delta/2}(y)$ be a $\delta/2$-neighbourhood of y. Because $\gamma(u)$ is compact, there are finite points $y_1=u$, y_2, $\cdots$, y_α on $\gamma(u)$ such that $\gamma(u)\subset N=\bigcup_{i=1}^{\alpha}N_{\delta/2}(y_i)$. Let α be the minimal integer such that $\gamma(u)\subset\bigcup_{i=1}^{\alpha}N_{\delta/2}(y_i)$. We can assume that y_1, $\cdots$, y_α are taken in proper order as time goes on. Take a point $x_i\in N_{\delta/2}(y_i)\cap N_{\delta/2}(y_{i+1})$ $(i=1,\cdots,\alpha)$ such that x_i belongs to the orbit arc $\widehat{y_iy_{i+1}}$ from y_i to y_{i+1}. Let I_i be a local cross section for Φ through x_i. Using the continuity of Φ, we may take n large enough so that $\gamma(u_n)\subset N$, $u_n\notin\gamma(u)$, $\gamma(u_n)\cap I_i=\{x_{ni}\}$ and if the orbit arc $\widehat{x_ix_{i+1}}\subset N_{\delta/2}(y_{i+1})$ then the orbit arc $\widehat{x_{ni}x_{ni+1}}\subset N_{\delta/2}(y_{i+1})$. Let $x_i=\Phi(x_{i-1}, t_{i-1})$, $x_{ni}=\Phi(x_{ni-1}, t_{ni-1})$ $(i=2,\cdots,\alpha+1)$ where $x_{\alpha+1}=x_1$, $x_{n\alpha+1}=x_{n1}$. Let T be the period of $\gamma(u)$, T_n be the period of

$\gamma(u_n)$. Then $\sum_{i=1}^{\alpha} t_i = T$, $\sum_{i=1}^{\alpha} t_{ni} = T_n$. Let $\tau_j = \sum_{i=1}^{j} t_i$, $\tau_{nj} = \sum_{i=1}^{j} t_{ni}$ $(j=1, \cdots, \alpha)$, $\tau_0 = \tau_{n0} = 0$. First we define a map $s_T: [0, T] \leftarrow [0, T_n]$ by $s_T(t) = \frac{\tau_{nj+1} - \tau_{nj}}{\tau_{j+1} - \tau_j}(t - \tau_j) + \tau_{nj}$, where $t \in [\tau_j, \tau_{j+1}]$ $(j=0, \cdots, \alpha)$. Then we define the map s: $R \to R$ by $s(t) = s_T(t') + KT_n$ for any $t \in R$, where $K \in \mathbf{Z}$ and $t = KT + t'$, $0 \leqslant t' < T$. It is obvious that s is a continuous map with $s(0) = 0$, and satisfies $d(\Phi(u, t), \Phi(u_n, s(t))) < \delta$. But $u_n \notin \gamma(u)$ and this contradicts the expansiveness of Φ.

The same idea as above can be applied when $\gamma(u)$ is a one-side periodic orbit.

We shall go on with the proof of the theorem for M^2 being each of the above cases, respectively.

Case (1): M^2 is a closed circular ring.

We may assume that Φ has no other periodic orbits except the boundary circles γ_1, γ_2. Hence, for any $x \in M^2 - (\gamma_1 \cup \gamma_2)$, $\omega(x) = \gamma_1$, $\alpha(x) = \gamma_2$, or $\omega(x) = \gamma_2$, $\alpha(x) = \gamma_1$. We now suppose that $\omega(x) = \gamma_1$, $\alpha(x) = \gamma_2$. Let $\delta > 0$ be given. Using the same method as above, we can obtain the neighbourhood N of γ_1, the points $x_i \in \gamma_1$ and the local cross sections I_i of Φ through x_i $(i=1, \cdots, \alpha)$. Take $x \in M^2 - (\gamma_1 \cup \gamma_2)$. Because $\gamma(x)$ will spirally approximate to γ with the increase of time t, there exists a large enough $T > 0$, which satisfies the following conditions:

(a) $\Phi(x, T)$ is an interior point of I_1;

(b) $\gamma^+(\Phi(x, T)) \subset N$;

(c) $\gamma^+(\Phi(x, T))$ will successively intersect the local cross sections I_1, I_2, $\cdots$, I_α, $I_{\alpha+1} = I_1$ in their interiors.

In the order of time, we write the intersection points as $x_m = \Phi(x, T)$, x_{m+1}, $\cdots$, respectively (where $x_{K\alpha+m+i} \in I_i$, $K \in \mathbf{Z}$, $i = 0, 1, \cdots, \alpha - 1$). By the continuity of Φ, there exists a $\eta > 0$ small enough such that $d(y, \Phi(x, T)) < \eta$ implies that $\gamma^+(y)$ will successively intersect the local cross sections I_1, $\cdots$, I_α, $I_{\alpha+1} = I_1$. From the compactness of the orbit arc $\Phi(x, [0, T])$, there exists points $x_0 = x$, x_1, x_2, $\cdots$, $x_m = \Phi(x, T)$ such that $x_{i+1} = \Phi(x_i, t_i)$ $(t_i > 0, i = 0, \cdots, m-1)$ and $\Phi(x, [0, T]) \subset \bigcup_{i=0}^{m} N_{\delta/2}(x_i) = N_1$. Using the continuity, the relations $d(\Phi(x, t), \Phi(y, t)) < \min(\delta/2, \eta)$ $(0 \leqslant t \leqslant T)$ and $\Phi(y, [0, T]) \subset N_1$ hold for any y close enough to x. Take the points $y_0 = y$, $y_i = N_{\delta/2}(x_i) \cap \gamma^+(y)$ $(i = 1, 2, \cdots, m-1)$, $y_{K\alpha+m+i} \in I_i \cap \gamma^+(y)$ $(i = 1, 2, \cdots, \alpha - 1; K = 0, 1, \cdots)$ which satisfy $y_{i+1} = \Phi(y_i, \beta_i)$, $\beta_i > 0$, for all $i \in \mathbf{Z}^+$. Let $\tau_j = \sum_{j=0}^{i-1} t_i$, $\sigma_j = \sum_{i=0}^{j-1} \beta_i$ $(j = 1, 2, \cdots)$, $\tau_0 = \sigma_0 = 0$. Now we define a map s_+: $R^+ \to R^+$ where $R^+ = [0, +\infty)$ by $s_+(t) = \frac{\sigma_{i+1} - \sigma_i}{\tau_{i+1} - \tau_i}(t - \tau_i) + \sigma_i$ for $t \in [\tau_i, \tau_{i+1}]$. It is easily seen that s_+ is a continuous map with $s_+(0) = 0$ and satisfies $d(\Phi(x, t), \Phi(y, s_+(t))) \leqslant \delta$ for any $t \geqslant 0$.

Now let $\Phi^*(x, t) = \Phi(x, -t)$ for any $(x, t) \in M^2 \times R$. Then the ω—limit set of the orbit $\gamma(x)$ under Φ^* is γ_2. The same proof for Φ and γ_1 as has been stated above can be used to Φ^* and γ_2. Hence, for any y close enough to x, there exists also continuous map s_+^*: $R^+ \to R^+$ with $s_+^*(0) = 0$ such that $d(\Phi^*(x, t), \Phi^*(y^*, s_+^*(t)) < \delta$ for all $t \geq 0$. Since both y and y^* are any points nearby x, we can take $y = y^*$. So, if we define a map $s: R \to R$ by

$$s(t) = \begin{cases} s_+(t), & t \geq 0, \\ -s_+^*(-t), & t \leq 0, \end{cases}$$

then s is a continuous map with $s(0) = 0$ and satisfies the relation $d(\Phi(x, t), \Phi(y, s(t)) < \delta$ for all $t \in R$. But $y \notin \gamma(x)$, this contradicts the assumption that Φ is expansive.

Case (2): M^2 is the torus.

There are two possibilities to consider.

(a) Φ has at least a periodic orbit γ.

By the proof of case (1), we can assume that Φ has the only periodic orbit γ, because any two periodic orbits non-homotopic to zero bound a closed circular ring in M^2 which is an invariant sets of Φ. Now we have $\omega(x) = \alpha(x) = \gamma$ for any $x \in M^2$. The following proof ls similar to that used in the case (1).

(b) Φ has no periodic orbit. In this case Φ. has non-trivial minimal set L which is a closure of nontrivial Possion stable orbits. Clearly, $\omega(x) = \alpha(x) = L$ for any $x \in M^2$. Let $\gamma(x)$ be an nontrivial Possion stable orbit. By [11] we see that there exists a cross circle C through x which is nonhomotopic to zero. It is obvious that $\gamma(y)$ must intersect C along positive and negative directions of $\gamma(y)$ for any $y \in M^2$. Thus the Poincaré map is difined on the whole circle C. Let $f: C \to C$ be the Poincare'map. As M^2 is orientable and by the continuity of Φ, it follows that f is an orientation-preserving self-homeomorphism. The definition of Poincare map shows that if $f(x) = y = \Phi(x, t_x)$ for any $x \in C$, then $t_x > 0$ is uniquely determined. Thus we may define a map $\psi: C \to (0, +\infty)$ by $\psi(x) = t_x$ for any $x \in C$. Clearly, ψ is a continuous map.

Let the flow Ψ on the space

$$C_f = \bigcup_{0 \leq t \leq \psi(y)} \{(y, t) \mid (y, \psi(y)) \sim (f(y), 0)\}$$

be the suspension of f under ψ. A map $h: M^2 \to C_f$ is defined by $h(x) = (y, t_y)$ for any $x \in M^2$, where $y \in C$, and $t_y \geq 0$ satisfies $x = \Phi(y, t_y)$ and $\Phi(y, [0, t_y]) \cap C = \{y\}$. It is easily seen that h is well defined and is a homeomorphism from M^2 to C_f mapping orbits of Φ onto orbits of Ψ. Thus h is a conjugacy between the flow Φ on M^2 and the flow Ψ on C_f. By the fact that expansiveness is conjugacy invariant and by Lemma 2, we see that Ψ is expansive, and so is f of C. However, from [6] we know that this is impossible. Hence there exist no expansive flows on the torun.

Case (3): M^2 is a Möbius trip.

It follows from Theorem 2 of [10] that Φ has the only one-side periodic orbit γ_1. By the topological properties of Möbius trip, we can assume that the boundary circle γ_2 is the only two-side periodic orbit of Φ. Therefore, for any $x\in M^2-(\gamma_1\cup\gamma_2)$, $\omega(x)=\gamma_1$, $\alpha(x)=\gamma_2$ or $\omega(x)=\gamma_2$, $\alpha(x)=\gamma_1$. The following proof is similar to that used above in case (1).

Case (4): M^2 is a Klein bottle.

It follows from [10] that Φ has at least a periodic orbit γ which is non-homotopic to zero. There are two possibilities to consider.

(a) Φ has a two-side periodic orbit γ.

If γ separates M^2 into two connected componets M_1^2, M_2^2, then, by the topogloical properties of Klein bottle, we know that both M_1^2 and M_2^2 are Möbius trips. Thus the rest is the same proof as in case (3).

If γ does not cut M^2 in two parts, i. e, $M^2-\gamma$ is connected. By the topological properties of Klein bottle[9] we can see that $M^2-\gamma$ is a circular ring. Thus the following proof is completely analogous to that in case (1).

(b) Φ has no two-side periodic orbit.

The topological properties of Klein bottle (see[9]) and Theorem 2 in [10] show that Φ has two one-side periodic orbits γ_1, γ_2, and does not have any periodic orbit other than γ_1, γ_2. Thus $\omega(x)=\gamma_1$, $\alpha(x)=\gamma_2$ or $\omega(x)=\gamma_2$, $\alpha(x)=\gamma_1$ for any $x\in M^2-(\gamma_1\cup\gamma_2)$. The following proof is similar to that in case (1).

So far, the proof of Theorem 2 is completed.

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