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A NECESSARY AND SUFFICIENT CONDITION FOR THE OSCILLATION OF HIGHER-ORDER NEUTRAL EQUATIONS WITH SEVERAL DELAYS**

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Abstract

Consider the higher-order neutral delay differential equation

$$\frac{d^n}{dt^n}\Big(x(t)+\sum_{i=1}^{l}p_i x(t-\tau_i)-\sum_{j=1}^{m}r_j x(t-\rho_j)\Big)+\sum_{k=1}^{N}q_k x(t-u_k)=0, \tag{A}$$

where the coefficients and the delays are nonnegative constants with $n\geqslant 2$ even. Then a necessary and sufficient condition for the oscillation of (A) is that the characteristic equation

$$\lambda^n+\lambda^n\sum_{i=1}^{l}p_i e^{-\lambda\tau_i}-\lambda^n\sum_{j=1}^{m}r_j e^{-\lambda\rho_j}+\sum_{k=1}^{N}q_k e^{-\lambda u_k}=0$$

has no real roots.

§1. Introduction

Neutral delay differential equations are differential equations in which the highest order derivative of the unknown function appears both with and without delays. The problem of oscillations of neutral equations is of both theoretical and practical interest. For example, the equations of this type appear in networks containing lossless transmission[3,10]. The oscillation theory of neutral equations has been extensively developed during the past few years[4,5,6,9].

In this paper, we consider the oscillations of higher-order neutral delay differential equations

$$\frac{d^n}{dt^n}\Big(x(t)+\sum_{i=1}^{l}p_i x(t-\tau_i)-\sum_{j=1}^{m}r_j x(t-\rho_j)\Big)+\sum_{k=1}^{N}q_k x(t-u_k)=0, \tag{1.1}$$

where the coefficients and the delays are nonnegative constants with $n\geqslant 2$ even.

Let $\phi\in C([t_0-T, t_0], R)$, where $T=\max\{\tau_i, \rho_j, u_k: 1\leqslant i\leqslant l, 1\leqslant j\leqslant m, 1\leqslant k\leqslant N\}$. By a solution of (1.1) with initial function ϕ at t_0, we mean a function $x\in C([t_0-T, \infty), R)$ such that $x(t)=\phi(t)$ for $t_0-T\leqslant t\leqslant t_0$,

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$$x(t)+\sum_{i=1}^{l}p_i x(t-\tau_i)-\sum_{j=1}^{m}r_j x(t-\rho_j)$$

is n-times continuously differentiable, and x satisfies (1.1) for all $t\geqslant t_0$. By using the method of steps, it follows that for every continuous function ϕ, there is a unique solution of (1.1) valid for $t\geqslant t_0$. For further questions on existence, uniqueness and continuous dependence, see Bellman and Cooke [1], Driver [2] and Hale [3].

As is customary, a solution is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. The characteristic equation of (1.1) is

$$\lambda^n+\lambda^n\sum_{i=1}^{l}p_i e^{-\lambda\tau_i}-\lambda^n\sum_{j=1}^{m}r_j e^{-\lambda\rho_j}+\sum_{k=1}^{N}q_k e^{-\lambda u_k}=0. \tag{1.2}$$

Our aim is to give a necessary and sufficient condition for all solutions of (1.1) to be oscillatory. We have

Theorem. *All solutions of* (1.1) *oscillate if and only if the characteristic equation* (1.2) *has no real roots.*

The proof of this theorem will be given in Section 3.

For the case $n=1$, the above result was proved recently by Grammatikopoulos, Sficas and Stavroulakis[5]. For the case that n is odd, the above result can be proved by using the similar arguments and the proof is omitted.

§2. Lammas

In this section we establish some useful lemmas which will be used in the proof of our main theorem.

In (1.1), without loss of generality we assume that $0<\tau_1<\tau_2<\cdots<\tau_l$, $0<\rho_1<\rho_2<\cdots<\rho_m$, $\tau_i\neq\rho_j(i=1, 2, \cdots, l;\ j=1, 2, \cdots, m)$, and $0\leqslant u_1<u_2<\cdots<u_N$. Let $P=\sum_{i=1}^{l}p_i$, $R=\sum_{j=1}^{m}r_j$ and $Q=\sum_{k=1}^{N}q_k$.

Lemma 1. *If $x(t)$ is a solution of* (1.1), *then each one of the following functions*

$$x(t-a), \int_{t-a}^{t-b}x(u)du,$$

$\dot{x}(t)$ (if $x(t)$ is continuously differentiable) is also a solution of (1.1), *where a and b are real numbers.*

The proof is trivial and is omitted.

Lemma 2. *If* (1.2) *has no real roots, then we have*

$$Q>0 \text{ with } \rho_m<\max\{\tau_l, u_N\}. \tag{2.1}$$

The proof is trivial and is omitted.

Lemma 3. *Assume that there is a nonoscillatory solution of* (1.1). *Then there is a nonoscillatory solution* $w(t)$ *of* (1.1) *such that either*

$$w(t)\in(I):=\{w(t)\in C^{2n}([T^*,\infty),R):(-1)^k w^{(k)}(t)>0,\ \lim_{t\to\infty}w^{(k)}(t)=0,\ k=0,1,2,\cdots,n\}$$

or

$$w(t)\in(II):=\{w(t)\in C^{2n}([T^*,\infty),R):w^{(k)}(t)>0,\ \lim_{t\to\infty}w^{(k)}(t)=\infty,\ k=0,1,2,\cdots,n\},$$

where $T^*\geqslant t_0$ *is sufficiently large.*

Proof As the negative of a solution of (1.1) is also a solution of the same equation, it suffices to consider that $x(t)$ is an eventually positive solution of (1.1). Set

$$z(t)=x(t)+\sum_{i=1}^{l}p_i x(t-\tau_i)-\sum_{j=1}^{m}r_j x(t-\rho_j), \tag{2.2}$$

and

$$w(t)=z(t)+\sum_{i=1}^{l}p_i z(t-\tau_i)-\sum_{j=1}^{m}r_j z(t-\rho_j). \tag{2.3}$$

By Lemma 1, $z(t)$ and $w(t)$ are solutions of (1.1). Then we have

$$z^{(n)}(t)=-\sum_{k=1}^{N}q_k x(t-u_k)<0, \tag{2.4}$$

$$w^{(n)}(t)=-\sum_{k=1}^{N}q_k z(t-u_k), \tag{2.5}$$

and so $z^{(n-1)}(t)$ is eventually strictly decreasing. Also all the derivatives of z of order less than or equal to $n-1$ are monotonic functions. From (2.4) it follows that either

$$\lim_{t\to\infty}z^{(n-1)}(t)=-\infty \tag{2.6}$$

or

$$\lim_{t\to\infty}z^{(n-1)}(t)=L, \tag{2.7}$$

L is finite.

If (2.6) holds, then

$$\lim_{t\to\infty}z^{(k)}(t)=-\infty,\ k=0,1,2,\cdots,n-1,$$

which imply

$$\lim_{t\to\infty}w^{(n)}(t)=\infty,$$

and so

$$\lim_{t\to\infty}w^{(k)}(t)=\infty,\ w^{(k)}(t)>0 \text{ eventually},\ k=0,1,2,\cdots,n.$$

Obviously, $z(t)\in C^n([t_0+T,\infty),R)$ and $w(t)\in C^{2n}([t_0+2T,\infty))$, that is, $w(t)\in(II)$.

If (2.7) holds, then integrating (2.7) from t_1 to t, with t_1 sufficiently large and letting $t\to\infty$ we find

$$L-z^{(n-1)}(t_1)=-\sum_{k=1}^{N}q_k\int_{t_1}^{\infty}x(s-u_k)ds,$$

which implies that $x\in L^1[t_1,\infty)$. Thus, from (2.2), $z\in L^1[t_1,\infty)$ and since z is monotonic, it follows that

$$\lim_{t\to\infty}z(t)=0, \tag{2.8}$$

and so $L=0$. As the function $z^{(n-1)}(t)$ decreases to zero, it follows that

$$z^{(n-1)}(t)>0. \tag{2.9}$$

Also (2.8) implies that consecutive derivatives of z must alternate signs and tend to zero as $t\to\infty$. Thus, in view of (2.9) and the fact that n is even we have

$$z(t)<0. \tag{2.10}$$

From (2.3) we have $w(t)\in L^1[t_1,\infty)$ and $w^{(n)}(t)>0$.

Using the similar arguments to $w(t)$ we obtain

$$(-1)^k w^{(k)}(t)>0,\ \lim_{t\to\infty}w^{(k)}(t)=0,\ k=0,1,2,\cdots,n.$$

Thus, $w(t)\in(I)$ and the proof is completed.

Lemma 4. *Assume that* (2.1) *holds, and that there is a nonoscillatory solution* $x(t)$ *of* (1.1). *Then there is a solution* $z(t)$ *of* (1.1) *which belongs to Class* (I) *or* (II), *such that the set*

$$\Lambda(z):=\{\lambda>0:-z^{(n)}(t)+\lambda^n z(t)\leqslant 0\}\neq\varnothing.$$

Proof By Lemma 3, we can assume that either $x(t)\in$ (I) or $x(t)\in$ (II). First, let $x(t)\in$ (I), and set

$$z(t)=-x(t)-\sum_{i=1}^{l}p_i x(t-\tau_i)+\sum_{j=1}^{m}r_j x(t-\rho_j). \tag{2.11}$$

It is easy to see that $z(t)\in(I)$. We consider the following two cases:

Case 1. $u_N\geqslant\rho_m$.

Noting that $x(t)$ is positive and decreasing, from (2.11) we have

$$z(t)<Rx(t-\rho_m)\leqslant Rx(t-u_N). \tag{2.12}$$

On the other hand,

$$z^{(n)}(t)=\sum_{k=1}^{N}q_k x(t-u_k)\geqslant q_N x(t-u_N). \tag{2.13}$$

Combining (2.12) and (2.13) we obtain

$$-z^{(n)}(t)+(q_N/R)z(t)\leqslant 0,$$

that is, $\lambda=(q_N/R)^{1/n}\in\Lambda(z)$.

Case 2. $\tau_l>\rho_m>u_N$.

As $z(t)$ and $x(t)$ are positive and decreasing, it follows that

$$z(t)<Rx(t-\rho_m) \tag{2.14}$$

and

$$\sum_{j=1}^{m} r_j x(t-\rho_j) > x(t) + \sum_{i=1}^{l} p_i x(t-\tau_i) > p_l x(t-\tau_l),$$

which implies that

$$x(t+(\tau_l-\rho_m)) > (p_l/R)x(t). \tag{2.15}$$

Let k be the first integer satisfying $\rho_m - u_N \leqslant k(\tau_l-\rho_m)$. Then combining (2.13), (2.14) and (2.15) we obtain

$$\begin{aligned} 0 &\geqslant -z^{(n)}(t) + q_N x(t-u_N) = -z^{(n)}(t) + q_N x(t-\rho_m + (\rho_m - u_N)) \\ &\geqslant -z^{(n)}(t) + q_N x(t-\rho_m + k(\tau_l-\rho_m)) \geqslant -z^{(n)}(t) + q_N (p_l/R)^k x(t-\rho_m) \\ &\geqslant -z^{(n)}(t) + (q_N p_l^k / R^{k+1}) z(t). \end{aligned}$$

Thus, $\lambda = (q_N p_l^k / R^{k+1})^{1/n} \in \Lambda(z)$.

Next, let $x(t) \in (II)$, and set

$$z(t) = -x(t) - \sum_{i=1}^{l} p_i x(t-\tau_i) + \sum_{j=1}^{m} r_j x(t-\rho_j).$$

It is easy to see that $z(t) \in (II)$, and so $x(t)$ and $z(t)$ are both positive and increasing. It follows that

$$z(t) < Rx(t-\rho_1) \tag{2.16}$$

and

$$x(t) < Rx(t-\rho_1). \tag{2.17}$$

Let k be the first integer satisfying $u_1 \leqslant k\rho_1$. Then

$$\begin{aligned} 0 &\geqslant -z^{(n)}(t) + q_1 x(t-u_1) \geqslant -z^{(n)}(t) + q_1 x(t-k\rho_1) \\ &\geqslant -z^{(n)}(t) + (q_1/R^{k-1}) x(t-\rho_1) \quad (\text{by}(2.17)) \\ &\geqslant -z^{(n)}(t) + (q_1/R^k) z(t). \quad (\text{by}(2.16)) \end{aligned}$$

Thus, $\lambda = (q_1/R^k)^{1/n} \in \Lambda(z)$ and the proof is completed.

Lemma 5. *Assume that* (2.1) *holds, and that there is a nonoscillatory solution* $x(t)$ *of* (1.1) *with* $x(t) \in$ (I) *or* $x(t) \in$ (II). *Then the set* $\Lambda(x)$ *has an upper bound which is independent of* x.

Proof We consider the following cases:

Case 1. $x(t) \in$ (I) with $\tau_l > \rho_m$.

Set

$$z(t) = -x(t) - \sum_{i=1}^{l} p_i x(t-\tau_i) + \sum_{j=1}^{m} r_j x(t-\rho_j), \tag{2.18}$$

and observe that $z(t) \in$ (I), and so $(-1)^k z^{(k)}(t)$ is positive and decreasing for $k=0$, 1, 2, $\cdots$, n. Then

$$\sum_{j=1}^{m} (-1)^k r_j x^{(k)}(t-\rho_j) > \sum_{i=1}^{l} (-1)^k p_i x^{(k)}(t-\tau_i),$$

which implies

$$R(-1)^k x^{(k)}(t-\rho_m) > p_l (-1)^k x^{(k)}(t-\tau_l).$$

Thus

$$(-1)^k x^{(k)}(t) > M(-1)^k x^{(k)}(t-w), \quad k=0, 1, 2, \cdots, n, \tag{2.19}$$

where $M=p_l/R$, $w=\tau_l-\rho_m>0$.

We now want to prove that $\lambda_0=-(1/w)\ \ln M\notin \Lambda(x)$. Otherwise, $\lambda_0\in\Lambda(x)$, which means that

$$-x^{(n)}(t)+\lambda_0^n x(t)\leqslant 0.$$

Set

$$y(t)=-x^{(n-1)}(t)+\lambda_0 x^{(n-2)}(t)+\cdots-\lambda_0^{n-2}\dot{x}(t)+\lambda_0^{n-1}x(t).$$

Then

$$\dot{y}(t)+\lambda_0 y(t)=-x^{(n)}(t)+\lambda_0^n x(t)\leqslant 0.$$

In view of (2.19) we have

$$y(t)>My(t-w). \tag{2.20}$$

Let $\phi(t)=e^{\lambda_0 t}y(t)$. Then

$$\dot{\phi}(t)=e^{\lambda_0 t}(\dot{y}(t)+\lambda_0 y(t))\leqslant 0,$$

and so $\phi(t)$ is decreasing. Thus, $\phi(t)\leqslant\phi(t-w)$, which implies that

$$y(t)\leqslant e^{-\lambda_0 w}y(t-w)=My(t-w).$$

It contradicts (2.20). Therefore, $\lambda_0=-(1/w)\ln M$ is an upper bound of $\Lambda(x)$.

Case 2. $x(t)\in(\mathrm{I})$ with $u_N>\rho_m$.

Let $z(t)$ be the function defined by (2.18). Then $z(t)\in(I)$.

It is easy to see that for every $k=0, 1, 2, \cdots, n$, $(-1)^k x^{(k)}(t)$ and $(-1)^k z^{(k)}(t)$ are positive and decreasing, and then

$$(-1)^k z^{(k)}(t)<(-1)^k Rx^{(k)}(t-\rho_m), \tag{2.21}$$

$$(-1)^{n+k}z^{(n+k)}(t)=\sum_{k=1}^{N}(-1)^k q_k x^{(k)}(t-u_k)\geqslant(-1)^k q_N x^{(k)}(t-u_N). \tag{2.22}$$

For $u_N>\rho_m$, there is a $b>0$ such that $u_N>\rho_m+nb$. By integrating (2.22) from t to $t+b$ we obtain

$$\begin{aligned}(-1)^{n+k}(z^{(n+k-1)}(t+b)-z^{(n+k-1)}(t))&\geqslant\int_t^{t+b}(-1)^k q_N x^{(k)}(s-u_N)ds\\&\geqslant(-1)^k q_N b x^{(k)}(t-(u_N-b)).\end{aligned}$$

As $(-1)^{n+k}z^{(n+k-1)}(t+b)<0$, it follows that

$$(-1)^{n+k-1}z^{(n+k-1)}(t)>q_N b(-1)^k x^{(k)}(t-(u_N-b)),$$

and after n steps we obtain

$$(-1)^k z^{(k)}(t)>q_N b^n(-1)^k x^{(k)}(t-(u_N-nb)),\ k=0, 1, 2, \cdots, n. \tag{2.23}$$

Combining (2.21) and (2.23) we have

$$(-1)^k x^{(k)}(t-\rho_m)>(q_N b^n/R)(-1)^k x^{(k)}(t-(u_N-nb)),$$

that is,

$$(-1)^k x^{(k)}(t)>(q_N b^n/R)(-1)^k x^{(k)}(t-(u_N-nb-\rho_m))$$

for $k=0, 1, 2, \cdots, n$.

Let $M=q_N b^n/R$, $w=u_N-nb-\rho_m>0$. Then, as in Case 1, we can show that $\lambda_0=-(1/w)\ln M$ is an upper bound of $\Lambda(x)$.

Case 3. $x(t)\in(\mathrm{II})$.

Set

$$z(t)=-x(t)-\sum_{i=1}^{l}p_ix(t-\tau_i)+\sum_{j=1}^{m}r_jx(t-\rho_j). \tag{2.24}$$

It is easy to see that $z(t)\in(\mathrm{II})$, and that $x^{(k)}(t)$and $z^{(k)}(t)$ are positive and increasing for $k=0,1,2,\cdots,n$. It follows that

$$x^{(k)}(t)<Rx^{(k)}(t-\rho_1),\ k=0,1,2,\cdots,n. \tag{2.25}$$

We now want to show that $\lambda_0=(1/\rho_1)\ln R$ is an upper bound of $\Lambda(x)$. Otherwise, $\lambda_0\in\Lambda(x)$, which means $-x^{(n)}(t)+\lambda_0^nx(t)\leqslant 0$. Set

$$y(t)=x^{(n-1)}(t)+\lambda_0x^{(n-2)}(t)+\cdots+\lambda_0^{n-1}x(t). \tag{2.26}$$

Then, from (2.25) and (2.26), we have

$$y(t)<Ry(t-\rho_1), \tag{2.27}$$

and

$$\dot{y}(t)-\lambda_0y(t)=x^{(n)}(t)-\lambda_0^nx(t)\geqslant 0.$$

Let $\phi(t)=e^{-\lambda_0t}y(t)$. Then

$$\dot{\phi}(t)=e^{-\lambda_0t}(y(t)-\lambda_0y(t))\geqslant 0,$$

and so $\phi(t)$ is increasing. Thus,

$$\phi(t)\geqslant\phi(t-\rho_1),$$

which implies that

$$y(t)\geqslant e^{\lambda_0\rho_1}y(t-\rho_1)=Ry(t-\rho_1).$$

It contradicts (2.27) and the proof of the lemma is completed.

§3. Main Result

Our Main result is the following

Theorem. *All solutions of* (1.1) *oscillate if and only if the characteristic equation* (1.2) *has no real roots.*

Proof The theorem will be proved in the contrapositive form: there is a nonoscillatory solution of (1.1) if and only if the characteristic equation (1.2) has a real root.

Assume first that (1.2) has a real root λ. Then, obviously, (1.1) has a nonoscillatory solution $x(t)=e^{\lambda t}$.

Assume, conversely, that there is a nonoscillatory solution of (1.1). Then we want to prove that (1.2) has a real root. Otherwise, assume that (1.2) has no real roots. Then, by Lemma 2, (2.1) holds. Let

$$F(\lambda)=\lambda^n+\lambda^n\sum_{i=1}^{l}p_ie^{-\lambda\tau_i}-\lambda^n\sum_{j=1}^{m}r_je^{-\lambda\rho_j}+\sum_{k=1}^{N}q_ke^{-\lambda u_k}.$$

Then we have $F(\infty)=F(-\infty)>0$ and

$$\alpha:=\min_{\lambda\in\mathbf{R}} F(\lambda)>0.$$

This implies

$$-\lambda^n-\lambda^n\sum_{i=1}^{l}p_ie^{-\lambda\tau_i}+\lambda^n\sum_{j=1}^{m}r_je^{-\lambda\rho_j}-\sum_{k=1}^{N}q_ke^{-\lambda u_k}\leqslant-\alpha \tag{3.1}$$

or

$$-\lambda^n-\lambda^n\sum_{i=1}^{l}p_ie^{\lambda\tau_i}+\lambda^n\sum_{j=1}^{m}r_je^{\lambda\rho_j}-\sum_{k=1}^{N}q_ke^{\lambda u_k}\leqslant-\alpha \tag{3.2}$$

for all $\lambda\in R$.

By Lemmas 3 and 4, if (1.1) has a nonoscillatory solution, then (1.1) has a nonoscillatory solution $x(t)$ which belongs to Class (I) or (II) with $\Lambda(x)\neq\varnothing$.

We now consider the following two cases:

Case 1. $x(t)\in(I)$ with $\Lambda(x)\neq\varnothing$.

Let $\lambda\in\Lambda(x)$. By Lemma 5, there is an upper bound λ_0 of $\Lambda(x)$ which is independent of x.

Let $T=\max\{\tau_l, u_N\}$, and set

$$y(t)=T_1x(t):=-x(t)-\sum_{i=1}^{l}p_ix(t-\tau_i)+\sum_{j=1}^{m}r_jx(t-\rho_j). \tag{3.3}$$

Obviously, $y(t)$ is a solution of (1.1) with $y(t)\in(I)$, and

$$y^{(n)}(t)=\sum_{k=1}^{N}q_kx(t-u_k). \tag{3.4}$$

Set

$$z(t)=T_2y(t):=y^{(n)}(t)-\lambda y^{(n-1)}(t)+\lambda^2y^{(n-2)}(t)-\cdots-\lambda^{n-1}\dot{y}(t). \tag{3.5}$$

It is easy to see that $z(t)$ is also a solution of (1.1) with $z(t)\in(I)$.

From (3.4) and (3.5), we have

$$z^{(n)}(t)+\lambda z^{(n-1)}(t)=y^{(2n)}(t)-\lambda^ny^{(n)}(t)$$
$$=\sum_{k=1}^{N}q_k(x^{(n)}(t-u_k)-\lambda^nx(t-u_k))\geqslant0$$

since $\lambda\in\Lambda(x)$. Let $\phi(t)=-e^{\lambda t}z^{(n-1)}(t)$. Then

$$\dot{\phi}(t)=-e^{\lambda t}(z^{(n)}(t)+\lambda z^{(n-1)}(t))\leqslant0,$$

and so $\phi(t)$ is positive and decreasing. Note that $z(t)\in(I)$ and $z(t)$ can be expressed as

$$z(t)=\int_t^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-2}}^\infty(-z^{(n-1)}(s))ds=\int_t^\infty dt_1\int_{t_1}^\infty dt_2\cdots\int_{t_{n-2}}^\infty e^{-\lambda s}\phi(s)ds,$$

which implies

$$z(t)\leqslant e^{-\lambda t}\phi(t)/\lambda^{n-1}. \tag{3.6}$$

Set

$$w(t)=T_3z(t):=z^{(n-1)}(t)+\sum_{i=1}^{l}p_iz^{(n-1)}(t-\tau_i)-\sum_{j=1}^{m}r_jz^{(n-1)}(t-\rho_j)$$
$$+\lambda^n\int_{t-T}^{t}z(u)du+\sum_{k=1}^{N}q_k\int_{t-T}^{t-u_k}z(u)du+\lambda^n\sum_{i=1}^{l}p_i\int_{t-T}^{t-\tau_i}z(u)du. \tag{3.7}$$

By Lemma 1, $w(t)$ is a solution of (1.1). We have

$$\dot{w}(t) = -\lambda^n(z(t-T)-z(t)) - \sum_{k=1}^{N} q_k z(t-T) - \lambda^n \sum_{i=1}^{l} p_i(z(t-T)-z(t-\tau_i)),$$

$$w^{(k)}(t) = -\lambda^n(z^{(k-1)}(t-T)-z^{(k-1)}(t)) - \sum_{k=1}^{N} q_k z^{(k-1)}(t-T)$$
$$-\lambda^n \sum_{i=1}^{l} p_i(z^{(k-1)}(t-T)-z^{(k-1)}(t-\tau_i))$$

for $k=2, 3, \cdots, n$. As $z(t)\in(I)$, it follows that $w(t)\in(I)$. Let

$$\mu = \alpha/(1+P+Q/\lambda^n+R)e^{\lambda T}. \tag{3.8}$$

We now show that

$$-w^{(n)}(t)+(\lambda^n+\mu)w(t)\leqslant 0. \tag{3.9}$$

In fact,

$$-w^{(n)}(t)+(\lambda^n+\mu)w(t)$$
$$=\lambda^n\Big(z^{(n-1)}(t-T)+\lambda^n\int_{t-T}^{t} z(u)\,du\Big) - \lambda^n \sum_{j=1}^{m} r_j z^{(n-1)}(t-\rho_j)$$
$$+\lambda^n \sum_{i=1}^{l} p_i\Big(z^{(n-1)}(t-T)+\lambda^n\int_{t-T}^{t-\tau_i} z(u)du\Big)$$
$$+\sum_{k=1}^{N} q_k\Big(z^{(n-1)}(t-T)+\lambda^n\int_{t-T}^{t-u_k} z(u)du\Big)$$
$$+\mu\Big(z^{(n-1)}(t)+\sum_{i=1}^{l} p_i z^{(n-1)}(t-\tau_i)-\sum_{j=1}^{m} r_j z^{(n-1)}(t-\rho_j)$$
$$+\lambda^n\int_{t-T}^{t} z(u)du+\sum_{k=1}^{N} q_k\int_{t-T}^{t-u_k} z(u)du+\lambda^n\sum_{i=1}^{l} p_i\int_{t-T}^{t-\tau_i} z(u)du\Big).$$

Noting that $-z^{(n-1)}(t)=\phi(t)e^{-\lambda t}$ with $\phi(t)$ positive and decreasing, by (3.6) we have

$$\lambda^n\int_{t-T}^{t} z(u)du\leqslant\phi(t-T)(e^{-\lambda(t-T)}-e^{-\lambda t}),$$

$$\lambda^n\sum_{k=1}^{N} q_k\int_{t-T}^{t-u_k} z(u)du\leqslant\sum_{k=1}^{N} q_k\phi(t-T)(e^{-\lambda(t-T)}-e^{-\lambda(t-u_k)})$$

and

$$\lambda^n\sum_{i=1}^{l} p_i\int_{t-T}^{t-\tau_i} z(u)du\leqslant\sum_{i=1}^{l} p_i\phi(t-T)(e^{-\lambda(t-T)}-e^{-\lambda(t-\tau_i)}).$$

Then

$$-w^{(n)}(t)+(\lambda^n+\mu)w(t)$$
$$\leqslant\phi(t-T)e^{-\lambda t}\Big(\Big(-\lambda^n-\lambda^n\sum_{i=1}^{l} p_i e^{\lambda\tau_i}+\lambda^n\sum_{j=1}^{m} r_j e^{\lambda\rho_j}-\sum_{k=1}^{N} q_k e^{\lambda u_k}\Big)$$
$$+\mu(1+P+Q/\lambda^n+R)e^{\lambda T}\Big)$$
$$\leqslant\phi(t-T)e^{-\lambda t}(-\alpha+\alpha)\quad\text{(by (3.2) and (3.8))}$$
$$=0,$$

and (3.9) holds, which means that

$$(\lambda^n+\mu)^{1/n}\in\Lambda(w).$$

Set

$$w(t)=Ux(t):=T_3(T_2(T_1x(t))),$$

and set $x_0=x$, $x_1=Ux_0$, and in general

$$x_k=Ux_{k-1},\ k=1,\ 2,\ \cdots$$

We observe that $x_k(t)\in(\mathrm{I})$ with $\Lambda(x_k)\neq\varnothing$, and that $\lambda\in\Lambda(x)=\Lambda(x_0)$ implies $(\lambda^n+\mu)^{1/n}\in\Lambda(w)=\Lambda(x_1)$ and after k steps we obtain

$$(\lambda^n+k\mu)^{1/n}\in\Lambda(x_k),\ k=1,\ 2,\ \cdots,$$

which is a contradiction since λ_0 is a common bound for all $\Lambda(x_k)$.

Case 2. $x(t)\in(\mathrm{II})$ with $\Lambda(x)\neq\varnothing$.

Let $\lambda\in\Lambda(x)$. By Lemma 5, there is an upper bound λ_0 of $\Lambda(x)$ which is independent of x. Let $b=\min\{\rho_1,\ \tau_1,\ u_1\}$, and set

$$y(t)=T_1x(t):=-x(t)-\sum_{i=1}^{l}p_ix(t-\tau_i)+\sum_{j=1}^{m}r_jx(t-\rho_j), \tag{3.10}$$

$$z(t)=T_2y(t):=y^{(n)}(t)+\lambda y^{(n-1)}(t)+\cdots+\lambda^{n-1}\dot{y}(t). \tag{3.11}$$

It is easy to see that $y(t)$ and $z(t)$ are solutions of (1.1), and that $y(t)\in(\mathrm{II})$, $z(t)\in(\mathrm{II})$. From (3.10), (3.11),

$$\begin{aligned}z^{(n)}(t)-\lambda z^{(n-1)}(t)&=y^{(2n)}(t)-\lambda^n y^{(n)}(t)\\&=\sum_{k=1}^{N}q_k(x^{(n)}(t-u_k)-\lambda^n x(t-u_k))\geqslant 0\end{aligned}$$

since $\lambda\in\Lambda(x)$.

Let $\phi(t)=e^{-\lambda t}z^{(n-1)}(t)$. Then

$$\dot{\phi}(t)=e^{-\lambda t}(z^{(n)}(t)-\lambda z^{(n-1)}(t))\geqslant 0,$$

and so $\phi(t)$ is increasing. Set

$$\begin{aligned}w(t)=T_3z(t):=&-z^{(n-1)}(t)-\sum_{i=1}^{l}p_iz^{(n-1)}(t-\tau_i)\\&+\sum_{j=1}^{m}r_jz^{(n-1)}(t-\rho_j)+\sum_{k=1}^{N}q_k\int_{t-u_k}^{t-b}z(u)du\\&+\sum_{i=1}^{l}p_i\lambda^n\int_{t-\tau_i}^{t-b}z(u)du.\end{aligned} \tag{3.12}$$

Then

$$\dot{w}(t)=\sum_{k=1}^{N}q_kz(t-b)+\lambda^n\sum_{i=1}^{l}p_i(z(t-b)-z(t-\tau_i)).$$

As $z(t)\in(\mathrm{II})$, it follows that

$$w^{(k)}(t)=\sum_{k=1}^{N}q_kz^{(k-1)}(t-b)+\lambda^n\sum_{i=1}^{l}p_i(z^{(k-1)}(t-b)-z^{(k-1)}(t-\tau_i))\to\infty,$$

as $t\to\infty$ for $k=1,\ 2,\ \cdots,\ n$. Obviously, $w(t)\to\infty$, as $t\to\infty$. Thus, $w(t)\in(\mathrm{II})$. Let

$$\mu=\alpha/(P+Q/\lambda^n+R). \tag{3.13}$$

We now show that

$$-w^{(n)}(t)+(\lambda^n+\mu)w(t)\leqslant 0. \tag{3.14}$$

In fact,

$$\begin{aligned}
&-w^{(n)}(t)+(\lambda^n+\mu)w(t)\\
&=-\lambda^n z^{(n-1)}(t)+\sum_{i=1}^{l}\lambda^n p_i\Big(-z^{(n-1)}(t-b)+\lambda^n\int_{t-\tau_i}^{t-b}z(u)du\Big)\\
&\quad+\sum_{j=1}^{m}\lambda^n r_j z^{(n-1)}(t-\rho_j)+\sum_{k=1}^{N}q_k\Big(-z^{(n-1)}(t-b)+\lambda^n\int_{t-u_k}^{t-b}z(u)du\Big)\\
&\quad+\mu\Big(-z^{(n-1)}(t)-\sum_{i=1}^{l}p_i z^{(n-1)}(t-\tau_i)+\sum_{j=1}^{m}r_j z^{(n-1)}(t-\rho_j)\\
&\quad+\sum_{k=1}^{N}q_k\int_{t-u_k}^{t-b}z(u)du+\sum_{i=1}^{l}\lambda^n p_i\int_{t-\tau_i}^{t-b}z(u)du\Big).
\end{aligned}$$

For $z(t)\in$ (II), there is a t^* such that for $t\geqslant t^*$,

$$z^{(k)}(t)>0,\ k=0,1,2,\cdots,n.$$

Then it is possible to extend the definition of $z^{(k)}(t)$ to $t<t^*$ such that for $k=0,1,2,\cdots,n$, $z^{(k)}(t)$ is continuous and increasing on $(-\infty,\infty)$ and $z^{(k)}(t)\to 0$ as $t\to-\infty$. Then $z(t)$ can be expressed as

$$z(t)=\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-2}}z^{(n-1)}(s)ds. \tag{3.15}$$

Noting that $z^{(n-1)}(t)=\phi(t)e^{\lambda t}$ with $\phi(t)$ increasing, we have by (3.15)

$$\begin{aligned}
\int_{t-u_k}^{t-b}z(u)du&\leqslant\phi(t-b)\int_{t-u_k}^{t-b}du\int_{-\infty}^{u}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-2}}e^{\lambda s}ds\\
&=\phi(t-b)(e^{\lambda(t-b)}-e^{\lambda(t-u_k)})/\lambda^n.
\end{aligned}$$

Similarly,

$$\int_{t-\tau_i}^{t-b}z(u)du\leqslant\phi(t-b)(e^{\lambda(t-b)}-e^{\lambda(t-\tau_i)})/\lambda^n.$$

Then

$$\begin{aligned}
&-w^{(n)}(t)+(\lambda^n+\mu)w(t)\\
&\leqslant\phi(t-b)e^{\lambda t}\Big(\Big(-\lambda^n-\sum_{i=1}^{l}p_i\lambda^n e^{-\lambda\tau_i}+\sum_{j=1}^{m}r_j\lambda^n e^{-\lambda\rho_j}-\sum_{k=1}^{N}q_k e^{-\lambda u_k}\Big)\\
&\quad+\mu(P+Q/\lambda^n+R)\Big)\\
&\leqslant\phi(t-b)e^{\lambda t}(-\alpha+\alpha)=0\quad(\text{by (3.1) and (3.13)}).
\end{aligned}$$

It follows that (3.14) holds, which means that

$$(\lambda^n+\mu)^{1/n}\in\Lambda(w).$$

Set $w(t)=Ux(t):=T_3(T_2(T_1x(t)))$, and let $x_0=x$, $x_1=Ux_0$, and in general

$$x_k=Ux_{k-1},\ k=1,2,\cdots$$

As in Case 1, we are led to a contradiction. This proves the theorem.

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