Chin. Ann. of Math.
12B: 2 (1991), 219—229.

ON AN INVERSE THEOREM OF APPROXIMATION

YANG LIHUA (杨力华)*

Abstract

The author gives some disagreement to the following result, which is published in [1]. Let $\{L_n(f()\}$ be mass-concerntative, $\phi_n \to 0+ (n\to\infty)$, $0<\alpha\leqslant 2$ and

$$C^{-1} \leqslant \frac{\phi_{n+1}}{\phi_n} \leqslant C \quad (n=1, 2, \cdots)$$

for some constrant $C>0$. Then for any $f\in C[-2a, 2a]$,

$$||L_n(f)-f||_{C[-a,a]} = O(\phi_n^\alpha)$$

implies $f\in \mathrm{Lip}^*\alpha$, where

$$\mathrm{Lip}*\alpha = \{f\in C[-2a, 2a] \mid \omega_2(f, \delta)_{[-2a,2a]} = O(\delta^\alpha)\}.$$

Then some similar results on $C_{2\pi}$ are given, and further some results on $C[-2a, 2a]$ are established by adding some proper conditions.

In this paper we give two counter examples to a lemma and a theorem by Professor Devore, which is an inverse theorem of approximation of continuous functions by "mass-concentrative" operators. Then we give some similar theorems on $C_{2\pi}$ and further establish some theorems on $C[-2a, 2a]$ by adding some proper conditions.

§ 1. Introduction

We state some terms at first.

Definition A. *Let $\{dU_n(t)\}$ be a sequence of positive Borel measures on $[-a, a]$ $(a>0)$, and*

$$\int_{-a}^{a} dU_n(t) = 1 \quad (n=1, 2, \cdots).$$

$\{dU_n(t)\}$ is called mass-concentrative if for any $\varepsilon>0$, there exists $A=A_\varepsilon>0$ and $N=N_\varepsilon>0$ such that

$$\int_{A\phi_n<|t|<a} t^2\, dU_n(t) \leqslant \varepsilon\phi_n^2$$

hold for $n>N$, where

Manuscript received October 18, 1988.
* Department of Mathematics, Hunan Normal University, Changsha, Hunan, China.

$$\phi_n^2=\int_{-a}^{a} t^2\,dU_n(t).$$

The sequence of operators on $C[-a, a]$

$$L_n[f, x)=\frac{1}{2}\int_{-a}^{a}[f(x+t)+f(x-t)]dU_n(t) \quad x\in[-a, a]$$

is also called mass-concentrative

For $f\in C[-2a, 2a]$ *we denote*

$$\|f\|_{\alpha,1}^*=\sup_{0<t\leqslant a} t^{-\alpha}\|\Delta_t^2 f(\cdot)\|_{C[-a,a]},$$

$$\|f\|_{\alpha,2}^*=\sup_{n\geqslant 1} \phi_n^{-\alpha}\|L_n(f)-f\|_{C[-a,a]},$$

where $\Delta_t^2 f(\cdot)=f(\cdot+t)+f(\cdot-t)-2f(\cdot)$, *and* $0<\alpha\leqslant 2$.

In [1 Lemma 8.2] R. A. Devore gave the following lemma.

Lemma A. *Let* $\phi_n\to 0\,(n\to\infty)$ *satissfying*

$$C^{-1}\leqslant\frac{\phi_n}{\phi_{n+1}}\leqslant C \quad (n=1, 2, \cdots)$$

for some constant $C>0$ *and* $\{L_n(f)\}$ *be mais-concentrative operators,* $0<\alpha\leqslant 2$. *Then there exists a positive constant* B_α *such that*

$$\|f\|_{\alpha,1}^*\leqslant B_\alpha^{-1}\|f\|_{\alpha,2}^*$$

for every $f\in C[-2a, 2a]$ *with* $\|f\|_{\alpha,1}^*<\infty$.

By using Lemma A the following theorem was proved in [1]

Theorem A. *Let* $\{L_n(f)\}$ *be mass-concentrative,* $\phi_n\to 0+\ (n\to\infty)$, $0<\alpha\leqslant 2$ *and*

$$C^{-1}\leqslant\frac{\phi_{n+1}}{\phi_n}\leqslant C \quad (n=1, 2, \cdots)$$

for some constant $C>0$. *Then for any* $f\in C[-2a, 2a]$,

$$\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$$

implies $f\in \mathrm{Lip}^*\alpha$, *where*

$$\mathrm{Lip}^*\alpha=\{f\in C[-2a, 2a]\,|\,\omega_2(f, \delta)_{[-2a,2a]}=O(\delta^\alpha)\}.$$

The author finds that both Lemma A and Theorem A are not true. We shall show this point in § 2.

But on $C_{2\pi}$ we find happily that a theorem similar to Theorem A can be established errorlessly. For further details please see § 3.

In § 4 we establish some similar theorems by adding some proper conditions (see Theorems 4—8).

§ 2. Two Counter Examples

The following counter example shows that Lemma A is not true.

Example 1. Let $\{dU_n(t)\}\equiv\left\{\left(B\left(\frac{1}{2}, n+1\right)\right)^{-1}(1-t^2)^n dt\right\}$ and

$$L_n(f, x)=\frac{1}{2}\int_{-1}^{1}[f(x+t)+f(x-t)]dU_n(t),\ f\in C[-2, 2],$$

where $B(p, q)$ is the Beta-function. Then

$$\phi_n^2=\int_{-1}^{1} t^2 dU_n(t)=\frac{1}{2n+3}\to 0 \quad (n\to\infty),$$

and

$$\int_{-1}^{1} t^4 dU_n(t)\leqslant\frac{3}{(2n+3)^2}=3\phi_n^4.$$

For any $\varepsilon>0$ let $A_\varepsilon=\sqrt{\frac{3}{\varepsilon}}$ and $N_\varepsilon=\left[\frac{3}{\varepsilon}\right]+1$. We have, for $n\in N$

$$\int_{A_\varepsilon\phi_n<|t|<1} t^2 dU_n(t)\leqslant 2(A_\varepsilon^2\phi_n^2)^{-1}\int_{A_\varepsilon\phi_n}^{1} t^4 dU_n(t)$$
$$\leqslant\frac{3\phi_n^4}{A_\varepsilon^2\phi_n^2}=\varepsilon\phi_n^2,$$

which shows that $\{L_n(f)\}$ is mass-concentrative. And since

$$\frac{1}{\sqrt{2}}\leqslant\frac{\phi_n}{\phi_{n+1}}\leqslant\sqrt{2},$$

all conditions of Lemma A are satisfied.

For $k=2, 3, \cdots$, we define

$$f_k(x)=\begin{cases}0, & -2\leqslant x\leqslant 2-k^{-1},\\ k^2(x-2+k^{-1}), & 2-k^{-1}\leqslant x\leqslant 2.\end{cases}$$

It is obvious that $f_k\in C[-2, 2]$ and for $0<\alpha\leqslant 2$,

$$\|f_k\|_{\alpha,1}^*\geqslant|\Delta_1^2 f_k(1)|=k,$$
$$\|f_k\|_{\alpha,1}^*=\sup_{0<t<1} t^{-\alpha}\|\Delta_t^2 f(\cdot)\|_{C[-1,1]}=\sup_{0<t<1} t^{-\alpha}|f_k(1+t)|$$
$$=\sup_{1-\frac{1}{k}<t<1} t^{-\alpha}k^2(t-1+k^{-1})\leqslant 2^\alpha k<\infty.$$

On the other hand since for $-1\leqslant x\leqslant 1-k^{-1}$,

$$L_n(f_k, x)-f_k(x)=\frac{2}{2B(1/2, n+1)}\int_0^1\Delta_t^2 f_k(x)(1-t^2)^n dt=0,$$

and for $1\geqslant x>-k^{-1}$,

$$|L_n(f_k, x)-f_k(x)|=\frac{1}{B(1/2, n+1)}\int_0^1 f_k(x+t)(1-t^2)^n dt$$
$$=\frac{1}{B(1/2, n+1)}\int_{2-x-k^{-1}}^{1} k^2(x+t-2+k^{-1})(1-t^2)^n dt$$
$$\leqslant\frac{k}{B(1/2, n+1)}\int_{1-k^{-1}}^{1}(1-t^2)^n dt$$
$$\leqslant\frac{k}{B(1/2, n+1)}\int_{1-k^{-1}}^{1}[1-(1-k^{-1})^2]^n dt$$
$$\leqslant\left(\frac{3}{4}\right)^n\left(n+\frac{1}{2}\right),$$

we have immediately

$$\|L_n(f_k)-f_k\|_{C[-1,1]}\leqslant\left(\frac{3}{4}\right)^n\left(n+\frac{1}{2}\right).$$

Consequently for $n\geqslant 2$ we have

$$\phi_n^{-\alpha}\|L_n(f_k)-f_k\|_{C[-1,1]}\leqslant\left(\frac{3}{4}\right)^n\left(n+\frac{1}{2}\right)(\sqrt{2n+3})^{\alpha}\leqslant 360,$$

which hold for $n=1$ too. Hence

$$\|f_k\|^*_{\alpha,2}\leqslant 360$$

for $k=2, 3, \cdots$.

Obviously if there exists a constant $B_\alpha>0$ such that

$$\|f_k\|^*_{\alpha,1}\leqslant B_\alpha^{-1}\|f_k\|^*_{\alpha,2}$$

then

$$k\leqslant B_\alpha^{-1}360$$

for all $k\geqslant 2$, which is impossible. So Lemma A is not true.

In [1] the proof of Theorem A depends on Lemma A and since Lemma A is wrong we naturally doubt the truth of Theorem A. The following counter example shows that Theorem A is not true either.

Example 2. Let

$$X_n(t)=\begin{cases}1, & |t|\leqslant n^{-1},\\ 0, & \frac{1}{n}<|t|\leqslant 1,\end{cases}$$

and

$$dU_n(t)=\frac{1}{2}nX_n(t)dt.$$

Then

$$\phi_n^2=\int_{-1}^{1}t^2\,dU_n(t)=\frac{1}{3n^2}\to 0\quad(n\to\infty),$$

and so

$$2\geqslant\frac{\phi_{n+1}}{\phi_n}\geqslant\frac{1}{2}.$$

For any $\varepsilon>0$, let $A=\sqrt{3}>0$. Then

$$\int_{A\phi_n<|t|<1}t^2\,dU_n(t)=2\int_{1/n}^{1}t^2\,dU_n(t)=0<\varepsilon$$

for all $n\geqslant 1$. Hence all the conditions of Theorem A are satisfied. We construct $f_k\in C\left[1+\frac{1}{k+1},\ 1+\frac{1}{k}\right]$ $(k=1, 2, \cdots)$ such that for $0<\alpha\leqslant 2$,

(1) $f_k\left(1+\frac{1}{k+1}\right)=f_k\left(1+\frac{1}{k}\right)=0,$

(2) $f_k\left(1+\frac{1}{2}\left(\frac{1}{k}+\frac{1}{k+1}\right)\right)=\frac{1}{k^{\alpha/2}},$

(3) $0\leqslant f_k(x)\leqslant\frac{1}{k^{\alpha/2}}$ for $x\in\left[1+\frac{1}{k+1},\ 1+\frac{1}{k}\right],$

(4) $\int_{1+\frac{1}{k+1}}^{1+\frac{1}{k}}f_k(t)dt\leqslant\frac{1}{k^{1+\alpha}}-\frac{1}{(1+k)^{1+\alpha}}.$

Then we define $f\in[-2, 2]$ by

$$f(x)=\begin{cases}0, & x\in[-2, 1],\\ f_k(x), & x\in\left[1+\frac{1}{k+1},\ 1+\frac{1}{k}\right]\end{cases}\quad (k=1, 2, \cdots).$$

Since

$$\|f\|^*_{\alpha,1}\geqslant t_k^{-\alpha}\Delta^2_{t_k}f(1)\quad \left(t_k=\frac{1}{2}\left(\frac{1}{k}+\frac{1}{k+1}\right)\right)$$
$$=k^{\alpha/2}\to\infty\quad (k\to\infty),$$

we have $\|f\|^*_{\alpha,1}=\infty$.

For $x\in[-1, 1]$ since

$$|L_n(f, x)-f(x)|=\left|\frac{n}{4}\int_{-\frac{1}{n}}^{\frac{1}{n}}\Delta^2_t f(x)\,dt\right|\leqslant\frac{n}{2}\int_k^{1+\frac{1}{n}}f(t)dt$$
$$=\frac{n}{2}\sum_{k=n}^{\infty}\int_{1+\frac{1}{k+1}}^{1+\frac{1}{k}}f(t)dt\leqslant\frac{n}{2}\sum_{k=n}^{\infty}\left(\frac{1}{k^{1+\alpha}}-\frac{1}{(1+k)^{1+\alpha}}\right]=\frac{1}{2n^{\alpha}},$$

we have for $n\geqslant1$,

$$\|f\|^*_{\alpha,2}=\sup_{n\geqslant1}\phi_n^{-\alpha}\|L_n(f)-f\|_{C[-1,1]}\leqslant\frac{1}{2}(\sqrt{3})^{\alpha}.$$

The results that $\|f\|^*_{\alpha,1}=\infty$ and $\|f\|^*_{\alpha,2}\leqslant\frac{1}{2}(\sqrt{3})^{\alpha}$ show that Theorem A is not true.

§ 3. Some Similar Theorems on $C_{2\pi}$

Although counter examples 1 and 2 show that both Lemma A and Theorem A are not true, we shall prove that the similar theorem on $C_{2\pi}$ can be establised correctly.

We introduce some similar terms at first.

Definition 1. *Let $\{dU_n(t)\}$ be positive Borel measures on $(-\infty, \infty)$ with period 2π satisfying*

$$\frac{1}{\pi}\int_{-\pi}^{\pi}dU_n(t)=1.$$

If $\psi_n^2=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2\frac{t}{2}\,dU_n(t)\to0$ $(n\to\infty)$ and for any $\varepsilon>0$ there exists $A_\varepsilon>0$ and $N_\varepsilon>0$ such that

$$\frac{1}{\pi}\int_{A_\varepsilon\psi_n<|t|<\pi}\sin^2\frac{t}{2}\,dU_n(t)\leqslant\varepsilon\psi_n^2$$

hold for $n>N_\varepsilon$, we say that $\{dU_n(t)\}$ is mass-concentrative and the following operators

$$1_n(f, x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x+t)+f(x-t)]dU_n(t)$$

on $C_{2\pi}$ are called mass-concentrative operators.

We denote

$$\|f\|^{**}_{\alpha,1}=\sup_{t>0} t^{-\alpha}\|\Delta^2_t f(\cdot)\|_{C_{2\pi}},$$

$$\|f\|^{**}_{\alpha,2}=\sup_{n\geqslant 1} \psi_n^{-\alpha}\|1_n(f)-f\|_{C_{2\pi}}$$

for $f\in C_{2\pi}$ and $0<\alpha\leqslant 2$.

A simple estimation gives the following theorem.

Theorem 1. *For* $\{1_n(f)\}$ *as above and* $0<\alpha\leqslant 2$,

$$\|f\|^{**}_{\alpha,2}\leqslant \frac{\pi^\alpha}{2}\left(\frac{1}{2^\alpha}+\pi^{2-\alpha}\right)\|f\|^{**}_{\alpha,1}$$

holds for all $f\in C_{2\pi}$.

Now we establish a theorem similar to Theorem A.

Theorem 2. *Let* $K_n(t)\in L^1_{2\pi}$, $K_n(t)\geqslant 0$, $\{K_n(t)dt\}$ *be mass-concentrative measures,* $\{\psi_n\}$, $\{1_n(f)\}$ *as above,* $0<\alpha\leqslant 2$, *and*

$$\frac{\psi_n}{\psi_{n+1}}\leqslant C \quad (n=1,\ 2,\ \cdots)$$

for some constant $C>0$. *Then there exists a constant* $B_\alpha>0$ *such that*

$$\|f\|^{**}_{\alpha,1}\leqslant B_\alpha^{-1}\|f\|^{**}_{\alpha,2}$$

holds for every $f\in C_{2\pi}$.

The proof can be completed by a method similar to that used in thecase of $C[-2a,\ 2a]$ and in this case no barriers or mistakes occur. We omit the details.

Theorems 1—2 give the following conclusion immediately

Corollary 1. *Under the condition of Theorem* 2 *we have, for* $f\in C_{2\pi}$,

$$\|1_n(f)-f\|_{C_{2\pi}}=O(\psi_n^\alpha)$$

if and only if $f\in \mathrm{Lip}^*\alpha$, *where*

$$\mathrm{Lip}^*\alpha=\{f\in C_{2\pi}\,|\,\omega_2(f,\ \delta)=O(\delta^\alpha)\}.$$

As an application we consider Jackson-type operators. Let

$$L_{n,r}(t)=\lambda_{n,r}^{-1}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2r},$$

$$\lambda_{n,r}=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2r}dt.$$

Then[2]

$$\psi_n^2=\frac{1}{\pi}\int_{-\pi}^{\pi}\sin^2(t/2)L_{n,r}(t)\,dt\approx n^{-2},$$

$$\lambda_{n,r}=n^{2r-1}\quad (r\geqslant 2).$$

It is not difficult to prove that $\{L_{n,r}(t)dt\}$ is mass-concentrative. Hence we have

Theorem 3. *Let* $r\geqslant 2$ *and* $\{L_{n,r}(t)dt\}$ *be as above. For Jackson-type operators*

$$L_n(f,\ x)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x-t)L_{n,r}(t)\,dt,\ f\in C_{2\pi},$$

we have for each $f\in C_{2\pi}$, $0<\alpha\leqslant 2$,

$$\|L_n(f)-f\|_{C_{2\pi}}=O(n^{-\alpha})$$

if and only if $f\in \mathrm{Lip}^*\alpha$.

§ 4. Some Results on C[$-2a$, $2a$]

In this section we shall establish some theorems on $C[-2a, 2a]$ by adding some proper conditions

Lemma 1. *Let $\{dU_n(t)\}$, a sequence of Borel measures on $[-a, a]$, be mass-concentrative, $\phi_n\to 0$ $(n\to\infty)$ and $0<\alpha\leqslant 2$. Then there esists a natural number $N_{1/2}$ such that for $n>N_{1/2}$*

$$\frac{1}{2}A_{1/2}^{\alpha-2}\phi_n^\alpha\leqslant\int_{-a}^{a}|t|^\alpha dU_n(t)\leqslant\left(A_{1/2}^\alpha+\frac{1}{2}A_{1/2}^{2-\alpha}\right)\phi_n^\alpha$$

hold.

Proof For $\varepsilon=1/2$ there exists a constant $A_{1/2}>0$ and $N_{1/2}>0$ such that for $n>N_{1/2}$.

$$\int_{A_{1/2}\phi_n<|t|<a}t^2\,dU_n(t)\leqslant\frac{1}{2}\phi_n^2$$

hold. Then

$$\begin{aligned}\int_{-a}^{a}|t|^\alpha dU_n(t)&=\int_{-A_{1/2}\phi_n}^{A_{1/2}\phi_n}+\int_{A_{1/2}\phi_n<|t|<a}\\&\leqslant(A_{/2}\phi_n)\int_{-a}^{a}dU_n(t)+(A_{1/2}\phi_n)^{\alpha-2}\cdot\frac{1}{2}\phi_n^2\\&\leqslant\left(A_{1/2}^\alpha+\frac{1}{2}A_{1/2}^{\alpha-2}\right)\phi_n^\alpha.\end{aligned}$$

The other side of our inequality can be proved similarly.

For the following mass-concentrative operators on $C[-2a, 2a]$ mentioned in § 1:

$$L_n(f, x)=\frac{1}{2}\int_{-a}^{a}[f(x+t)+f(x-t)]dU_n(t),$$

we can establish the following theorem.

Theorem 4. *Let $K_n(t)\in L^1[-a, a]$, $K_n(t)\geqslant 0$, $\{K_n(t)dt\}$ be mass-concentrative and*

$$C^{-1}\leqslant\frac{\phi_n}{\phi_{n+1}}\leqslant C\quad(n=1, 2, \cdots)$$

for some constant $C>0$. For any $f\in C[-2a, 2a]$, if there exists a positive number $\eta<a$ such that the extension $\tilde{f}$ on $(-\infty, \infty)$ with period $2a$ of

$$g(x)=f(x)-\frac{1}{2a}[f(a)-f(-a)],\quad x\in C[-a, a]$$

belongs to $\mathrm{Lip}^*\alpha_{[a-\eta, a+\eta]}$, $(0<\alpha\leqslant 2)$, *then*

$$\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$$

implies $f\in\mathrm{Lip}^*\alpha_{[-a,a]}$, *where*

$$\mathrm{Lip}^*\alpha_{[-a,a]}=\{f\in C[-a, a]\,|\,\omega_2(f, \delta)=O(\delta^\alpha)\}.$$

Proof For $|x|\leqslant a-\frac{1}{2}\eta$ we have

$$|L_n(\tilde{f},x)-\tilde{f}(x)|\leqslant\left|\frac{1}{2}\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}}\Delta_t^2f(x)K_n(t)dt\right|+\frac{16}{\eta^2}\|f\|_{C[-2a,2a]}\int_{\frac{1}{2}\eta<|t|<a}t^2K_n(t)dt$$

$$\leqslant\frac{1}{2}\left|\int_{-a}^{a}\Delta_t^2f(x)K_n(t)dt\right|+\frac{24}{\eta^2}\|f\|_{C[-2a,2a]}\int_{\frac{1}{2}\eta<|t|<a}t^2K_n(t)dt$$

$$=O(1)[\|f\|_{\alpha,2}^*\phi_n^\alpha+\phi_n^\alpha]+O(\phi_n^\alpha),$$

and for $a\geqslant|x|>a-\frac{1}{2}\eta$ we have

$$|L_n(\tilde{f},x)-\tilde{f}(x)|\leqslant\frac{1}{2}\int_{-\frac{1}{2}\eta}^{\frac{1}{2}\eta}|\Delta_t^2\tilde{f}(x)|K_n(t)dt+\frac{16}{\eta^2}\|f\|_{C[-2a,2a]}\int_{\frac{1}{2}\eta<|t|<a}t^2K_n(t)dt$$

$$=O(1)\left[\int_{-\frac{1}{2}\eta}^{\frac{1}{2}\eta}|t|^\alpha K_n(t)dt+\phi_n^\alpha\right]=O(\phi_n^\alpha).$$

Hence

$$\|L_n(\tilde{f})-\tilde{f}\|_{C[-a,a]}=O(\phi_n^\alpha).$$

Using Corollary 1 (we may assume $a=\pi$ without lossing generality) we have $\tilde{f}\in\mathrm{Lip}^*\alpha$ and consequently $g\in\mathrm{Lip}^*\alpha_{[-a,a]}$, then $f\in\mathrm{Lip}^*\alpha_{[-a,a]}$. This proves our Theorem 4.

Theorem 5. *Let $\{K_n(t)dt\}$ be as in Theorem 4, $0<\alpha\leqslant2$. Then for $f\in C[-2a,2a]$ having twice continuous derivatives on some left neighborhood of a and some right neighborhood of $-a$,*

$$\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$$

implies $f\in\mathrm{Lip}^*\alpha_{[-a,a]}$.

Proof Choose proper constants α, β such that for $h(x)=f(x)+\alpha x^3+\beta x^2$

$$h_+'(-a)=h_+'(a),\ h_+''(-a)==h_-''(a)$$

hold Then by Theorem 4 we see that $\|L_n(h)-h\|_{C[-a,a]}=O(\phi_n^\alpha)$ implies $h\in\mathrm{Lip}^*\alpha_{[-a,a]}$. Since. $\|L_n(h)-h\|_{C[-a,a]}=O(\phi_n^\alpha)$ is equivalent to $\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$ we see that $\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$ implies $h\in\mathrm{Lip}^*\alpha_{[-a,a]}$, and so $f\in\mathrm{Lip}^*\alpha_{[a,a]}$. The proof ends.

For $0<\alpha<1$ or $1<\alpha<2$ we can weaken the conditions of f in Theorem 5, that is, we have the following theorem.

Theorem 6. *Let $0<\alpha<1$ or $1<\alpha<2$, $\{K_n(t)dt\}$ be as in Theorem 4. Then for each $f\in C[-2a,2a]$, if $f\in\mathrm{Lip}^*\alpha_{[-a,-a+\eta]}\cap\mathrm{Lip}^*\alpha_{[a-\eta,a]}$ for some $\eta>0$,*

$$\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$$

implies $f\in\mathrm{Lip}^*\alpha_{[-a,a]}$.

We prove the following lemma at first.

Lemma 2. *Let $0<\alpha<1$ or $1<\alpha<2$, $0<\eta<a$, $f\in\mathrm{Lip}^*\alpha_{[-a,-a+\eta]}\cap\mathrm{Lip}^*\alpha_{[a-\eta,a]}$. Then there exists a function $F\in\mathrm{Lip}^*\alpha_{[-2a,2a]}$ such that $F(x)=f(x)$ for $x\in[-a,-a+\eta]\cup[a-\eta,a]$.*

Proof For $0<\alpha<1$ since $\mathrm{Lip}^*\alpha_{[-a,-a+\eta]}=\mathrm{Lip}\alpha_{[-a,-a+\eta]}$, where $\mathrm{Lip}\ \alpha_{[-a,-a+\eta]}=\{f\in C[-a,-a+\eta]\mid\omega(f,\delta)=O(\delta^\alpha)\}$, we have $f\in\mathrm{Lip}\alpha_{[-a,-a+\eta]}$. Similarly we have $f\in\mathrm{Lip}\alpha_{[a-\eta,a]}$. Define

$$F(x)=\begin{cases} f(-a), & x\in[-2a,-a];\\ f(a), & x\in[a,2a],\\ f(x), & x\in[-a,-a+\eta]\cup[a-\eta,a],\\ f(-a+\eta)+\dfrac{f(a-\eta)-f(a+\eta)}{2(a-\eta)}(x+a-\eta), & x\in[-a+\eta,a-\eta].\end{cases}$$

It is obvious that $F\in\mathrm{Lip}\alpha_{[-2a,2a]}$ and then $F\in\mathrm{Lip}^*\alpha_{[-2a,2a]}$.

For $1<\alpha<2$, since $f\in\mathrm{Lip}^*\alpha_{[-a,-a+\eta]}$, f' exists on $[-a,-a+\eta]$ and $f'\in\mathrm{Lip}(\alpha-1)_{[-a,-a+\eta]}$. Similarly, f' exists on $[a-\eta,a]$ and $f'\in\mathrm{Lip}(\alpha-1)_{[a-\eta,a]}$. Choose proper constants C_0, C_1, C_2, C_3 such that for $g(x)=C_0+C_1x+C_2x^2+C_3x^3$

$$g(\pm(a-\eta))=f(\pm(a-\eta)),\ g'(\pm a-\eta))=f'(\pm(a-\eta))$$

hold. Define

$$F(x)=\begin{cases} f(-a)+f'_+(-a)(x+a), & x\in[-2a,-a],\\ f(a)-f'_-(a)(x-a), & x\in[a,2a],\\ f(x), & x\in[-a,-a+\eta]\cup[a-\eta,a],\\ g(x), & x\in[-a+\eta,a-\eta].\end{cases}$$

It is obvious that $F'\in\mathrm{Lip}(\alpha-1)_{[-2a,2a]}$, and so $F\in\mathrm{Lip}^*\alpha_{[-2a,2a]}$. Our lemma is proved.

Now we turn to the proof of Theorem 6.

For $f\in\mathrm{Lip}^*\alpha_{[-a,-a+\eta]}\cap\mathrm{Lip}^*\alpha_{[a-\eta,a]}$ let $F\in\mathrm{Lip}^*\alpha_{[-2a,2a]}$ as above. Then $f\in\mathrm{Lip}^*\alpha_{[-a,a]}$ is equivalent to $h(x)=f(x)-F(x)\in\mathrm{Lip}^*\alpha_{[-a,a]}$. Since $\|F\|^*_{\alpha,2}\leqslant\|F\|^*_{\alpha,1}<\infty$, $\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$ is equivalent to $\|L_n(h)-h\|_{C[-a,a]}=O(\phi_n^\alpha)$. However $\|L_n(h)-h\|_{C[-a,a]}=O(\phi_n^\alpha)$ implies $h\in\mathrm{Lip}^*\alpha_{[-a,a]}$ because $h\in C[-2a,2a]$ satisfies all the conditions of Theorem 5. Therefore, $\|L_n(f)-f\|_{C[-a,a]}=O(\phi^\alpha)$ implies $f\in\mathrm{Lip}^*\alpha_{[-a,a]}$. Our Theorem 6 is proved.

For $K_n(t)\in L^1[-a,a]$, $K_n(t)\geqslant0$ we consider the following operators on $C\left[-\frac{1}{2}a,\frac{1}{2}a\right]$

$$\tilde{L}_n(f,x)=\int_{-\frac{1}{2}a}^{\frac{1}{2}a}f(t)K_n(x-t)dt.$$

We have the following theorem.

Theorem 7. *Let $0<\alpha\leqslant2$, $K_n(t)\geqslant0$ be even functions, $\{K_n(t)dt\}$ be mass-concentrative measures on $[-a,a]$, and there exist a constant $C>0$ such that*

$$C^{-1}\leqslant\frac{\phi_n}{\phi_{n+1}}\leqslant C\quad(n=1,2,\cdots).$$

Then for each $f\in C\left[-\frac{1}{2}a,\frac{1}{2}a\right]$ having twice continuous derivatives on $\left[-\frac{1}{2}a,\right.$

$-\frac{1}{2}a+\eta\Big]$ *and* $\Big[\frac{1}{2}a-\eta,\ \frac{1}{2}a\Big]$ *for some positive number* $\eta<\frac{1}{4}a$

$$\|\tilde{L}_n(f)-f\|_{C[-\frac{1}{2}a,\frac{1}{2}a]}=O(\phi_n^\alpha)$$

implies $f\in \mathrm{Lip}^*\alpha_{[-\frac{1}{2}a,\frac{1}{2}a]}$.

Proof Denote

$$C_n=\int_{-\frac{1}{2}(a-\eta)}^{\frac{1}{2}(a-\eta)}K_n(t)dt,\quad \tilde{K}_n(t)=\frac{1}{C_n}K_n(t);\quad \tilde{\phi}_n^2=\int_{-\frac{1}{2}(a-\eta)}^{\frac{1}{2}(2-\eta)}t^2K_n(t)dt.$$

It is easy to prove that there exists a constant $\tilde{C}>0$ such that

$$\tilde{C}^{-1}\leqslant\frac{\tilde{\phi}_n}{\tilde{\phi}_{n+1}}\leqslant\tilde{C},\ \lim_{n\to\infty}C_n=1,\ \lim_{n\to\infty}\frac{\tilde{\phi}_n}{\phi_n}=1,$$

and $\{\tilde{K}_n(t)dt\}$ is mass-concentrative on $\Big[-\frac{1}{2}(a-\eta),\ \frac{1}{2}(a-\eta)\Big]$. Let

$$\tilde{f}(x)=\begin{cases}f(x), & |x|\leqslant\frac{1}{2}a-\frac{1}{4}\eta,\\ f\Big(\frac{1}{2}a-\frac{1}{4}\eta\Big), & x\in\Big[\frac{1}{2}a-\frac{1}{4}\eta,\ a-\eta\Big],\\ f\Big(-\frac{1}{2}a+\frac{1}{4}\eta\Big), & x\in\Big[-a+\eta,\ -\frac{1}{2}a+\frac{1}{4}\eta\Big].\end{cases}$$

Then for $|x|\leqslant\frac{1}{2}(a-\eta)$ we have

$$\begin{aligned}\tilde{L}_n(f,x)-f(x)=&\frac{1}{2}C_n\int_{-\frac{1}{2}(a-\eta)}^{\frac{1}{2}(a-\eta)}[\tilde{f}(x+t)+\tilde{f}(x-t)-2\tilde{f}(x)]\tilde{K}_n(t)dt\\ &-\frac{1}{2}C_n\int_{\frac{1}{4}\eta<|t|\leqslant\frac{1}{2}(a-\eta)}[\tilde{f}(x+t)+f(x-t)-2f(x)]\tilde{K}_n(t)dt\\ &-\int_{\frac{1}{4}\eta<|t|<a}f(x)K_n(t)dt+\Big(\int_{x-\frac{1}{2}a}^{-\frac{1}{4}\eta}\int_{\frac{1}{4}\eta}^{x+\frac{1}{2}a}\Big)f(x-t)K_n(t)dt\\ =&C_n[L_n(\tilde{f},x)-\tilde{f}(x)]+R_n(x),\end{aligned}$$

where

$$L_n(\tilde{f},x)=\frac{1}{2}\int_{-\frac{1}{2}(a-\eta)}^{\frac{1}{2}(a-\eta)}[\tilde{f}(x+t)+\tilde{f}(x-t)]\tilde{K}_n(t)dt,$$

$$\|R_n(\cdot)\|_{C[-\frac{1}{2}(a-\eta),\frac{1}{2}(a-\eta)]}=O(\phi_n^2).$$

Hence

$$\|\tilde{L}_n(f)-f\|_{C[-\frac{1}{2}a,\frac{1}{2}a]}=O(\phi_n^\alpha)$$

implies $\|L_n(\tilde{f})-\tilde{f}\|_{C[-\frac{1}{2}(a-\eta),\frac{1}{2}(a-\eta)]}=O(\tilde{\phi}_n^\alpha)$. Consequently, by Theorem 5 we have $\tilde{f}\in \mathrm{Lip}^*\alpha_{[-\frac{1}{2}(a-\eta),\frac{1}{2}(a-\eta)]}$ and so $f\in \mathrm{Lip}^*\alpha_{[-\frac{1}{2}(a-\eta),\frac{1}{2}(a-\eta)]}$. Since $f\in \mathrm{Lip}^*\alpha_{[-\frac{1}{2}a,-\frac{1}{2}(a-2\eta)]}$ $\cap\,\mathrm{Lip}^*\alpha_{[\frac{1}{2}(a-2\eta),\frac{1}{2}a]}$, we get at last $f\in \mathrm{Lip}^*\alpha_{[-\frac{1}{2}a,\frac{1}{2}a]}$, which is what we need.

Similarly by Theorem 6 we can obtain

Theorem 8. *Let* $0<\alpha<1$ *or* $1<\alpha<2$ *and* $\{K_n(t)dt\}$ *be as in Theor-em 7. For* $f\in C\left[-\frac{1}{2}a,\ \frac{1}{2}a\right]$, *if* $f\in \text{Lip}^*\alpha_{[-\frac{1}{2}a,-\frac{1}{2}a+\eta]}\cap \text{Lip}^*\alpha_{[\frac{1}{2}a-\eta,\frac{1}{2}a]}$ *for some positive number* $0<\eta<\frac{1}{4a}$, *then*

$$\|\tilde{L}_n(f)-f\|_{C[-\frac{1}{2}a,\frac{1}{2}a]}=O(\phi_n^\alpha)$$

implies $f\in \text{Lip}^*\alpha_{[-\frac{1}{2}a,\frac{1}{2}a]}$.

As an application we can obtain two inverse theorems of approximation of the following operators

$$L_n^*(f,\ x)=C_n\int_{-1/2}^{1/2}f(t)\,[1-(x-t)^2]^n dt,\ x\in[-1/2,\ 1/2],$$

where

$$C_n\int_{-1/2}^{1/2}(1-t^2)^n dt=1,$$

by Theorems 7—8. We omit the satatements.

Remarks. The following questions are still open.

(i) Under the conditions of Theorem A, does

$$\|L_n(f)-f\|_{C[-a,a]}=O(\phi_n^\alpha)$$

imply $f\in \text{Lip}^*\alpha_{[-a,a]}$?

(ii) Are Theorems 6 and 8 also true for $\alpha=1$ or 2?

(iii) Is Lemma 2 also true for $\alpha=1$ or 2?

References

[1] Devore, R. A., The approximation of continuous functions by positive operators, Springer-Verlag, Berlin /Heidelberg/New York, 1972, 254—262.
[2] Lorentz, G. G., Approximation of functions, Holt, Rinehart and Winston, New York, 1966, 57.
[3] Schumaker, L., Spline functions: Basic theory, John Wiley & Sons New York. Chickester, Brisbane, Toronto, 1981, 57—58.