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ESTIMATE ON LOWER BOUND OF THE FIRST EIGENVALUE OF A COMPACT RIEMANNIAN MANIFOLD

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Abstract

The author gives an optimum estimate of the first eigenvalue of a compact Riemannian manifold. It is shown that let M be a compact Riemannian manifold, then the first eigenvalue λ_1 of the Laplace operator of M satisfies $\lambda_1+\max\{0, -(n-1)K\}\geqslant\pi^2/d^2$ where d is the diameter of M and $(n-1)K$ is the negative lower bound of the Ricci curvature of M.

§ 1. Introduction

The purpose of the note is to prove the following

Theorem. *Let M be a compact Riemannian manifold. Then the first eigenvalue λ_1 of the Laplace operator of M satisfies*

$$\lambda_1+\max\{0, -(n-1)K\}\geqslant\pi^2/d^2,$$

where d is the diameter of M and $(n-1)K$ is the negative lower bound of the Ricci curvature of M.

As well known, the Poincaré inequality plays a very important role in analysis of manifold and in fact a lower bound of the first eigenvalue gives an upper bound of the constant in the Poincaré inequality. It is very desirable to find a good lower estimate of the first eigenvalue. Li and Yau([1] p.216) proved that

$$\lambda_1+\max\{0, -(n-1)K\}\geqslant\pi^2/(4d^2).$$

Later, Zhong and Yang[2] improved the result and proved $\lambda_1\geqslant\pi^2/d^2$ if the Ricci curvature of M is non-negative. Hence, Theorem gives the optimum estimate of the eigenvalue on the compact manifold.

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§ 2. Preliminaries

Let M be an n-dimansional Riemannian manifold and $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field on M, $\{w_1, \cdots, w_n\}$ being its dual coframe. Then the structural equations of M are

$$dw_i = -\sum w_{ij} \wedge w_j,$$

$$dw_{ij} = -\sum w_{ik} \wedge w_{kj} + \frac{1}{2} \sum R_{ijkl} w_k \wedge w_l$$

for $1 \leqslant i, j, k \leqslant n$. Let f be a differential function on M. The covariant derivatives f_i, f_{ij} and f_{ijk} of f are successively defined by

$$df = \sum f_i w_i,$$

$$\sum f_{ij} w_j = df_i - \sum f_j w_{ji},$$

$$\sum f_{ijk} w_k = d f_{ij} - \sum f_{kj} w_{ki} - \sum f_{ik} w_{kj}.$$

From the structural equations we have $f_{ij} = f_{ji}$ and the Ricci identity

$$f_{ijk} = f_{ikj} + \sum f_l R_{lijk}.$$

The Laplacian of f is defined as

$$\Delta f = \sum f_{ii}.$$

Now assume that h is an eigenfunction corresponding to the first eigenvalue λ_1 on M, i. e.,

$$\Delta h = -\lambda_1 h. \tag{1}$$

Without lose the generality we may suppose $B = \max\{h: x \in M\} > 0$, $-b = \min\{h: x \in M\} < 0$ and $B > -b$. Put

$$u = \frac{2}{B+b}(h - B) + 1. \tag{2}$$

We have $\max u = 1$ and $\min u = -1$ on M. At that time, (1) becomes

$$\Delta u = -\lambda_1 (u + a) \tag{3}$$

where $a = (B-b)/(B+b)$, $0 \leqslant a < 1$. Letting $u = \sin v$, we can define a continuous function F on $(-\pi/2, \pi/2)$ by

$$F(v_0) = \max_{v(x)=v_0} |\nabla v|^2(x) \text{ for } v_0 \in (-\pi/2, \pi/2). \tag{4}$$

By substituting $u/(1+\varepsilon)$ for u in (2), and at last letting $\varepsilon \to 0$, the domains of $|\nabla v|^2$ and $F(v)$ can be expanded to the whole M and the Closed interval $[-\pi/2, \pi/2]$ respectively. By the compactness of M it is obvious to see that for each $v_0 \in [-\pi/2, \pi/2]$, there exists $x_0 \in M$ such that $v(x_0) = v_0$ and $|\nabla v|^2(x_0) = F(v_0)$.

§ 3. Gradient Estimates

First of all we set up an elementary estimation.

Lemma 1. *Let M be a compact Riemannian manifold and its Ricci curvature possess a negative lower bound $(n-1)K$. Then we have*

$$|\nabla v|^2 \leqslant \lambda_1\left(1+a-\frac{(n-1)K}{\lambda_1}\right) \tag{5}$$

and

$$\lambda_1 \geqslant \pi^2 \Big/ \left(1+a-\frac{(n-1)K}{\lambda_1}\right)d^2, \tag{6}$$

where $0 \leqslant a < 1$ and d is the diameter of M.

Proof Study the properties of the following function at the maximum points

$$\{|\nabla v|^2 - F(v_0)\}\cos^2 v, \tag{7}$$

where $F(v_0) = \max\{F(v): -\pi/2 \leqslant v \leqslant \pi/2\}$, but we omit the calculation. One can find the details in [2].

In order to gain an accuracy of the estimate in Lemma 1, we introduce a continuous function g on $[-\pi/2, \pi/2]$ defined by

$$F(v) = \lambda_1 A(1+ag(v)), \tag{8}$$

where we denote $A = 1-(n-1)K/\lambda_1$. Thus $g(v) \leqslant 1/A$. obviously, we only need to consider the case where $a>0$. For comparable function y with g the following is true.

Lemma 2. *Assume that function $y \in C^0[-\pi/2, \pi/2] \cap C^2(-\pi/2, \pi/2)$ satisfies*

I. *$y(v) \geqslant g(v)$ for $v \in [-\pi/2, \pi/2]$.*

II. *There exists $v_0 \in (-\pi/2, \pi/2)$ such that $y(v_0) = g(v_0) \geqslant -1$ and $y'(v_0) \geqslant 0$. Then*

$$y(v_0) \leqslant \frac{1}{A}\sin v_0 - y'(v_0)\sin v_0 \cos v_0 + \frac{1}{2}y''(v_0)\cos^2 v_0. \tag{9}$$

Proof Consider the function $G(x)$ on M defined by

$$G(x) = [|\nabla v|^2 - \lambda_1 A(1+ay(v(x)))\}\cos^2 v(x). \tag{10}$$

Then $G(x) \leqslant 0$ and $G(x)$ attains the maximum at $x = x_0$. Thus from the maximum principle, we have

$$G(x_0) = 0 \text{ and } \Delta G(x_0) \leqslant 0. \tag{11}$$

On the other hand, differentiating (10) directly, we get

$$G_j = 2\sum u_i u_{ij} + \lambda_1 A\{2(1+ay)\sin v - ay'\cos v\}u_j \tag{12}$$

and

$$\Delta G = 2\sum_{ij} u_{ij}^2 - 2\lambda_1 \sum u_i^2 + 2\sum R_{ij}u_iu_j + \lambda_1 A\{2(1+ay) + 3ay' \operatorname{tg} v - ay''\}\sum u_i^2$$
$$-\lambda_1^2 A\{2(1+ay)u - ay'\cos v\}(u+a). \tag{13}$$

According to the assumtion of $y(v_0)$, we get

$$\sum u_i^2(x_0) = \lambda_1 A(1+ay(v_0))\cos^2 v_0 > 0. \tag{14}$$

Combining (11) with (14) and using the Schwarz inequality, we have at $v_0 = v(x_0)$

$$\sum_{ij} u_{ij}^2 \geqslant \lambda_1^2 A^2\left\{(1+ay)\sin v - \frac{1}{2}ay'\cos v\right\}^2. \tag{15}$$

Substituting (15) into (13), from (11) it follows that at $v_0=v(x_0)$

$$0\geqslant\lambda_1^2A^2a(1+ay)\left\{\frac{2}{a}\sin^2v+2y+y'\sin v\cos v-y''\cos^2v+\frac{a(y')^2\cos^2v}{2(1+ay)}\right\}-\lambda_1^2A\{2(1+ay)\sin v-ay'\cos v\}(\sin v+a). \tag{16}$$

Here the main point is that we are in the process of deriving a uniform estimate of $y(v_0)$ in all different cases of v_0.

I. If $\sin v_0+a>0$ and $2(1+ay(v_0))\sin v_0\leqslant ay'(v_0)\cos v_0$, one can prove from (16) that at $v_0=v(x_0)$

$$\begin{aligned}0\geqslant&\lambda_1^2A^2(1+ay)\{2\sin^2v+2ay+2ay'\sin v\cos v-ay''\cos^2v\}\\&-2\lambda_1^2A\sin v(1+ay)(\sin v+a)\\=&\lambda_1^2A^2(1+ay)\left\{2\left(1-\frac{1}{A}\right)\sin^2v-\frac{2a\sin v}{A}\right.\\&\left.+2ay+2ay'\sin v\cos v-ay''\cos^2v\right\},\end{aligned} \tag{17}$$

by using $\qquad a(y'(v_0))^2\cos^2v_0/2(1+ay(v_0)\geqslant y'(v_0)\sin v_0\cos v_0$

and

$$\lambda_1^2Aay'\cos v_0(\sin v_0+a)\geqslant0,\ \text{where } 1>a>0,\ y(v_0)=g(v_0)\leqslant1/A.$$

II. If $2(1+ay(v_0))\sin v_0>ay'(v_0)\cos v_0$, then $\sin v_0>0$. Taking off $a(y'(v_0))^2\cos^2v_0/2(1+ay(v_0))$ and adding A in front of the last $\sin v_0$ from (16), we have at $v_0=v(x_0)$

$$\begin{aligned}0\geqslant&\lambda_1^2A^2(1+ay)\{2\sin^2v+2ay+ay'\sin v\cos v-ay''\cos^2v\}\\&-\lambda_1^2A\{2(1+ay)\sin v-ay'\cos v\}(A\sin v+a).\end{aligned} \tag{18}$$

Since $A\sin v_0+a\geqslant A\sin v_0(1+ay(v_0))\geqslant0$, we have

$$0\geqslant\lambda_1^2A^2(1+ay)\{2ay+2ay'\sin v\cos v-ay''\cos^2v\}.$$

So (9) still holds.

III. If $\sin v_0+a\leqslant0$, then $\sin v_0<0$. We can give up $a(y'(v_0))^2\cos^2v_0/2(1+ay(v_0))$ and obtain at $v_0=v(x_0)$

$$\begin{aligned}0\geqslant&(1+ay)(2\sin^2v+2ay+ay'\sin v\cos v-ay''\cos^2v)\\&-\{2(1+ay)\sin v-ay'\cos v\}(\sin v+a)\\\geqslant&a(1+ay)(2y-2\sin v+2y'\sin v\cos v-y''\cos^2v).\end{aligned} \tag{19}$$

Thus (9) is true. The proof is complete.

Lemma 3. *Suppose*

$$f(v)=\frac{4(v+\sin v\cos v)-2\pi\sin v}{\pi A\cos^2v}\quad for\ v\in(-\pi/2,\pi/2)$$

and $f(\pm\pi/2)=\pm1/A$. *Then*

$$f(v)\in C^0[-\pi/2,\pi/2]\cap C^\infty(-\pi/2,\pi/2)\ and\ f'(v)\geqslant0\ for\ v\in(-\pi/2,\pi/2).$$

In particular

$$g(v)\leqslant f(v)\ for\ v\in[-\pi/2,\pi/2]. \tag{20}$$

Hence

$$|\nabla v|(x)\leqslant\lambda_1 A(1+af(v(x)))\ \ for\ v\in[-\pi/2,\ \pi/2]. \tag{21}$$

Proof We prove the last conclusion and the others just are routine work. Consider the differential equation

$$A\cos^2 vy''-2A\sin v\cos vy'-2Ay+2\sin v=0. \tag{22}$$

One can make use of transformation $y=z/\cos^2 v$ to simplify (22). Then it is not difficult to obtain solution $f(v)$ of (22) which is an odd function.

Futhermore, if (20) does not hold. It follows from (8) that $g(\pm\pi/2)\leqslant f(\pm\pi/2)$ because $F(\pm\pi/2)=0$. Then there exists a real number c such that

$$c=g(v_0)-f(v_0)=\max\{g(v)-f(v):\ v\in(-\pi/2,\ \pi/2)\}>0. \tag{23}$$

Denote $y=f(v)+c$ and apply Lemma 2 to y. From (9) and (22) it follows that

$$y(v_0)=f(v_0)+c\leqslant\frac{\sin v_0}{A}-\sin v_0\cos v_0 y'(v_0)-\frac{1}{2}y''(v_0)\cos^2 v_0=f(v_0). \tag{24}$$

This contradicts the assumption of c and (21) is derived from (8) immediatly. This ends the proof.

§ 4. Estimate of Lower Bound of Eigenvalues

Following the method of Zhong and Yang, we may give

Proof of Theorem From Lemma 3, we have

$$\frac{|\nabla v|}{(1+af(v)))^{1/2}}\leqslant(\lambda_1 A)^{1/2}. \tag{25}$$

Since $-1/A\leqslant f\leqslant 1/A$, we have $|af|<1$. Using the Taylor expansions, we have

$$\frac{1}{(1+af)^{1/2}}+\frac{1}{(1-af)^{1/2}}=2\left\{1+\sum_{p=1}^{\infty}\frac{1\cdot3\cdots(4p-1)}{2\cdot4\cdots(4p)}(af)^{2p}\right\}\geqslant 2. \tag{26}$$

Taking $x_1,\ x_2\in M$ such that $v(x_1)=\pi/2$, $v(x_2)=-\pi/2$, one can join x_1 and x_2 with a geodesic L whose length equals the distance d' on M between x_1 and x_2. Then $d'\leqslant d$. Integrating both sides of (26) along L, we obtain

$$(A\lambda_1)^{1/2}d\geqslant(A\lambda_1)^{1/2}d'\geqslant\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dv}{(1+af)^{1/2}}=\int_0^{\frac{\pi}{2}}\left(\frac{1}{(1+af)^{1/2}}+\frac{1}{(1-af)^{1/2}}\right)dv\geqslant\pi. \tag{27}$$

If the Ricci curvature of M is non-negative, we may take $A=1$ in the preceding estimate and may carry on the same treatment. This completes the proof.

References

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[2] Zhong, J. Q. & Yang, H. C., On the estimate of the first eigenvalue of a compact Riemannian manifold, *Sci, Sinica, Ser. A*, **12** (1984), 1265—1273.