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THE EXTENDED CARLESON MEASURE AND THE DERIVATIVES OF L^p FUNCTIONS**

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Abstract

A characteristic of the extended Carleson measure expressed by the derivatives of L^p functions in R_+^{n+1} is obtained.

One of the important discovery in Carleson's consideration of the interpolation problems in H^p spaces is the Carleson measure [1]. It is well-known that the characteristic of Carleson measure can be represented by some inequalities of the integrals of L^p or H^p functions [2,3]. [4] gave the inequalities for the extended Carleson measure. [5] (Theorem 3.1) gave a necessary and sufficient condition of the extended Carleson measure by the derivatives of H^p functions in the unit disk. We obtain an analogue in R_+^{n+1}. Our method is appropriate to the case in [5] and may be more natural than that of [5].

If Q is a cube of side length h in R^n, we call $S_h=Q\times(0, h]$ a Carleson domain in the upper half space R_+^{n+1}. If μ is a positive measure in R_+^{n+1}, and there are two positive numbers C and α such that

$$\mu(S_h)\leqslant Ch^\alpha$$

for every $S_h\in R_+^{n+1}$, then μ is called an extended Carleson measure (or α-Carleson measure).

Theorem 1. *Let μ be a positive measure on R_+^{n+1}. Then*

$$\mu(S_h)\leqslant Ch^{n+pr} \tag{1}$$

for every S_h if and only if

$$\left\{\int_{R_+^{n+1}}\left(\left|\frac{\partial^r u}{\partial y^r}\right|^p+\sum_{i=1}^n\left|\frac{\partial^r u}{\partial x_i^r}\right|^p\right)d\mu\right\}^{\frac{1}{p}}\leqslant C\|f\|_p, \tag{2}$$

for any $f\in L^p$, where u is the Poisson integral of f, r is a non-negative integer and $p\geqslant 2$.

(Here and afterwards C denotes constants which are different for current cases)

Proof For $r=0$ the result of Theorem 1 is classical, so we assume $r\geqslant 1$.

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Necessity. Firstly we prove it for the case $r=1$.

Divide R_+^{n+1} into a union of cubes $R_+^{n+1}=\bigcup Q_j^k$, where if $(x, y)\in Q_j^k$, $2^k\leqslant y\leqslant 2^{k+1}$ $(k=0, \pm1, \pm2, \cdots)$, and Q_j^k $(j=0, 1, 2, \cdots)$ is a partition of the strip $2^k\leqslant y\leqslant 2^{k+1}$ into cubes of side length 2^k, the coordinates of their top points being dyadic numbers. Then we have

$$\int_{R_+^{n+1}}\left|\frac{\partial u}{\partial y}\right|^p d\mu=\sum_{j,k}\int_{Q_j^k}\left|\frac{\partial u}{\partial y}\right|^p d\mu\leqslant\sum_{j,k}\sup_{(x,y)\in Q_j^k}\left|\frac{\partial u}{\partial y}\right|^p \mu(Q_j^k) \tag{3}$$

and

$$\int_{R_+^{n+1}}\left|\frac{\partial u}{\partial x_i}\right|^p d\mu=\sum_{j,k}\int_{Q_j^k}\left|\frac{\partial u}{\partial x_i}\right|^p d\mu\leqslant\sum_{j,k}\sup_{(x,y)\in Q_j^k}\left|\frac{\partial u}{\partial x_i}\right|^p \mu(Q_j^k) \tag{4}$$

$$(i=1, 2, \cdots, n).$$

Let $Q_{j'}^{\prime k}(j=0, 1, \cdots)$ denote the partition of the strip $0<y\leqslant 2^{k+1}(k=0, \pm1, \cdots)$ into cubes of side length 2^{k+1}, the coordinates of their top points being dyadic numbers. We have

$$\mu(Q_j^k)\leqslant\mu(Q_{j'}^{\prime k})\leqslant C(2^{k+1})^{n+p} \tag{5}$$

for $Q^k\subset Q_{j'}^{\prime k}$.

In order to estimate $\sup\limits_{Q_j^k}\left|\frac{\partial u}{\partial y}\right|^p$ and $\sup\limits_{Q_j^k}\left|\frac{\partial u}{\partial x_i}\right|^p$, we use the fact that $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial x_i}$ are harmonic functions. For any $(x, y)\in Q_j^k$, by $B_{2^{k-1}}(x, y)$ we denote the ball with centre (x, y) ard radius 2^{k-1}, and by $2Q_j^k$ we denote the cube with the same centre of Q_j^k and side length two times to that of Q_j^k. Then $B_{2^{k-1}}(x, y)\subset 2Q_j^k$ and $2Q_j^k\subset R_+^{n+1}$. Therefore the mean value theorem gives

$$\begin{aligned}\left|\frac{\partial u}{\partial y}(x, y)\right|^p&\leqslant\frac{1}{|B_{2^{k-1}}(x, y)|}\int_{B_{2^{k-1}}(x,y)}\left|\frac{\partial u}{\partial y'}(x', y')\right|^p dx'dy'\\&\leqslant\frac{C}{2^{(k-1)(n+1)}}\int_{2Q_j^k}\left|\frac{\partial u}{\partial y'}(x', y')\right|^p dx'dy'.\end{aligned} \tag{6}$$

Substituting (5) and (6) into (3) we obtain

$$\begin{aligned}\int_{R_+^{n+1}}\left|\frac{\partial u}{\partial y}\right|^p d\mu&\leqslant C\sum_{j,k}\frac{2^{(k+1)(n+p)}}{2^{(k-1)(n+1)}}\int_{2Q_j^k}\left|\frac{\partial u}{\partial y'}(x', y')\right|^p dx'dy'\\&\leqslant C\sum_{j,k}\int_{2Q_j^k}y^{p-1}\left|\frac{\partial u}{\partial y}\right|^p dxdy\\&\leqslant C\int_{R_+^{n+1}}y^{p-1}\left|\frac{\partial u}{\partial y}\right|^p dxdy.\end{aligned} \tag{7}$$

Similarly,

$$\int_{R_+^{n+1}}\left|\frac{\partial u}{\partial x_i}\right|^p d\mu\leqslant C\int_{R_+^{n+1}}y^{p-1}\left|\frac{\partial u}{\partial x_i}\right|^p dxdy. \quad (i=1, 2, \cdots, n). \tag{8}$$

Let

$$G_p(x)=\left\{\int_0^\infty y^{p-1}\left|\frac{\partial u}{\partial y}\right|^p dy\right\}^{\frac{1}{p}},$$

$$G_\infty(x)=\sup_{y>0}\left|y\frac{\partial u}{\partial y}\right|.$$

Then

$$G_\infty(x) \leqslant C(Mf)(x) \quad \text{(see [6, p. 79])}$$

(Mf is the Hardy–Littlewood maximal function).

So we have

$$\|G_2(x)\|_p \leqslant A_p \|f\|_p$$

and

$$\|G_\infty(x)\|_p \leqslant C\|(Mf)(x)\|_p \leqslant B_p \|f\|_p.$$

Therefore

$$G_p(x) \leqslant G_2^{2/p}(x) G_\infty^{1-2/p}(x)$$

for $p \geqslant 2$. With Hölder's inequality we obtain

$$\|G_p(x)\|_p \leqslant \|G_2(x)\|_p^{2/p} \|G_\infty(x)\|_p^{1-2/p} \leqslant C\|f\|_p. \tag{9}$$

Substituting (9) into (7), we obtain

$$\int_{R_+^{n+1}} \left|\frac{\partial u}{\partial y}\right|^p d\mu \leqslant C \int_{R^n} dx \int_0^\infty y^{p-1} \left|\frac{\partial u}{\partial y}\right|^p dy \leqslant C\|f\|_p^p. \tag{10}$$

Similarly, from (8) we have

$$\int_{R_+^{n+1}} \left|\frac{\partial u}{\partial x_i}\right|^p d\mu \leqslant C\|f\|_p^p \quad (i=1,\ 2,\ \cdots,\ n). \tag{11}$$

Therefore (2) holds for $r=1$.

For $r>1$ the proof is in the same way as above. Instead of $G_p(x)$ and $G_\infty(x)$ we use

$$G_{p,r}(x) = \left\{\int_0^\infty \left|y^r \frac{\partial^r u}{\partial y^r}\right|^p \frac{dy}{y}\right\}^{1/p}$$

and

$$G_{\infty,r}(x) = \sup_{y>0} \left|y^r \frac{\partial^r u}{\partial y^r}\right|$$

respectively. Then we have

$$\|G_{2,r}(x)\|_p \leqslant A'_p \|f\|_p \text{ (see [6, p. 86)} \tag{12}$$

and

$$\|G_{\infty,r}(x)\|_p \leqslant C\|(Mf)(x)\|_p \leqslant B'_p \|f\|_p \text{ (see [6, p. 80]).} \tag{13}$$

Moreover, in the expressions of $G_{2,r}(x)$ and $G_{\infty,r}(x)$ we use $\frac{\partial^r u}{\partial x_i^r}$ instead of $\frac{\partial^r u}{\partial y^r}$. The inequalities corresponding to (12) and (13) hold. Therefore (2) holds.

Sufficiency. We choose a harmonic function

$$u(x,\ y) = \frac{y_0}{[|x-x_0|^2 + (y+y_0)^2]^{(n+1)/2}},$$

where $(x_0,\ y_0) \in R_+^{n+1}$ is fixed. The boundary value f of u satisfies

$$\|f\|_p = s y_0^{-n(1-1/p)},\ s = \int_{R^n} \frac{dx}{(1+|x|^2)^{(n+1)p/2}}.$$

If $r=1$, we consider an h-box q in R^n so that

$$q = \{x_{0,1}+h \leqslant x_1 \leqslant x_{0,1}+2h,\ x_{0,i} \leqslant x_i \leqslant x_{0,i}+h,\ (i=2,\ \cdots,\ n)\},$$

where $(x_{0,1}, x_{0,2}, \cdots, x_{0,n}) = x_0$. Let $S_h = q\times(0, h]$ and choose $y = h$. Write

$$u(x, y) = h[(x_1 - x_{0,1})^2 + v]^{-(n+1)/2},$$

where

$$v = \sum_{i=2}^{n}(x_i - x_{0,i})^2 + (y+h)^2.$$

For $(x, y)\in S_h$ we have the following estimates

$$v \leqslant (n-1)h^2 + (2h)^2 = (n+3)h^2,$$
$$(x_1 - x_{0,1})^2 + v \leqslant (n+7)h^2$$

and

$$|u(x, y)| \geqslant C/h^n. \tag{14}$$

Denote $(n+1)/2 = \beta$. Then

$$\frac{\partial u}{\partial x_1} = -2\beta u(x_1 - x_{0,1})[(x_1 - x_{0,1})^2 + v]^{-1}.$$

So we have

$$\left|\frac{\partial u}{\partial x_1}\right| \geqslant \frac{2\beta h u}{(n+7)h^2} = \frac{Cu}{h}. \tag{15}$$

With the assumption (2) for $r=1$, the above u and f satisfy

$$\left\{\int_{S_h}\left|\frac{\partial u}{\partial x_1}\right|^p d\mu\right\}^{1/p} \leqslant C\|f\|_p = Ch^{-n(1-1/p)}.$$

On the other hand, (14) and (15) imply

$$\left\{\int_{S_h}\left|\frac{\partial u}{\partial x_1}\right|^p d\mu\right\}^{1/p} \geqslant \left\{\int_{S_h}\left|\frac{Cu}{h}\right|^p d\mu\right\}^{1/p} \geqslant \left\{\frac{C}{h^{(n+1)p}}\,\mu(S_h)\right\}^{1/p}.$$

Therefore

$$\mu(S_h) \leqslant Ch^{n+p}.$$

If $r=2$, we have

$$\frac{\partial^2 u}{\partial x_1^2} = -2\beta u[(x_1 - x_{0,1})^2 + v]^{-2}[-(2\beta+1)(x_1 - x_{0,1})^2 + v].$$

Consider the h-box

$$q = \{x_{0,1} + kh \leqslant x_1 \leqslant x_{0,1} + (k+1)h,\ x_{0,i} \leqslant x_i \leqslant x_{0,i} + h (i=2, \cdots, n)\},$$

where k is an integer such that

$$k^2(2\beta+1) - (n+3) = \gamma > 0.$$

Then $\frac{\partial^2 u}{\partial x_1^2}$ remains positive or negative in $S_h = q\times(0, h]$ and so

$$\left|\frac{\partial^2 u}{\partial x_1^2}\right| \geqslant \frac{2\beta\gamma u h^2}{[(n+7)h]^2} = \frac{Cu}{h^2}.$$

Consequently it follows from (2) that

$$\mu(S_h) \leqslant Ch^{n+2p}.$$

In the general case, if $r = 2m+1$, we have

$$\frac{\partial^r u}{\partial x_1^r} = C_{2m+1} u[(x_1 - x_{0,1})^2 + v]^{-(2m+1)} \prod_{j=1}^{m}[R_j^{(r)}(x_1 - x_{0,1})^2 + S_j^{(r)} v](x_1 - x_{0,1});$$

if $r = 2m$, we have

$$\frac{\partial^r u}{\partial x_1^r}=C_{2m}u[(x_1-x_{0,1})^2+v]^{-2m}\prod_{j=1}^{m}[R_j^{(r)}(x_1-x_{0,1})^2+S_j^{(r)}v],$$

where $R_j^{(r)}$'s and $S_j^{(r)}$'s are constants depending on n, p and r. If we choose an integer k sufficiently large such that

$$|R_j^{(r)}|k^2-|S_j^{(r)}|(n+3)>0 \quad (j=1,\ 2,\ \cdots,\ m),$$

then we have

$$\left|\frac{\partial^r u}{\partial x_1^r}\right|\geqslant\frac{Cu}{h^r}$$

in the Carleson domain $S_h=q\times(0,\ h]$, where

$$q=\{x_{0,1}+kh\leqslant x_1\leqslant x_{0,1}+(k+1)h,\ x_{0,i}\leqslant x_i\leqslant x_{0,i}+h \quad (i=2,\ \cdots,\ n)\}.$$

This inequality combined with inequality (2) implies (1).

The proof is completed.

As remarked in [5] the restriction $p\geqslant 2$ in Theorem 1 can not be canceled.

Theorem 2. *Let μ be a positive measure on R_+^{n+1}. Then*

$$\mu(S_h)\leqslant Ch^{qn/p+rq} \tag{16}$$

for every S_h if and only if

$$\left\{\iint_{R_+^{n+1}}\left(\left|\frac{\partial^r u}{\partial y^r}\right|^q+\sum_{i=1}^{n}\left|\frac{\partial^r u}{\partial x_i^r}\right|^q\right)d\mu\right\}^{1/q}\leqslant C\|f\|_p \tag{17}$$

for every $f\in L^p$, where $2\leqslant p\leqslant q<\infty$, r is a non-negative integer and u is the Poisson integral of f.

Proof We observe that Theorem A of [5] is valid for harmonic functions. So (16) implies that

$$\left\{\iint_{R_+^{n+1}}\left|\frac{\partial^r u}{\partial y^r}\right|^q d\mu\right\}^{1/q}\leqslant C\left\{\iint_{R_+^{n+1}}\left|\frac{\partial^r u}{\partial y^r}\right|^p y^{rp-1}dxdy\right\}^{1/p}. \tag{18}$$

Combining (18) with the inequality

$$\left\{\iint_{R_+^{n+1}}\left|\frac{\partial^r u}{\partial y^r}\right|^p y^{rp-1}dxdy\right\}^{1/p}\leqslant C\|f\|_p,$$

we obtain

$$\left\{\iint_{R_+^{n+1}}\left|\frac{\partial^r u}{\partial y^r}\right|^q d\mu\right\}^{1/q}\leqslant C\|f\|_p.$$

Hence (17) holds.

The proof of the converse is similar to that of Theorem 1.

References

[1] Carleson, L., *Ann. Math.*, **76**: 3 (1962), 547—559

[2] Garnett, J. B., Bounded analytic functions, Acad. Press, 1981.

[3] Hörmander, L., *Math. Scand.*, **20** (1967), 65—78.

[4] Duren, P., *Bull. Amer. Math. Soc.*, **75** (1969), 143—146.

[5] Luecking, D. H. *Amer. J. Math.*, **107** (1985), 85—111.

[6] Stein, E. M., Singular integrals and differentiability properties of functions, Princeton Univ., Prests, 1970.