Chin. Ann. of Math.
12B: 3(1991), 272—281.

ON n– WIDTHS OF PERIODIC FUNCTIONS

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Abstract

Let $\tilde{B}_p^{r,1}=\{f: f^{(r-1)}$ is abs. cont. on $I=[a, b]$, f is periodic with period $H(=b-a)$, $f(x_1)=0$, $\|f^{(r)}\|_p\leqslant 1\}$, where x_1 is any fixed point in $[a, b]$. The author finds the Kolmogorov, Gel'fand, linear, and Bernstein n-widths of $\tilde{B}_p^{r,1}$ in $L^p(I)$ for n odd, $\infty>p>1$. The optimal subspaces and operators are also found.

§ 1. Introduction

Let X be a normed linear space and K a subspace of X. The n-width of K relative to X, in the sense of Kolmogorov, is given by

$$d_n(K, X)=\inf_{X_n}\sup_{a\in K}\inf_{x\in X_n}\|a-x\|_X,$$

where X_n runs over the totality of n-dimensional subspaces of X. If there exists an n-dimensional subspace X_n^*, for which

$$d_n(K, X)=\sup_{a\in K}\inf_{x\in X_n^*}\|a-x\|_X,$$

then X_n^* is said to be optimal.

The linear n-width of K in X is defined by

$$\delta_n(K, X)=\inf_{P_n}\sup_{x\in K}\|x-P_n(x)\|,$$

where the P_n ranges over the set of continuous linear operators of X into X of rank at most n, i.e., whose range is of dimension at most n. If $\delta_n(K, X)=\sup\{\|x-P_n^*(x)\| \mid x\in K\}$ and rank $P_n^*\leqslant n$, then P_n^* is said to be optimal for $\delta_n(K, X)$.

The Gel'fand n-width of K in X is given by

$$d^n(K, X)=\inf_{L_n}\sup_{x\in K\cap L_n}\|x\|,$$

where L_n ranges over all closed subspaces of codimension at most n. If $d^n(K, X)=\sup\{\|x\| \mid x\in K\cap L_n^*\}$ and codim $L_n^*\leqslant n$, then L_n^* is said to be optimal for $d^n(K, X)$.

The Bernstein n-width of K in X is given by

$$b_n(K, X)=\sup_{X_{n+1}}\sup\{\lambda \mid \lambda S(X_{n+1})\subseteq K\},$$

Manuscript received November, 9, 1988.
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where $S(X_{n+1})=\{x|x\in X_{n+1},\ \|x\|\leqslant 1\}$ and X_{n+1} ranges over all subspaces of dimension at least $n+1$. If $\dim X^*_{n+1}\geqslant n+1$, and $b_n(K,\ X)S(X^*_{n+1})\subseteq K$, then X^*_{n+1} is said to be optimal for $b_n(K,\ X)$.

The Sobolev space $\widetilde{W}^r_p[a,\ b]$ of periodic real-valued functions on $I=[a,\ b](b-a=H,\ H>0)$ is defined by $\widetilde{W}^r_p=\widetilde{W}^r_p(I)=\{f|f^{(r-1)}$ abs. cont., $f^r\in L^p(I),\ f^{(i)}(a)=f^{(i)}(b)$ for $i=\overline{0,\ r-1}\}$. Set $\widetilde{B}^r_p=\{f|f\in\widetilde{W}^r_p(I),\ \|f^{(r)}\|_p\leqslant 1\}$. One of the important problems in the study of n-width has been the determination of the n-width and the characterization of optimal subspaces and operators when $A=\widetilde{B}^r_p$ and $X=L^q(I)$ for $p,\ q\geqslant 1$. In [1] Buslaev and Tichomirov claimed that $d_{2L-1}(\widetilde{B}^r_p,\ L^q(I))=d_{2L}(\widetilde{B}^r_p,\ L^q(I))$ for $L=1,\ 2,\ \cdots$ and $p\geqslant q$, but there are no results on linear, Bernstein's and Gel'fand's n-widths.

It has been conjectured that the four n-widths (or three of them) are equal for all $p,\ q$ such that $\infty\geqslant p\geqslant q\geqslant 1$. This conjecture was proved for some special cases, i. e., 1) $p=\infty$ and $q\in[1,\ \infty]$, 2) $p\in[1,\ \infty]$ and $q=1$, 3) $p=q=2$ [2]. In this paper, we study tho $(2m-1)$-widths of class $\widetilde{B}^{r,1}_p$ in $L^p(I)$, prove that the four widths are equal to some quantity, and find the corresponding optimal subspaces and operators. The motivation of this work is the paper [3] by Pinkus, where the author proved the conjecture for non-periodic cases.

§ 2. Preliminaries and Definitions

Given an interval $I=[a,\ b],\ b-a=H,\ H>0$.

Definition 1. *Let* $f\in L^1(I)$. We say that $S(f,\ I)=S(f)=n$, *if there exist* $n+1$ *disjoint ordered intervals* $\{I_j\}_1^{n+1}$*(by ordered we mean that* $x<y$ *for all* $x\in I_j,\ y\in I_{j+1},\ j=1,\ \cdots,\ n$*) whose union is* $[a,\ b]$ *and such that* $\varepsilon(-1)^j f\geqslant 0$ a. e. *on* $I_j,\ j=1,\ \cdots,\ n+1$, *with constant* $\varepsilon=1$ *or* -1, *and* meas $\{x:x\in I_j,\ f(x)\neq 0\}>0,\ j=1,\ \cdots,\ n+1$. *If no such* n *exists, we set* $S(f)=\infty$.

Let $y=(y_1,\ \cdots,\ y_s)\in R^s\setminus\{0\}$. The number $S_c(y)$ of cyclic variations of sign of y is given by $S_c(y)=\max S^-(y_i,\ y_{i+1},\cdots,\ y_s,\ y_1,\cdots,\ y_i)=S^-(y_k,\cdots,\ y_s,\ y_1,\ \cdots,\ y_k)$, where k is any integer for which $y_k\neq 0$.

This definition of cyclic variations is of sign only for discrete numbers, the following definition is for functions.

Definition 2. *Let* $f\in L^1(I)$, *f is an H-periodic function, the number* $S_c(f,\ I)$ *of cyclic variations of sign of f is given by* $S(f,\ I)$ *if* $S(f,\ I)$ *is even, and by* $S(f,I)+1$ *if* $S(f,\ I)$ *is odd, where* $S(f,\ I)$ *is defined in Definition* 1.

Definition 3. *Let* $g_i\in C[I],\ i=1,\cdots,\ S$. *We say that g is a WT-system on I if*

$$G\begin{pmatrix}g_1,\ \cdots,\ g_s\\ 1,\ \cdots,\ s\end{pmatrix}=\det(g_j(y_i))^s_{i,j=1}\geqslant 0$$

for all choices of $a\leqslant y_1<\cdots<y_s\leqslant b$.

Lemma 2.1. *If* $U=\{g_i\}_1^s$ *is a WT-system on* I, *then*

$$S\left(\sum_1^s a_i g_i\right)\leqslant s-1,$$

and conversely, if the above inequality holds, then either U *or* $\tilde{U}=\{g_1,\cdots,g_{s-1},-g_s\}$ *is a WT-system on* I *(see [5]).*

Definition 4. *Set* $\mu_r(x-y)=(b-a)^{r-1}B_r\left(\frac{x-y}{b-a}\right)$ *where* $B_r(x)$ *is the Bernoulli polynomial of degree* r *on* $[0, 1]$ *(see [4]).* μ_r *has the following properties*

$$\mu_r(x-y)=(-1)^r\mu_r(y-x),\ \mu_r\in C^{r-2}(R^1)(r\geqslant 2), \tag{1}$$

$$\int_J \mu_r(x-y)dy=0,\ \frac{d\mu_r(x-y)}{dx}=\mu_{r-1}(x-y), \tag{2}$$

$$\mu_1(x-y)=\begin{cases}-\dfrac{1}{2}-\dfrac{x-y}{H} & x<y\leqslant b,\\ \dfrac{1}{2}-\dfrac{x-y}{H} & a\leqslant y<x.\end{cases} \tag{3}$$

For $f\in\tilde{B}_p^r$, the following formulas are valid

$$f(x)=\frac{1}{b-a}\int_I f(y)dy+(\mu_r * f^{(r)})(x), \tag{4}$$

where

$$(f_1 * f_2)(x)=\int_I f(x-y)f_2(y)dy.$$

If f vanishes at x_1, then

$$f(x)=\int_I V(x, y)f^{(r)}(y)dy, \tag{5}$$

where $V(x, y)=\mu_r(x-y)-\mu_r(x_1-y)$.

Let $y(t)=c+(\mu_r*\varphi)(t)$. Then

$$y^{(r)}(t)=\varphi(t)-\frac{1}{H}\int_I\varphi(y)dy.$$

Definition 5. *A real* H*-periodic, continuous function* G *has proyerty* B, *if for every choice of* $a\leqslant y_1<\cdots<y_L<a+H$ *and for each* L, *the subspace*

$$X_L=\left\{\{b+\sum_{i=1}^L b_i G(\cdot-y_i):\sum_{i=1}^L b_i=0\right\}$$

is of dimension L, *and is a WT-system for* L *odd.*

The proof of the following lemma can be found in [2].

Lemma 2.2. *G has property B. Let* $\phi\in L^1[a, b]$ *be such that* $\phi\perp 1$, meas $\{x|\phi(x)=0\}=0$, *and* ϕ *has* $2m$ *sign changes on* $[a, b)$ *(considered cyclically). By this we mean that there exist points* $a\leqslant y_1<y_2<\cdots<y_{2m}<b$, *for which* $s\phi(y)(-1)^i>0$ a.e. $y\in(y_i, y_{i+1})$, $i=\overline{0, 2m}$, *where* $y_0=a$ *and* $y_{2m+1}=b$, *and* $\varepsilon=1$ *or* -1. *Assume that* $\psi(x)=c+(G*\phi)(x)$ *has* $2m$ *zercs at* $a\leqslant x_1<x_2<\cdots<x_{2m}<b$. *Then for any given data* $\{c_i\}_1^{2m}$, *there exists a unique function* f *in*

$$X_{2m}=\left\{b+\sum_1^{2m}b_iG(\cdot-y_i)\ \Big|\sum_1^{2m}b_i=0\right\}$$

such that $f(x_i)=c_i$, $i=\overline{1,\ 2m}$.

Lemma 2.3. *For each* $r\geqslant 2$, $\mu_r(t-r)$ *satisfies property B.*

Lemma 2.4. *Let A be a closed, convex, centrally symmetric subset of a normed linear space X. Then*

$$\delta_n(A,\ X)\geqslant d_n(A,\ X),\ d^n(A,\ X)\geqslant b_n(A,\ X).$$

We now define a class of functions D as follows

$$D=\{\varphi:\varphi\in L^p(I),\ \varphi(x+h)=-\varphi(x),\ x\in I,\ \varphi(2a-x)=-\varphi(x)\}. \tag{6}$$

If $\varphi\in D$, then the function $(\mu_r*\varphi)(x)$ enjoys the following properties

$$(\mu_r*\varphi)(x+h)=-(\mu_r*\varphi)(x), \tag{7}$$

$$(\mu_r*\varphi)(2a-x)=(-1)^{r+1}(\mu_r*\varphi)(x); \tag{8}$$

$$(\mu_r*\varphi)(\beta+\varkappa h)=0,\ \varkappa=1,\ \cdots,\ 2m, \tag{9}$$

where

$$\beta=\begin{cases}a \text{ if } r \text{ even,}\\ a+h/2 \text{ if } r \text{ odd.}\end{cases} \tag{10}$$

In fact, (7) and (8) simply follow from (6), and they imply (9).

By the same reason, the function

$$E(x)=\int_I\mu_r(x-y)\operatorname{sgn}\left(\sin\frac{\pi(y-a)}{h}\right)dy$$

vanishes at $x_\varkappa=\beta+\varkappa h$, $\varkappa$ being an integer. Let $Z(E)$ (or $Z_c(E,\ I)$) be the number of zeros of E on $[a,\ b)$, counting multiplicities, then $2m\leqslant Z(E)\leqslant S_c(\operatorname{sgn}E^{(r)},\ I)=2m$; therefore, $Z(E)=2m$. $\{x_\varkappa\}_1^{2m}$ are simple zeros of E on $[a,\ b)$ and $E(x)$ does not vanish elsewhere.

Set $y_\varkappa=a+\varkappa h$, $T(x,\ y)=\mu_r(x-y)-\mu_r(x-y_1)$. According to Lemma 2, we conclude that the matrix

$$T_{2m}=\begin{pmatrix}1 & \cdots & \cdots & 1\\ T(x_1,\ y_2) & T(x_2,\ y_2) & \cdots & T(x_{2m},\ y_2)\\ \cdots & & & \cdots\\ T(x_1,\ y_{2m}) & T(x_2,\ y_{2m}) & \cdots & T(x_{2m},\ y_{2m})\end{pmatrix}$$

is non-singular.

Let $L(x,\ y)=T(x,\ y)-T(x_1,\ y)$; By a simple calculation, the matrix

$$L_{2m}=\begin{pmatrix}L(x_2,\ y_2) & \cdots & L(x_2,\ y_{2m})\\ L(x_3,\ y_2) & \cdots & L(x_3,\ y_{2m})\\ \cdots & & \cdots\\ L(x_{2m},\ y_2) & \cdots & L(x_{2m},\ y_{2m})\end{pmatrix} \tag{11}$$

is also non-singular, and

$$\det L_{2m}=\det T_{2m}\neq 0, \tag{12}$$

where

$$x_\varkappa=\beta+\varkappa h,\ y_\varkappa=a+\varkappa h,\ \varkappa\in\mathbf{Z}. \tag{13}$$

We have

Theorem1. *Let* $1<p<\infty$, $r\geqslant 2$. *There is a function* $\hat{\varphi}$ *in* D, $\|\hat{\varphi}\|_p=1$, *which solves the following extremal problem*

$$\sup\{\|\mu_r*\varphi\|_p: \|\varphi\|_p\leqslant 1, \varphi\in D\}=\sup\left\{\frac{\|\mu_r*\varphi\|_p}{\|\varphi\|_p}, \varphi\in D\right\}=\|\mu_r*\hat{\varphi}\|_p. \tag{14}$$

Denote the extremal value by

$$\eta=\|\mu_r*\hat{\varphi}\|_p. \tag{15}$$

Set

$$\bar{L}(x, y)=\det L\begin{pmatrix} x, & x_2, & \cdots, & x_{2m} \\ y, & y_2, & \cdots, & y_{2m}\end{pmatrix}\Big/\det L_{2m}, \ M(x, y)=|\bar{L}(x, y)|.$$

Then

$$\eta=\sup\{\|\bar{L}*\varphi\|_p: \|\varphi\|_p\leqslant 1, \varphi\in D\}=\|\bar{L}*\hat{\varphi}\|_p \tag{16}$$

$$=\sup\left\{\|M*\psi\|_p: \psi(x)=\varphi(x)\operatorname{sgn}\left(\sin\frac{\pi(x-a)}{h}\right), \varphi\in D\right\}$$

$$=\|M*\hat{\psi}\|_p, \ \hat{\psi}(x)=\hat{\varphi}(x)\operatorname{sgn}\left(\sin\frac{\pi(x-a)}{h}\right). \tag{17}$$

The function $\hat{\varphi}$ *satisfies the equation*

$$\int_I \mu_r(x-y)N(x)dx=J(y), \ y\in I, \ \hat{f}(x)=(\mu_r*\hat{\varphi})(x), \\ N(x)=|\hat{f}(x)|^{p-1}\operatorname{sgn}\hat{f}(x), \ J(y)=\eta^p|\hat{\varphi}(y)|^{p-1}\operatorname{sgn}\hat{\varphi}(y), \tag{18}$$

and J *is continuous on* I. *There are* $\lambda_1, \lambda_2\in\{-1, 1\}$ *such that*

$$\operatorname{sgn}(\lambda_1\hat{\varphi}(y))=\operatorname{sgn}\left(\sin\left(\frac{\pi(y-a)}{h}\right)\right), \ y\in I, \tag{19}$$

$$\operatorname{sgn}(\lambda_2\hat{f}(x))=\operatorname{sgn}\left(\sin\frac{\pi(x-\beta)}{h}\right), \ x\in I, \tag{20}$$

Proof From the weak compactness theorem, there exists a function $\hat{\varphi}$ in D which solves the extremal problem (14). Assume $\varphi\in D$. Then, for any $t\in R^1$, the function $\hat{\varphi}+t\varphi$ is also in D. Set $G(t)=\|\mu_r*(\hat{\varphi}+t\varphi)\|_p/\|\hat{\varphi}+t\varphi\|_p$; since $G'(0)=0$ we have $\int_I F(y)\varphi(y)dy=0$ for any $\varphi\in D$, where $F(y)=\int_I \mu_r(x-y)N(x)dx-J(y)$, $F(y)$ is in D; thus $\int_I F(y)^2dy=0$, i.e., $F(y)=0$ a.e. for $y\in I$. $\int_I \mu_r(x-y)N(x)dx$ is continuous on I; by modifying the value of $\hat{\varphi}(y)$ on a set of measure zero, $\hat{\varphi}(y)$ (and $J(y)$) will also be continuous. Therefore we have (18). Assume $\varphi\in D$, and set $f=\mu_r*\varphi$. From (9), $f(x_\kappa)=0$ and

$$(L*\varphi)(x)=f(x), \text{ for any } \varphi\in D, \tag{21}$$

$$(\bar{L}*\varphi)(x)=f(x)-R_{2m-1}(f(\cdot))(x)=f(x), \tag{22}$$

where $R_{2m-1}(g)$ is the interpolation from $\operatorname{span}\{L(\cdot,y_2), \cdots, L(\cdot,y_{2m})\}$ to $g(\cdot)$ at $\{x_i\}_2^{2m}$. Therefore

$$(L*\varphi)(x)=f(x)=(\bar{L}*\varphi)(x), \text{ for any } \varphi\in D \text{ and } x\in I, \tag{23}$$

(16) follows from (23) and (14).

By a simple calculation, $\bar{L}(x, y)$ can also be presented as

$$\bar{L}(x, y)=T(x, y)+\sum_{j=1}^{2m} b_j T(x_j, y),\ 1+\sum_{j=1}^{2m} b_j=0. \tag{24}$$

Similarly, then we also have

$$\bar{L}(x, y)=T(x, y)+\sum_{j=1}^{2m} d_j T(x, y_j),\ 1+\sum_{j=1}^{2m} d_j=0. \tag{25}$$

From (25) and Lemma 2.3, Lemma 2.1, we have

$$2m\leqslant S_c(\bar{L}(\cdot, y), I)\leqslant 2m,\ y \text{ fixed}. \tag{26}$$

Similarly,

$$2m\leqslant S_c(\bar{L}(x, \cdot), I)\leqslant 2m,\ x \text{ fixed}. \tag{27}$$

We conclude that ($\bar{L}(x, y)$ (y fixed) can only have simple zeros at $x=\{x_j\}_1^{2m}$. Similarly, when x is fixed, $\bar{L}(x, y)$ only has simple zeros at $y=\{y_j\}_1^{2m}$. Thus from (13) we have

$$\bar{L}(x, y)=\lambda \operatorname{sgn}\left(\sin\left(\frac{\pi(x-\beta)}{h}\right)\right)|\bar{L}(x, y)|\operatorname{sgn}\left(\sin\left(\frac{\pi(y-\alpha)}{h}\right)\right), \tag{28}$$

where $\lambda=1$ or -1.

From (28) we have (17). Now we prove (19), (20).

Since $\hat{\varphi}\in D$, from (7) and (18) $\hat{f}\perp 1$, $N\perp 1$, then $2m\leqslant S_c(\hat{\varphi}, I)=S_c(J, I)\leqslant Z(J)\leqslant Z(J^{(r-1)})\leqslant S_c(N, I)=S_c(\hat{f}, I)\leqslant Z(\hat{f})\leqslant Z(\hat{f}^{(r-1)})\leqslant S_c(\hat{\varphi}, I)$, where $Z(g)$ denotes the number of zeros of g on $[a, b)$. We conclude that J, $\hat{f}$ only have simple zeros. Thus, (20) follows. Since J has no zero interval, this implies that $\hat{\varphi}$ has no zero interval. From (17) and $M(x, y)\geqslant 0$, $\hat{\psi}(x)$ must keep constant sign on I. Therefore, $\hat{\psi}(x)$ changes sign only at $y_\varkappa(\varkappa=1, \cdots, 2m)$, these are the only zeros (in I) of function $J(y)$, (19) is true.

Let

$$\omega=\sup_{\varphi\in L^p(I)}\frac{\|\bar{L}*\varphi\|_p}{\|\varphi\|_p}. \tag{29}$$

We have

Theorem 2. $\infty>p>1$, *there exists a function* $\varphi_0\in L^p(I)$ *and a function* $f_0=\bar{L}*\varphi_0$ *satisfying* $\omega=\|f_0\|_p/\|\varphi_0\|_p$, f_0 *satisfies*

$$\int_I \bar{L}(x, y)|f_0(x)|^{p-1}\operatorname{sgn} f_0(x)dx=\omega^p|\varphi_0(y)|^{p-1}\operatorname{sgn}\varphi_0(y), \tag{30}$$

$$\varepsilon\hat{\varphi}(y)\varphi_0(y)\geqslant 0 \text{ a.e.},\ \varepsilon=1 \text{ or or } -1, \tag{31}$$

$$\left|\int_I \bar{L}(x, y)\hat{\varphi}(y)dy\right|=\int_I M(x, y)|\hat{\varphi}(y)|dy=|\hat{f}(x)|, \tag{32}$$

$$\int_I \bar{L}(x, y)|(\bar{L}*\hat{\varphi})(x)|^{p-1}\operatorname{sgn}(\bar{L}*\hat{\varphi})(x)dx=J(y), \tag{33}$$

$$\omega=\eta. \tag{34}$$

Proof From the compactness consideration, there is a function φ_0 in $L^p(I)$, φ_0 attains the extremal value in (29). Equation (30) follows from a proof parallel to the one of Theorem 1. From (28), we have

$$\int_I\left|\int_I \bar{L}(x,\ y)\varphi_0(y)dy\right|^p dx=\int_I\left|\int_I |\bar{L}(x,\ y)|\ \Big|\varphi_0(y)|\operatorname{sgn}\Big(\varphi_0(y)\sin\frac{\pi(y-a)}{h}\Big)dydx.$$

Since φ_0 attains the extremal value, $\operatorname{sgn}\Big(\varphi_0(y)\sin\dfrac{\pi(y-a)}{h}\Big)$ must be a constant; thereby, (31) is true. (32) follows from (19), (28), (23) and (18).

From the definition of $\bar{L}(x, y)$ and $J(y_j)=0$ we have

$$\int_I \bar{L}(x,\ y)N(x)dx=\int_I L(x,\ y)N(x)dx=J(y). \tag{35}$$

Combine it with (32) and (18) we obtain (33).

Set

$$\tilde{\varphi}_0(x)=\varphi_0(x)/\|\varphi_0\|_p,\ \tilde{f}_0(x)=(\bar{L}*\tilde{\psi}_0)(x),\ \hat{f}(x)=(\mu_r*\hat{\varphi})(x),$$
$$H_0(y)=\omega^p|\tilde{\varphi}_0(y)|^{p-1}\operatorname{sgn}\tilde{\psi}_0(y).$$

From (19), (20), (28), it follows that

$$|(\bar{L}*\tilde{\varphi}_0)(x)|=|(M*|\tilde{\varphi}_0|)(x)|,\ \operatorname{sgn}(\bar{L}*\tilde{\varphi}_0)(x)=\varepsilon\lambda_1\lambda_2\lambda\operatorname{sgn}\hat{f}(x). \tag{36}$$

Combining it with (30) we have

$$\int_I M(x,\ y)\,|(M*|\tilde{\varphi}_0|)(x)|^{p-1}dx=|H_0(y)|. \tag{37}$$

From (23), (28)

$$|\hat{f}(x)|=\int_I M(x,\ y)\,|\hat{\varphi}(y)|dy=(M*|\hat{\varphi}|(x),\ \lambda\lambda_1\lambda_2=1. \tag{38}$$

Since $\hat{f}=\bar{L}*\hat{\varphi}$, from (38), (33) and (28) we have

$$\int_I M(x,\ y)\,|(M*|\hat{\varphi}|)(x)|^{p-1}dx=|J(y)|. \tag{39}$$

From (31), $\tilde{\varphi}_0(y)$ changes sign at $\{y_j\}_1^{2m}$. Then $H_0(y_j)=0$, $J(y)$ has simple zeros at $\{y_j\}_1^{2m}$, and does not vanish elsewhere. Therefore $|H_0(y)/J(y)|<\infty$ for $y\in I$, there is a constant $\hat{a}$ such that $\hat{a}=\inf\{a: |H_0(y)|\leqslant a|J(y)|,\ y\in I\}$. Then

$$\omega^{\frac{p}{p-1}}|\tilde{\varphi}_0\ (y)|\leqslant\hat{a}^{\frac{1}{p-1}}\eta^{\frac{p}{p-1}}|\hat{\varphi}(y)|$$

for $y\in I$, and $\omega^{\frac{p}{p-1}}(M*|\tilde{\varphi}_0|)\ (x)\leqslant\hat{a}^{\frac{1}{p-1}}\eta^{\frac{p}{p-1}}(M*|\hat{\varphi}|)(x)$ for $x\in I$. Combining them with (39), (37), we get $|H_0(y)|\leqslant\hat{a}\Big(\dfrac{\eta}{\omega}\Big)^p|J(y)|$ for $y\in I$. Thus we have $\eta\geqslant\omega$. From (14), (15), (9) and (29) we have $\omega\geqslant\eta$; (34) is true.

set

$$\varphi_j(y)=|\hat{\varphi}(y)|\Omega_j(y),\ \Omega_j(y)=\begin{cases}1,\ y\in(y_j,\ y_{j+1}),\\ 0,\ \text{otherwise},\end{cases}$$

$j=1,\ \cdots,\ 2m,\ y_{2m+1}=y_1+H.$

$$g_j(h)=\begin{cases}\displaystyle\int_I \mu_r(x-y)\varphi_j(y)dy,\ j=1,\ \cdots,\ 2m,\\ 1, \qquad\qquad j=2m+1.\end{cases} \tag{40}$$

Define

$$F_{2m}=\left\{g_\alpha: g_\alpha(x)=\sum_{j=1}^{2m+1}\alpha_j g_j(x),\ \sum_{j=1}^{2m}\alpha_j=0\right\}. \tag{41}$$

Let $\int_{y_j}^{y_j+1}|\hat{\varphi}(y)|^p dy=d_j\ (j=\overline{1,\ 2m})$. We consider the extremal problem

$$\inf_{g_\alpha\in F_{2m}}\frac{\|g_\alpha\|_p}{\|\varphi_\alpha\|_p},\quad g_\alpha(x)=\alpha_{2m+1}+\int_I\mu_r(x,\ y)\varphi_\alpha(y)dy, \tag{42}$$

$\varphi_\alpha(y)=\sum_1^{2m}\alpha_j\varphi_j(y)$, $\sum_1^{2m}\alpha_j=0$. Evidently, there is a function $\varphi*(y)=\sum_1^{2m}\alpha_j^*\varphi_j(y)$, and a constant α_{2m+1}^* such that the function $g^*=\sum_1^{2m}\alpha_i^*g_j+\alpha_{2m+1}^*$ solves the problem (42).

$$\inf_{g_\alpha\in F_{2m}}\frac{|g_\alpha\|_p}{\|\varphi_\alpha\|_p}=\frac{\|g^*\|_p}{\|\varphi^*\|_p}=\xi. \tag{43}$$

Set $F(\alpha)=F(\alpha_1\cdots,\ \alpha_{2m+1})=\|g_\alpha\|_p/\|\varphi_\alpha\|_p$, differentiate $F(\alpha)$ with respect to $\alpha_\varkappa$ and set $\alpha=\alpha^*$, we have $\left.\frac{\partial F(\alpha)}{\partial\alpha_\varkappa}\right|_{\alpha=\alpha^*}=0$ for $\varkappa=1,\cdots,\ 2m+1$. Then

$$\int_I g_\varkappa(x)\,|g^*(x)|^{p-1}\operatorname{sgn}g^*(x)dx=\xi^p d_\varkappa|\alpha_\varkappa^*|^{p-1}\operatorname{sgn}\alpha_\varkappa^*,\quad \varkappa=\overline{1,\ 2m}.$$

$$\int_I|g(x)|^{p-1}\operatorname{sgn}g^*(x)dx=0. \tag{44}$$

From (18) we have

$$\int_I g_\varkappa(x)\,|\hat{f}(x)|^{p-1}\operatorname{sgn}\hat{f}(x)dx=\eta^p d_\varkappa(-1)^\varkappa,\quad \varkappa=\overline{1,\ 2m}$$

$$\int_I|\hat{f}(x)|^{p-1}\operatorname{sgn}\hat{f}(x)dx=0. \tag{45}$$

Set $E(x)=|\hat{f}(x)|^{p-1}\operatorname{sgn}\hat{f}(x)-|g^*(x)|^{p-1}\operatorname{sgn}g^*(x)$, $E\perp 1$, let

$$E_\varkappa=\int_I g_\varkappa(x)E(x)dx=d_\varkappa|\eta^p(-1)^\varkappa-\xi^p|\alpha_\varkappa^*|^{p-1}\operatorname{sgn}\alpha_\varkappa^*|.$$

If $\eta>\xi$ and $|\alpha_j^*|\leqslant 1$ $(j=1,\ \cdots,\ 2m)$, then $E_c(E_1,\ \cdots,\ E_{2m})=2m$; thereby, $2m\leqslant S_c\left(\int_I\mu_r(x-\cdot)E(x)dx,\ I\right)\leqslant Z_c\left(\int_I\mu_r(x-\cdot)E(x)dx\ \ I\right)\leqslant S_c(E(\cdot),\ I)=S_c(\hat{f}(\cdot)-g^*(\cdot),\ I)\leqslant S_c(\hat{\varphi}(\cdot)-\varphi^*(\cdot),\ I)$. Normalize $\{\alpha_j^*\}_1^{2m}$ so that $|\alpha_j^*|\leqslant 1$ for all j and $\alpha_\varkappa^*=(-1)^\varkappa$. Moreover, with no loss of generality, we assume that $\operatorname{sgn}\hat{\varphi}(y)=(-1)^j$ for $y\in(y_j,\ y_{j+1})$. Then, $S_c(\hat{\varphi}(\cdot)-\varphi^*(\cdot),\ I)\leqslant 2m-2$; it leads to a contradiction $2m\leqslant 2m-2$. Thus we conclude $\eta\leqslant\xi$. Since

$$\hat{f}(x)=\lambda_2\sum_{\varkappa=1}^{2m}(-1)^\varkappa g_\varkappa(x),$$

we have $\xi=\eta$ and the following

Theorem 3. *Let F_{2m} be defined as in* (40), (41). *Then*

$$\inf_{g_\alpha\in F_{2m}}(\|g_\alpha\|_p/\|\varphi_\alpha\|_p)=\eta,$$

where η is the number defined in (14), (15).

We now define a class of functions $\hat{B}_p^r$:

$$\hat{B}_p^r=\{f:\ f=\mu_r*\varphi,\ \varphi\perp 1,\ \|\varphi\|_p\leqslant 1,\ f(\beta+h)=0\},$$

where β is defined as in (10). We have the following

Theorem 4. *If $1<p<\infty$, then*

$$\delta_{2m-1}(\hat{B}_p^r, L^p(I))=d_{2m-1}(\hat{B}_p^r, L^p(I))=d^{2m-1}(\hat{B}_p^r, L^p(I))$$
$$=b_{2m-1}(\hat{B}_p^r, L^p(I))=\|\hat{f}\|_p=\eta. \tag{46}$$

(a) *R_{2m-1} is an optimal rank $2m-1$ operator for $\delta_{2m-1}(\hat{B}_p^r, L^p(I))$, R_{2m-1} is defined as in* (22).

(b) $F_{2m}=\mathrm{Span}\left\{g_{am}=\alpha_{2m+1}+\sum_1^{2m}\alpha_j g_j,\ \sum_1^{2m}\alpha_j=0\right\}$ *is optimal for $b_{2m-1}(\hat{B}_p^r, L^p(I))$.*

(c) $Z_{2m-1}=\mathrm{Span}\{Z\colon Z\in\hat{B}_p^r,\ Z(x_j)=0,\ j=2,\ \cdots,\ 2m\}$ *is optimal for $d^{2m-1}(\hat{B}_p^r, L^p(I))$.*

(d) $X_{2m-1}=\mathrm{Span}\{\mu_r(x_2-y)-\mu_r(x_1-y),\ \cdots\mu_r(x_{2m}-y)-\mu_r(x_1-y)\}$ *is optimal for $d_{2m-1}(\hat{B}_p^r, L^p(I))$,.*

Proof From (29), (34), (21) and (22) we have

$$\eta\geqslant\sup_{f\in\hat{B}_p^r}\|f-R_{2m-1}(f)\|_p\geqslant\delta_{2m-1}(\hat{B}_p^r, L^p(I)). \tag{47}$$

From Theorem 3,

$$b_{2m-1}(\hat{B}_p^r, L^p(L))\geqslant\eta. \tag{48}$$

Then Theorem 4 follows from (47), (48) and Lemma 2.4.

Let x_1 be any point in the interval $I_c=[c, d]$. The subclass $\tilde{B}_p^{r,1}$ of $\tilde{B}_p^r$ is defined by $\tilde{B}_p^{r,1}=\{f\colon f^{(r-1)}$ abs. cont. on $I_c=[c, d]$, f is periodic with period $H=d-c$, $\|f^{(r)}\|_{p,[c,d]}\leqslant 1$, $f(x_1)=0\}$ where

$$\|g\|_{p,[c,d]}=\left(\int_c^d|g(x)|^p dx\right)^{1/p}.$$

Set

$$h=H/2m,\quad a=\begin{cases}x_1-h, & r \text{ even},\\ x_1-\dfrac{3h}{2}, & r \text{ odd},\end{cases}\qquad b=a+H,$$

$\|g\|_p=\|g\|_{p,[a,b]}=\left(\int_a^b|g(x)|^p dx\right)^{1/p}$. Let β be defined as in (10). The subclass $\tilde{B}_p^r$ is defined as in Theorem 4. Since each function f in $\tilde{B}_p^{r,1}$ (or in $\tilde{B}_p^r$) may be regarded as the function on the real line, and $\|g\|_{p,[a,b]}=\|g\|_{p,[c,d]}$, thereby $\tilde{B}_p^{r,1}=\tilde{B}_p^r$, we therefore have the following

Theorem 5. *If $1<p<\infty$, then $\delta_{2m-1}(\tilde{B}_p^{r,1}, L^p)=d_{2m-1}(\tilde{B}_p^{r,1}, L^p)=d^{2m-1}(\tilde{B}_p^{r,1}, L^p)=b_{2m-1}(\tilde{B}_p^{r,1}, L^p)=\|\hat{f}\|_p=\eta$. The optimal subspace and operators are described as in Theorem 4.*

Acknowledgement. I would like to acknowledge Mr.Chun Li for his helfpul discussions and the simplification for the proof of Theorem 1. During the Spring semester of 1988, as a Visiting Professor at the Center for Approximation Theory at Texas A & M University, I behefited by many discussions with Professor Charles Chui and his graduate students. I also appreciate Ms. Robin Bronson for her very

nice typing.

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