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HANKEL OPERATORS ON HARDY SPACES AND SCHATTEN CLASSES

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Abstract

For a holomorphic function f on the unit ball B^N of $\mathbf{C}^N$, it is proved that the reduced Hankel operator R_f on Hardy space $H^2(B^N)$ is of Schatten class S_p for $p\geqslant 1$ if and only if f is in a corresponding Sobolev space.

§ 0. Introduction

Let $B=B^N$ be the open unit ball in $\mathbf{C}^N$ with the normalized Lebesgue measure dm on it, and $S=\partial B^N$ the unit sphere with the normalized area measure $d\sigma$ on it. Denote by $H(B)$ the set of holomorphic functions on B. Let $H^p=H^p(B)$ be the Hardy space, $p\geqslant 1$, and P be the orthogonal projection from $L^2(S, d\sigma)$ to H^2. For $f\in H(B)$, the reduced Hankel operator R_f on H^2 is defined by

$$R_f g=Pf\bar{g},\ g\in H^2,\ f\bar{g}\in L^2(S,\ d\sigma).$$

R_f is conjugated linear, it can be defined to be linear, but we will adopt this definition. In [4] and [6], it was proved that R_f is bounded and compact if and onlyif $f\in$ BMOA and $f\in$ VMOA, respectively, and R_f is Hilbert-Schmidt if and only if f is in a certain Sobolev space of holomorphic functions. Let S_p be the Schatten class of operators on a Hilbert space. Many authors have studied the problem of characterizing Hankel operators on various function spaces belonging to Schatten classes (see [8, 9, 11,15]). In this paper, using the methods originated from Peller [15], Rochberg [9],and developed by the author[12], we characterize those holomorphic functions f for which R_f are of Schatten classes S_p. As a consequence of our result, when $N=1$,we obtain the results of Peller and Rochberg. Our main result is the following

Theorem *Let $f\in H(B),p\geqslant 1$. Then R_f belongs to the Schatten class S_p if and only if*

$$\sum_{|\alpha|=N+1}\int_B\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\right|^p(1-|z|^2)^{(p-1)(N+1)}dm(z)<\infty.$$

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§ 1. Notations and Preliminaries

1.1. Some inequalities

Let $\beta>-1$. Denote $dm_\beta(z)=C_\beta(1-|z|^2)^\beta dm(z)$, where C_β is a positive constant suc h that $m_\beta(B)=1$. The weighted Bergman space $L_a^p(\beta)$ is the closed subspace of $L^p(B, dm_\beta)$ consisting of analytic functions on B. Let $\rho(z, \omega)$ be the pseudohyperbolic distance between z and ω. For $0<r<1$, let $E(z, r)$ be the pseudohyperbolic ball with radius z and center r. Suppose $\varepsilon>0$. $\{a_n\}_1^\infty \subset B$. If $\inf\{\rho(a_n, a_m\}, n\neq m\}\geqslant\varepsilon$, then $\{a_n\}$ is called ε separated. Let $0<\delta<1$. If

$$\bigcup_n E(a_n,\delta)=B,$$

then $\{a_n\}$ is called δ dense. If both are satisfied then $\{a_n\}$ is called an $\varepsilon-\delta$ lattice. We have the following (see [7])

Lemma 1. *Let $p\geqslant1$, $\beta>-1$. Then there are sufficient small $\varepsilon,\delta>0$, such that if $\{a_{k1}^\infty\}$, is an $\varepsilon-\delta$ lattice, then*

$$C\|f\|^p_{L_a^p(\beta)}\leqslant\sum_{k=1}^\infty|f(a_k)|^p(1-|a_k|^2)^{N+1+\beta}\leqslant C_1\|f\|^p_{L_a^p(\beta)}$$

holds for all $f\in L_a^p(\beta)$, where the constants C and C_1 do not depend on f.

Remark. It is easy to check that Lemma 1 holds for all $f\in H(B)$, that is, the sum in the above inequality is finite if and only if $f\in L_a^p(\beta)$.

Define an operator T on $L^p(B, dm_{\beta_1})$ by the following

$$Tf(z)=\int_B\frac{|f(\omega)|}{|(1-\langle z\omega\rangle^{N+1+\beta}|}\,dm_\beta(\omega),z\in B.$$

Then we have [2,7]

Lemma 2. *Let $p\geqslant1$, β, $\beta_1>-1$.* If $\beta+1>\dfrac{\beta_1+1}{p}$, *then*

$$\|Tf\|_{L^p(B,\ dm_{\beta_1})}\leqslant C\|f\|_{L^p(B,dm_{\beta_1})}.$$

1.2. Some formulae

In [14], Zhu studied the Gleason's problem in the Bergman spaces. We will extend Zhu's methods to the Hardy spaces setting.

Let $f\in H(B)$, $\alpha=(\alpha_1, \cdots, \alpha_N)$, with each α_i being nonnegative integer, i. e. $\alpha\in Z_+^N$, for $z=(z_1\cdots z_N)\in B$, we will write $|\alpha|=\sum_1^N\alpha_i$ and

$$z^\alpha=z_1^{\alpha_1}\cdots z_N^{\alpha_N},\ \frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)=\frac{\partial^{|\alpha|}f(z)}{\partial z_1^{\alpha_1}\cdots\partial z_N^{\alpha_N}},$$

where we let $\dfrac{\partial^0 f}{\partial z^0}(z)=f(z)$.

Let n be a positive integer and $g\in H(B)$. Then it holds that (by a direct calculation)

$$g(z)=\sum_{|\alpha|<n}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)z^\alpha+\sum_{|\alpha|=n}z^\alpha g_\alpha(z),$$

where

$$g_\alpha(z)=\int_0^1 s_1\,ds_1\int_0^1 s_2\,ds_2\cdots\int_0^1 s_{n-1}ds_{n-1}\int_0^1\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(s_1\cdots s_n z)ds_n.$$

Assume further that $g\in L_a^q(\beta)$, $q\geqslant 1$. We can write

$$g(z)=\int_B\frac{g|\omega|}{(1-\langle z,\ \omega\rangle)^{N+1+\beta}}\,dm_\beta(\omega).$$

Direct computation shows that

$$\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)=(N+1+\beta)\cdots(N+|\alpha|+\beta)\int_B\bar{w}^\alpha g(\omega)dm_\beta(\omega), \tag{1.2}$$

$$g_\alpha(z)=\int_B\frac{\bar{w}^\alpha g(\omega)Q_\alpha(z\omega)}{(1-\langle z,\ \omega\rangle)^{N+1+\beta}}\,dm_\beta(\omega),\ |\alpha|=n, \tag{1.3}$$

$$\frac{Q_\alpha(z,\ \omega)}{(1-\langle z,\ \omega\rangle)^{N+1+\beta}}=C'_\alpha\int_0^1 ds_1\cdots\int_0^1 s_{n-1}ds_{n-1}\int_0^1\frac{1}{(1-s_1\cdots s_n\langle z\omega\rangle)^{N+1+\beta+n}}ds_n,\ |\alpha|=n.$$

From these equalities we see that there hold $|Q_\alpha(z,\ \omega)|\leqslant C$, $|\alpha|=n$, and the following

Lemma 3. *Let $\beta>-1$, $q>1$ and n be a positive integer. Then for any $g\in L_a^q(\beta)$, written as above, we have*

$$\left|\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)\right|\leqslant C\|g\|_{L_a^q(\beta)},\ |\alpha|<n,$$

$$\|g_\alpha\|_{L_a^q(\beta)}\leqslant C\|g\|_{L_a^q(\beta)},\ |\alpha|=n.$$

Let $f,\ g\in H(\beta)$. We define the pairing $(f,\ g)$ by

$$(f,\ g)=\lim_{r\to 1}\int_S f(r\xi)\,\overline{g(r\xi)}\,d\sigma(\xi), \tag{1.4}$$

if the limit exists.

Suppose $g\in L_a^q(n-1)$, $q>1$. By Lemma 3 we have $g_\alpha\in L_a^q(n-1)$, and thus

$$g_\alpha(z)=C_{n-1}\int_B\frac{g_\alpha(\omega)(1-|\omega|^2)^{n-1}}{(1-\langle z\omega\rangle)^{N+1+n-1}}\,dm(z). \tag{1.5}$$

Assume $f\in H^2$. Then

$$f(z)=\int_S\frac{f(\omega)}{(1-\langle z\omega\rangle)^N}\,d\sigma(\omega).$$

Taking derivatives we have

$$\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)=N(N+1)\cdots(N+|\alpha|-1)\int_S\bar{\omega}^\alpha f(\omega)d\sigma(\omega), \tag{1.6}$$

$$\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)=N(N+1)\cdots(N+|\alpha|-1)\int_S\frac{\bar{\omega}^\alpha f(\omega)}{(1-\langle z\omega\rangle)^{N+|\alpha|}}\,d\sigma(\omega). \tag{1.7}$$

Then by definition (1.4) and (1.5), (1.6), (1.7)

$$(f,\ g)=\sum_{|\alpha|<n}\int_S f(z)\frac{1}{\alpha!}\overline{\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)}\ \bar{z}^\alpha d\sigma(z)+\sum_{|\alpha|=n}\lim_{r\to 1}r^{|\alpha|}\int_S f(rz)\overline{g_\alpha(rz)}\bar{z}^\alpha d\sigma(z)$$

$$=\sum_{|\alpha|<n}C_\alpha\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\overline{\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)}+\sum_{|\alpha|=n}\lim_{r\to 1}r^{|\alpha|}\int_S f(rz)\ \overline{g_\alpha(rz)\bar{z}^\alpha}\ d\sigma(z).$$

However, we have for α, $|\alpha|=n$,

$$\begin{aligned}&\int_S \bar{z}^\alpha f(rz)\,\overline{g_\alpha(rz)}\,d\sigma(z)\\ &=C_n\int_S f(rz)\bar{z}^\alpha\int_B \frac{\overline{g_\alpha(\omega)}\,(1-|\omega|^2)^{n-1}}{(1-r\langle\omega,z\rangle)^{N+n}}\,dm(\omega)\\ &=C_n\int_B (1-|\omega|^2)^{n-1}\,\overline{g_\alpha(\omega)}\,dm(\omega)\int_S \frac{f(rz)\bar{z}^\alpha}{(1-r\langle\omega,z\rangle)^{N+n}}\,d\sigma(z)\\ &=C_\alpha\int_B (1-|\omega|^2)^{n-1}\,\overline{g_\alpha(\omega)}\,\frac{\partial^{|\alpha|}f_r(r\omega)}{\partial\omega^\alpha}\,dm(\omega),\end{aligned}$$

where $f_r(\omega)=f(r\omega)$. So

$$\lim_{r\to 1} r^{|\alpha|}\int_S z^\alpha f(rz)\,\overline{g_\alpha(rz)}\,d\sigma(z)=C_\alpha\int_B (1-|z|^2)^{n-1}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\,\overline{g_\alpha(z)}\,dm(z)$$

if the integral in the right exists. Thus

$$(f,\ g)=\sum_{|\alpha|<n} C_\alpha\,\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\,\overline{\frac{\partial^{|\alpha|}g}{\partial z^\alpha}}(0)+\sum_{|\alpha|=n} C_\alpha\int_B (1-|z|^2)^{n-1}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\,g_\alpha(z)\,dm(z), \tag{1.8}$$

where C_α are positive constants.

1.3. Interpolation

Let (A_0, A_1) be a quasi-Banach couple, i. e. A_0, A_1 are quasi-Banach spaces imbedded in a common quasi-Banach space. Let $0<\theta<1$. Denote by $(A_0, A_1)_\theta$ the interpolation space between A_0 and A_1. Let H be a Hilbert space. Denote by $B(H)$ and S_p, $p>0$, the space of bounded operators on H and the Schatten class of operators on H, respectively. We let S_∞ be the class of compact operators.

Let (X, μ) be a measure space. Then we have

Lemma 4. *Let* $1\leqslant p_0<p<\infty$. *Then*

$$(S_{p_0}, S_\infty)_\theta=(S_{p_0}, B(H))_\theta=S_p,\quad (L^{p_0}(\mu), L^\infty(\mu))_\theta=L^p(\mu),$$

where $\frac{1}{p}=\frac{1-\theta}{p_0}$ *(see* [3]).

§ 2. Sobolev Spaces of Holomorphic Functions

In this section, we define some spaces of holomorphic functions and identify their duals and preduals. We will use those results in Section 3 to study Hankel operators.

Let $\eta>0$. Define

$$B_\eta=\{f\in H(B),\ \|f\|_{B_\eta}=\sup_{z\in B}(1-|z|^2)^\eta|f(z)|<\infty\}$$

and $B_{\eta,0}$ to be the closure in B_η of polynomials. For $p\geqslant 1$, let us also define

$$A_p=\left\{\ f\in H(B),\ \ \|f\|^p_{A_p}=\sum_{|\alpha|<N+1}\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\right|^p\right.$$

$$+\sum_{|\alpha|=N+1}\int_B\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\right|^p(1-|z|^2)^{(N+1)(p-1)}dm(z)<\infty\Big\}.$$

In [4], a large class of Sobolev spaces of holomorphic functions, $W^p_{q,s}$, is defined,

$$W^p_{q,s}=\left\{f\in H(B),\ \|f\|_{p,q,s}=\sum_{|\alpha|\le s}\left\|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\right\|_{L^p_a(q-1)}<\infty\right\}.$$

Also recall that if $f\in H(B)$, $\beta>-1$, n is a positive integer, then $f\in L^p_a(\beta)$ if and only if $\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\in L^p_a(\beta+pn)$ for all α with $|\alpha|=n$. (In [14], the result was proved for $\beta=0$, it can be extended to any $\beta>-1$ by using much the same techniques, see [13]). Thus it is easy to see that with $q=(N+1)(p-1)$, $|\alpha|=N+1$, $f\in W^p_{q+1,N+1}$ if and only if $\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\in L^p_q(q)$, $|\alpha|=N+1$. So $A_p=W^p_{q+1,N+1}$, and their norms are equivalent. By Theorems 2.4, 2.5, and 2.7, we have, $f\in A_p$ if and only if for α, $|\alpha|=N+1$, $\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\in L^p_a(q+p)$. We write it as a lemma,

Lemma 5. *Let $p\ge 1$. Then*

$$A_p=W^p_{(N+1)(p-1)+1,N+1}=W^p_{(N+1)(p-1)+p+1,N+2}\subset BMOA.$$

We now identify the predual of A_1.

Proposition 6. $A_1=(B_{N,0})^*$. *More precisely, if $f\in A_1$, $g\in B_{N,0}$, then (g,f) defined by (1.4) exists. and the pairing (g,f) for $f\in A_1$ and $g\in B_{N,0}$ gives an isomorphism between A_1 and $(B_{N,0})^*$.*

Proof First we claim that

$$|(g,f)|\le C\|g\|_{B_N}\|f\|_{A_1}\tag{2.1}$$

holds for all $f\in A_1$ and $g\in B_N$. So (gf), for $g\in B_{N,0}$, is a bounded linear functional on $B_{N,0}$. Let us prove (2.1). Since $g\in B_N\subset L^1_a(N)$, we have

$$g(z)=C_N\int_B\frac{g(\omega)(1-|\omega|^2)^N}{(1-\langle z\omega\rangle)^{2N+1}}dm(\omega).$$

Taking derivatives and also writing g as in (1.1), we have

$$\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)=C_N(2N+1)\cdots(2N+|\alpha|)\int_B\bar{\omega}^\alpha g(\omega)(1-|\omega|^2)^N dm(\omega),$$

$$g_\alpha(z)=C_N\int_{B^N}\frac{g(\omega)\theta_\alpha(z\omega)(1-|\omega|^2)^N}{(1+\langle z\omega\rangle)^{2N+1}}dm(\omega),\ |\alpha|=N+1$$

with $|\theta_\alpha(z,\omega)|\le C$, z, $w\in B$, $|\alpha|=N+1$. Thus we get

$$\sum_{|\alpha|<N+1}\left|\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)\right|\le C\|g\|_{B_N},\tag{2.2}$$

and for α, $|\alpha|=N+1$,

$$|g_\alpha(z)|\le C\int_B\frac{(1-|\omega|^2|g(\omega)|}{|(1-\langle z\omega\rangle)|^{2N+1}}dm(\omega)\le C\|g\|_{B_N}\int_B\frac{1}{|(1-z\omega\rangle)^{2N+1}|}dm(\omega)$$
$$\le C\|g\|_{B_N}\frac{1}{(1-|z|^2)^N},$$

where the last ineqality is obtained by Proposition 1.4.10 in [10]. So we have

$$\sum_{|\alpha|=N+1}\|g_\alpha\|_{B_N}\leqslant C\|g\|_{B_N}. \tag{2.3}$$

By (1.8) (with $n=N+1$), it is easy to check that

$$|(f, g)|\leqslant C\|f\|_{A_1}\|g\|_{B_N}. \tag{2.4}$$

Thus (2.1) holds.

Now suppose $\varphi\in(B_{N,0})^*$. Noticing that if $g\in B_{N,0}$, then

$$\lim_{|z|\to1} g(z)(1-|z|^2)^N=0,$$

$$E=\{(1-|z|^2)^N g(z),\ z\in B\,|\,g\in B_{N,0}\}\subset C(\overline{B}).$$

Define a linear functional ψ on E by

$$\psi(g_1)=\varphi(g),\ g_1(z)=(1-|z|^2)^N g(z),\ g\in B_{N,0}.$$

Then we see that $\|\psi\|=\|\varphi\|$. The Hahn–Banach theorem and the Riesz theorem then imply that there is a Borel measure μ on $\overline{B}$ with $\mu(\partial B)=0$ such that

$$\varphi(g)=\int_B(1-|z|^2)^N g(z)d\mu(z)$$

and $\|\mu\|=\|\varphi\|$.

Define a function f by

$$f(z)=\int_B\frac{(1-|\omega|^2)^N}{(1-\langle z,\omega\rangle)^N}\,d\bar{\mu}(\omega). \tag{2.5}$$

Then $f\in H(B)$. Taking derivatives, we have

$$\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)=N(N+1)\cdots(N+|\alpha|-1)\int_B\frac{\bar{\omega}^\alpha(1-|\omega|^2)^N}{(1-\langle z,\omega\rangle)^{N+|\alpha|}}\,d\mu(\omega).$$

Thus

$$\sum_{|\alpha|<N+1}\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\right|\leqslant C\|\mu\|=C\|\varphi\|,$$

and for α, $|\alpha|=N+1$,

$$\begin{aligned}\int_B\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\right|dm(z)&\leqslant C\int_B dm(z)\int_B\frac{(1-|\omega|^2)^N}{|(1-\langle z,\omega\rangle)^{2N+1}|}\,d|\mu|(\omega)\\&\leqslant C\int_B(1-|\omega|^2)^N d|\mu|(\omega)\int_B\frac{1}{|(1-\langle z,\omega\rangle)^{2N+1}|}\,dm(z)\\&\leqslant C\int_B d|\mu|(\omega)=C\|\mu\|=C\|\varphi\|,\end{aligned}$$

where we have used Proposition 1.4.10 in [10], that

$$\int_B\frac{1}{|(1-\langle z,\omega\rangle)^{2N+1}|}\,dm(z)\leqslant C\,\frac{1}{(1-|\omega|^2)^N}.$$

So $f\in A_1$, and $\|f\|_{A_1}\leqslant C\|\varphi\|$.

For any polynomials g, we have

$$\begin{aligned}\varphi(g)&=\int_B(1-|\omega|^2)^N g(\omega)d\mu(\omega)=\lim_{r\to1}\int_B(1-|\omega|^2)^N g(r^2\omega)d\mu(\omega)\\&=\lim_{r\to1}\int_B(1-|\omega|^2)^N d\mu(\omega)\int_S\frac{g(rz)}{(1-r\langle\omega,z\rangle)^N}\,d\sigma(z)\\&=\lim_{r\to1}\int_S g(rz)d\sigma(z)\int_B\frac{(1-|\omega|^2)^N}{(1-r\langle\omega,z\rangle)^N}\,d\mu(\omega)\end{aligned}$$

$$=\lim_{r\to 1}\int_S g(rz)\overline{f(rz)}\,d\sigma(z)=(g,\ f). \tag{2.6}$$

Since polynomials are dense in $B_{N,\,0}$, and by (2.1), we see that holds for all $g\in B_{N,0}$ and $\|f\|_{A_1}\leqslant C\|\varphi\|$. Again by (2.1), we have

$$C\|f\|_{A_1}\leqslant\|\varphi\|\leqslant C_1\|f\|_{A_1}.$$

This finishes the proof.

Proposition 7. $(A_1)^*=B_N$. *More precisely, the pairing* $(f,\ g)$ *for* $f\in A_1,\ g\in B_N$, *gives an isomorphism between* $(A_1)^*$ *and* B_N.

Proof The inequality (2.1) implies that for every $g\in B_N$, $(f,\ g)$ for $f\in A_1$ induces a bounded linear functional on A_1. Thus $B_N\subset(A_1)^*$.

Now assume $\varphi\in(A_1)^*$. For every α, $|\alpha|=N+1$, we have

$$\left\{\frac{\partial^{|\alpha|}f}{\partial z^\alpha},\ f\in A_1\right\}\subset L^1_a(0).$$

It is easy to see that there exist a set $\{d_\alpha\ |\alpha|<N+1\}$ of complex numbers and a set $\{\varphi_\alpha,\ |\alpha|=N+1\}$ of functionals on $L^1_a(0)$, such that

$$\varphi(f)=\sum_{|\alpha|<N+1}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\bar{d}_\alpha+\sum_{|\alpha|=N+1}\varphi_\alpha\Big(\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\Big)$$

with

$$\sum_{|\alpha|<N+1}|d_\alpha|+\sum_{|\alpha|=N+1}\|\varphi_\alpha\|\leqslant C\|\varphi\|.$$

By the proof of Proposition 4.5 in [7], we see that there exists $\{h_\alpha,|\alpha|=N+1\}\subset B_N$, such that

$$\varphi_\alpha\Big(\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\Big)=\int_B\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\,\overline{h_\alpha(z)}(1-|z|^2)^N dm(z),\ f\in A_1,$$

with $C^{-1}\|\varphi_\alpha\|\leqslant\|h_\alpha\|_{B_N}\leqslant C\|\varphi_\alpha\|$.

Define a function g by

$$g(z)=\sum_{|\alpha|<N+1}\frac{d_\alpha}{c_\alpha\alpha!}z^\alpha+\sum_{|\alpha|=N+1}\frac{1}{C_\alpha}z^\alpha h_\alpha(z),$$

where C_α are the constants in (1.8). By (1.8), we have

$$\begin{aligned}\varphi(f)&=\sum_{|\alpha|<N+1}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\bar{d}_\alpha+\sum_{|\alpha|=N+1}\int_B\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\,\overline{h_\alpha(z)}\,(1-|z|^2)^N dm(z)\\&=\sum_{|\alpha|<N+1}C_\alpha\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\,\overline{\frac{\partial^{|\alpha|}g}{\partial z^\alpha}}(0)+\sum_{|\alpha|=N+1}C_\alpha\int_B\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\,\overline{g_\alpha(z)}\,(1-|z|^2)^N dm(z)\\&=(f,\ g)\end{aligned}$$

and

$$C^{-1}\|g\|_{B_N}\leqslant\|\varphi\|\leqslant C\|g\|_{B_N},$$

where the first inequality is obtained by

$$\|g\|_{B_N}\leqslant C\Big(\sum_{|\alpha|<N+1}|d_\alpha|+\sum_{|\alpha|=N+1}\|h_\alpha\|_{B_N}\Big)\leqslant C\|\varphi\|.$$

This completes the proof.

Theorem 1. *For* $p>1$,

$$\frac{1}{q}+\frac{1}{p}=1,$$

and $s=qN-N-1$, *we have* $A_p=(L_a^q(s))^*$. *More precisely, the pairing* $(g,\ f)$ *for* $f\in A_p$, $g\in L_a^q(s)$, *gives an isomorphism between* $(L_a^q(s))^*$ *and* A_p.

Proof Suppose $f\in A_p$, and g is a polynomial. By Lemma 5, $f\in H^2$. Thus (g, f) exists. Writing g as in (1.1) and using Lemma 3, we have

$$\sum_{|\alpha|<N+1}\left|\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)\right|^q+\sum_{|\alpha|=N+1}\|g\|^q_{L_a^q(s)}\leqslant C\|g\|^q_{L_a^q(s)}.$$

By (1.8) and the above inequality, and noticing that

$$N=\frac{(p-1)(N+1)}{p}+\frac{s}{q},$$

it is not difficult to check that

$$|(g, f)|=|(f, g)|\leqslant C\|f\|_{A_p}\|g\|_{L_a^q(s)}. \tag{2.8}$$

Since polynomials are dense in $L_a^q(s)$, we see that (f,g) exists for all $g\in L_a^q(s)$, and it defines a bounded linear functional on $L_a^q(s)$. Thus $A_p\subset(L_a^q(s))^*$.

Now suppose $\varphi\in(L_a^q(s))^*$. Theorem 2.1 in [7] then implies that there exists $h\in L_a^p(p-1)(N+1))$, such that

$$\varphi(g)=\int_B g(z)\,\overline{h(z)}\,(1-|z|^2)^N dm(z)$$

with

$$C\|\varphi\|\leqslant\|h\|_{L_a^p((p-1)(N+1))}\leqslant C_1\|\varphi\|.$$

Define a function f by

$$f(z)=\int_B\frac{h(\omega)(1-|\omega|^2)^N}{(1-\langle z,\omega\rangle)^N}\,d_m(\omega).$$

Then $f\in H(B)$, and

$$\frac{\partial^{|\alpha|}f}{\partial z^\alpha}=N\,(1+N)\cdots(N+|\alpha|-1)\int_B\frac{\bar\omega^\alpha h(\omega)(1-|\omega|^2)^N}{(1-\langle z,\omega\rangle)^{N+|\alpha|}}\,d_m(\omega).$$

Thus by Lemma 3, we have $f\in A_p$, and

$$\begin{aligned}\|f\|^p_{A_p}&=\sum_{|\alpha|<N+1}\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\right|^p+\sum_{|\alpha|=N+1}\left\|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\right\|^p_{L_a^p((p-1)(N+1))}\\&\leqslant C\|h\|^p_{L_a^p((p-1)(N+1))}\leqslant C\|\varphi\|^p.\end{aligned}$$

Morever, for any polynomial g,

$$\begin{aligned}\varphi(g)&=\int_B g(z)\,\overline{h(z)}\,(1-|z|^2)^N dm(z)\\&=\lim_{r\to1}\int_B g(r^2z)\,\overline{h(z)}\,(1-|z|^2)^N dm(z)\\&=\lim_{r\to1}\int_B\bar h(z)\,(1-|z|^2)^N dm(z)\int_S\frac{g(r\omega)}{(1-\langle rz,\ \omega\rangle)^N}\,d\sigma(\omega)\\&=\lim_{r\to1}\int_S g(r\omega)d\sigma(\omega)\int_B\frac{\bar h(z)(1-|z|^2)^N}{(1-\langle rz,\ \omega\rangle)^N}\,dm(z)\\&=\lim_{r\to1}\int_S g(r\omega)\,\overline{f(r\omega)}\,d\sigma(\omega)=(g, f).\end{aligned}$$

Thus, since polynomials are dense in $L_a^2(p-1)(N+1)$, and by (2.8), we have $\varphi(g) = (g, f)$ for all $g\in L_a^p((p-1)(N+1)$. Furthermore

$$C\|\varphi\|\leqslant\|f\|_{A_p}\leqslant C_1\|\varphi\|.$$

This finishes the proof of the theorem.

§ 3. The Reduced Hankel Operators

In this section, we will prove the main theorem of the paper.

Let $f\in H(B)$, and R_f be the reduced Hankel operator on H^2. For an operator A on a Hilbert space, we will denote by $\|A\|_p$ its norm in S_p.

Theorem 2. *Let $f\in A_1$. Then $R_f\in s$, and $\|R_f\|_1\leqslant C\|f\|_{A_1}$.*

Proof By the proof of Proposition 4.5 in [7], we can find a δ dense sequence $\{a_n\}_{n=1}^{\infty}\subset B$, such that

$$\|g\|_{B_N}\leqslant C\sup_n|g(a^n)(1-|a_n|^2)^N| \tag{3.1}$$

holds for all $g\in B_N$.

Define an operator T, $B_{N,0}\to l^\infty$, by

$$Tg(h)=(1-|a_n|^2)^N g(a_n),\ g\in B_{N,\,0}.$$

Notice that for every $g\in B_{N,0}$, $\lim_{n\to\infty}(1-|a_n|^2)^N g(a_n)=0$. Thus $T:B_{N,0}\to l_0^\infty$, where $l_0^\infty=\{\{a_n\}_1^\infty, \lim_n a_n=0\}$. Then (3.1) implies that

$$C\|g\|_{B_{N,0}}\leqslant\|Tg\|\leqslant\|g\|_{B_{N,0}}.$$

Thus T is bounded and bounded below. So $T^*:l'=(l_0^\infty)^*\to A_1=(B_{N,0})^*$ is onto. Similar arguments as in [11] and [7] imply that for every $\{C_n\}_1^\infty\in l'$,

$$T^*\{C_n\}(z)=\sum_{n=1}^{\infty}C_n\frac{(1-|a_n|^2)^N}{(1-\langle z,\,a_n\rangle)^N}.$$

Let $K_a(z)=\dfrac{1}{(1-\langle z,\,a\rangle)^N}$. Then $R_{K_a}=K_a\otimes K_a$, where

$$K_a\otimes K_a(g)=(K_a,\,g)K_a,\ g\in H^2.$$

In fact, if $g,\,h\in H^\infty$, we have

$$\begin{aligned}(R_{K_a}g,\,h)&=(PK_a\bar g,\,h)=(K_a\bar g,\,h)=(K_a,\,gh)=\bar g(a)\bar h(a)\\&=((K_a g)K_a,\,h)=(K_a\otimes K_a(g),\,h).\end{aligned}$$

Because H^∞ is dense in H^2, we have $R_{K_a}=K_a\otimes K_a$, and $\|R_{K_a}\|_p^2=\dfrac{1}{(1-|a|^2)^N}$, $p\geqslant1$.

Now let $f\in A_1$. Then there exists $\{C_n\}_1^\infty\in l'$, such that $f=T^*\{C_n\}$, i.e.

$$f=\sum_{n=1}^{\infty}C_n(1-|a_n|^2)^N K_{a_n}.$$

Thus

$$R_f=\sum_{n=1}^{\infty}C_n(1-|a_n|^2)^N K_{k_n}\otimes K_{k_n}$$

and

$$\|R_f\|_1 \leqslant \sum_{n=1}^{\infty} |C_n| < \infty.$$

That is, $R_f \in S_1$

$$\|R_f\|_1 \leqslant \|\{C_n\}\|_{l'} \tag{3.2}$$

for every $\{C_n\}_1^\infty$, such that $f = T^*\{C_n\}$. Since $A_1 = T^* l'$, we have $A_1 \cong l'/\mathrm{Ker}\ T^*$. So for every $f \in A_1$,

$$(\|f\|_{A_1} \leqslant \inf\{\|a\|_{l'},\ T_a = f\} \leqslant C_1 \|f\|_{A_1}.$$

By (3.2), we have $\|R_f\|_1 \leqslant C\|f\|_{A_1}$. This proves the theorem.

The Hardy space H^2 has an orthonormal basis $\{e_\alpha,\ \alpha \in Z_+^N\}$, $e_\alpha = b_\alpha z^\alpha$,

$$b_\alpha = \frac{(n+|\alpha|-1)!}{\alpha!(n-1)!}.$$

Let us define a linear map W, $S_\infty \to H(B)$. For an operator T on H^2 such that except for finite numbers of indices α, β, $(Te_\alpha, e_\beta) = 0$, we let

$$\omega T(z) = \sum_{r \in Z_+^N} \sum_{\alpha+\beta=r} b_\alpha b_\beta (Te_\beta, e_\alpha) z^r,\ z \in B. \tag{3.3}$$

Then direct computation shows that

$$\mathrm{tr}\ (TR_\varphi) = (\varphi, \omega T), \tag{3.4}$$

where $(\varphi, \omega T)$ is defined by (1.4). The following proposition ensures that ωT can be defined for all $T \in S_\infty$.

Proposition 8. *Let $q > 1$. Then ω, $S_q \to L_q^a(qN - N - 1)$ is continuous.*

Proof Recall that for $\varphi \in H(B)$, R_φ is compact if and only if $\varphi \in VMOA$ and $\|R_\varphi\| \sim \|\varphi\|_{VMOA}$. Then by (3.4) we have

$$|(\varphi, \omega T)| = |\mathrm{tr}(TR_\varphi)| \leqslant \|T\|_1 \|R_\varphi\| \leqslant C\|T\|_1 \|\varphi\|_{VMOA}.$$

Since $(H^1)^* = VMOA$, we have

$$\|\omega T\|_{H^1} \leqslant C\|T\|_1. \tag{3.5}$$

Similarly, using Proposition 7 that $B_N = (A_1)^*$ and Theorem 2, we have

$$\|\omega T\|_{B_N} \leqslant C\|T\|. \tag{3.6}$$

Thus ωT can be defined for all compact operators T, since T can be approximated by finite rank operators. Thus we have proved that

$$\omega,\ S_\infty \to B_N,\ S_1 \to H^1 \tag{3.7}$$

is continous.

By Theorem 2.5 in [4], for every $f \in H^1$ and α, $|\alpha| = 1$, $\dfrac{\partial^{|\alpha|} f}{\partial z^\alpha} \in L_a^1(0)$,

$$\int_B \left| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \right| dm(z) \leqslant C\|f\|_{H^1}. \tag{3.8}$$

On the other hand, if $f \in B_N$, then we can write

$$f(z) = C_N \int_B \frac{f(\omega)(1-|\omega)|^2)^N}{(1-\langle z, \omega\rangle)^{2N+1}} dm(\omega).$$

Thus for α, $|\alpha|=1$,

$$\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)=(2N+1)C_N\int_B\frac{f(\omega)(1-|\omega|^2)^N}{(1-\langle z\omega\rangle)^{2N+2}}dm(\omega).$$

Then by Proposition 1.4.10 in [10], it holds that

$$\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)\right|\leqslant|C\|f\|_{B_N}\int_B\frac{1}{|(1-\langle z\omega\rangle)^{2N+2}|}dm(\omega)$$

$$=C\|f\|_{B_N}\frac{1}{(1-|z|^2)^{N+1}}.$$

So

$$\left\|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\right\|_{B_{N+1}}=\sup_{z\in B}\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)(1-|z|^2)^{N+1}\right|\leqslant C\|f\|_{B_N}. \tag{3.9}$$

Let μ be the measure on B defined by $d\mu(z)=(1-|z|^2)^{-N-1}dm(z)$. We define maps U_α, $T\to\frac{\partial^{|\alpha|}(\omega T)}{\partial z^\alpha}(z)(1-|z|^2)^{N+1}$. Then for α, $|\alpha|=1$, (3.5)—(3.9) imply that U_α, $S_\infty\to L^\infty(\mu)$, $S_1\to L^1(\mu)$ is continuous. Thus

$$U_\alpha.\ S_q=(S_1,S_\infty)_\theta\to L^q(\mu)=(L^1(\mu),L^\infty(\mu))_\theta,\ \frac{1}{q}=1-\theta$$

is continuous, that is,

$$\int_B\left|\frac{\partial^{|\alpha|}(\omega T)}{\partial z^\alpha}(z)\right|^q(1-|z|^2)^{q(N+1)-(N+1)}dm(z)\leqslant C\|T\|_{S_q}.$$

By Theorem 2.5 in [4] (see also [13]), we have for all $T\in S_q$, $\omega T\in L_a^q(qN-N-1)$. So ω, $S_q\to L_a^q(qN-N-1)$ is continuous. Therefore

$$\|\omega T\|_{L_a^q(qN-N-1)}\leqslant C\|T\|_q.$$

This proves the proposition.

Theorem 3. *If $\varphi\in A_p$, $p>1$, then $R_\varphi\in S_p$*

Proof Suppose $\varphi\in A_p$, $\frac{1}{p}+\frac{1}{q}=1$. Then by (3.4), Proposition 8 and Theorem 1, we have

$$|\operatorname{tr}(TR_\varphi)|=|(\varphi,\ \omega T)|\leqslant C\|\varphi\|_{A_p}\|\omega T\|_{L_a^q(qN-N-1)}\leqslant C\|\varphi\|_{A_p}\|T\|_q.$$

Since $(S_q)^*=S_p$, we have $R_\varphi\in S_p$ and $\|R_\varphi\|_p\leqslant C\|\varphi\|_{A_p}$. This proves the theorem.

Theorem 4. *Let $p\geqslant1$, If $R_\varphi\in S_p$, then $\varphi\in A_p$.*

Proof Assume $R_\varphi\in S_p$, $p\geqslant1$, so R_φ is compact and $\varphi\in VMOA$. For any α, $|\alpha|=N+2$, we will prove that $\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\in L_a^p((p-1)(N+1)+p)$, i. e.

$$\varphi\in W^p_{(p-1)(N+1)+p+1,\,N+2}.$$

By Lemma 5 we have $\varphi\in A_p$.

By Lemma 4, we can find sufficient small ε, $\delta>0$ and an $\varepsilon-\delta$ lattice $\{a_n\}_{n=1}^\infty\subset B$, such that for all $f\in H(B)$,

$$\sum_{n=1}^\infty|f(a_n)|^2(1-|a_n|^2)^{N+2}\leqslant C\int_B|f(z)|^2(1-|z|^2)dm(z), \tag{3.10}$$

$$\int_B|f(z)|^p(1-|z|^2)^{(p-1)(N+1)+p}dm(z)\leqslant C\sum_{n=1}^\infty|f(a_n)|^p(1-|a_n|^2)^{p(N+2)}. \tag{3.11}$$

Also recall that if $f\in H^2$, then for α, $|\alpha|=1$, $\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\in L_a^2(1)$, and

$$\sum_{|\alpha|=1}\int_B\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\right|^2(1-|z|^2)dm(z)\leqslant C\|f\|_{H^2} \tag{3.12}$$

(see Theorem 2.5 in [4]).

Now fix α, $|\alpha|=N+2$. Let γ, ν, $\tau\in Z_+^N$ such that $|\gamma|=|\nu|=1$, $\alpha=\gamma+\nu+\tau$. Define two operators T_γ, T_ν, $l^2\to H^2$ by

$$T_\mu e_n(z)=\frac{z^\mu(1-|a_n|^2)^{(N+2)/2}}{(1-\langle z a_n\rangle)^{N+1}},\ \mu=\gamma,\ \nu,$$

wh $\}_1^\infty$ is an orthonormal basis of l^2. Let us also define an operator S_τ on H^2 by $S_\tau f(z)=z^\tau f(z)$, $f\in H^2$.

We claim that T_γ, T_ν are bounded. In fact, if $f\in H^\infty$, then

$$\begin{aligned}f(n)&=(T_\nu^*f,\ e_n)=(f,\ T_\nu^{e_n})\\&=\int_S f(z)\,\frac{\bar z^\nu(1-|a_n|^2)^{(N+2)/2}}{(1-\langle a_n,\ z\rangle)^{N+1}}\,d\sigma(z)\\&=\frac{1}{N\cdots(N+|\nu|-1)}\,\frac{\partial^{|\nu|}f}{\partial z^\nu}(a_n)\,(1-|a_n|^2)^{(N+2)/2}.\end{aligned}$$

So by (3.10), we have

$$\begin{aligned}\sum_{n=1}^\infty|T_\nu^*f(n)|^2&=\sum_{n=1}^\infty\frac{1}{N^2\cdots(N+|\nu|-1)^2}\left|\frac{\partial^{|\nu|}f}{\partial z^\nu}(a_n)\right|^2(1-|a_n|^2)^{n+2}\\&\leqslant C\int_B\left|\frac{\partial^{|\nu|}f}{\partial z^\nu}(z)\right|^2(1-|z|^2)dm(z)\leqslant C\|f\|_{H^2}.\end{aligned}$$

So T_ν^*, consequently, T_ν is bounded. Similarly T_γ is bounded.

Because $\varphi\in VMOA\subset H^2$, we can write

$$\varphi(z)=\int_S\frac{\varphi(\omega)}{(1-\langle z\omega\rangle)^N}\,d\sigma(\omega).$$

Taking derivative, we see that

$$\begin{aligned}(1-|a_n|^2)^{N+2}\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}(a_n)&=N\cdots(N+|\alpha|-1)\ (1-|a_n|^2)^{N+2}\cdot\int_S\frac{\bar\omega^\alpha\varphi(\omega)}{(1-\langle z\omega\rangle)^{2N+2}}\,d\sigma(\omega)\\&=N\cdots(N+|\alpha|-1)\,(T_\gamma^*R_\varphi S_\tau T_\nu e_n,\ e_n).\end{aligned} \tag{3.13}$$

Since $R_\varphi\in S_p$, we have $T_\gamma^*R_\varphi S_\tau T_\nu\in S_p$ and

$$\sum_{n=1}^\infty|(T_\gamma^*R_\varphi S_\tau T_\nu e_n,\ e_n)|^p\leqslant\|T_\gamma\|^p\|T^\nu\|^p\|S_\tau\|^p\|R_\varphi\|_p<\infty.$$

By (3.11) and (3.13), it follows that

$$\begin{aligned}&\int_B\left|\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}(z)\right|^p(1-|z|^2)^{(p-1)(N+1)+p}dm(z)\\&\leqslant C\sum_{n=1}^\infty\left|\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}(a_n)\right|^p(1-|a_n|^2)^{p(N+2)}\leqslant C\|R_\varphi\|_p^p<\infty.\end{aligned}$$

Thus $\frac{\partial^{|\alpha|}\varphi}{\partial z^\alpha}\in L_a^p((p-1)(N+1)+p)$. This proves the theorem.

Combining Theorems 2, 3, and Theorem 4, we have proved our main theorem.

Theorem 5. *Let $f\in H(B)$, $p\geqslant1$. Then $R_f\in S_p$ if and only if $f\in A_p$.*

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