Chin. Ann. of Math.
12B: 3(1991), 301—305.

A REMARK ON THE UNIQUENESS OF THE HARMONIC MAPS**

BIAN BAOJUN (边保军)*

Abstract

This paper studies the uniqueness of the harmonic maps from R^n into S^{n-1}. At first it is proved that the map $\psi(x)=\frac{x}{|x|}:B^n\to S^{n-1}$ is the unique energy minimizing harmonic map. Then the uniqueness of the harmonic maps with identity boundary value from B^n to S^{n-1} follows.

In this paper we consider the uniqueness of the harmonic maps from R^n to S^{n-1}. We prove that $\psi=\frac{x}{|x|}:R^n\to R^{n-1}$ is the unique harmonic map with identity boundary value for all $n\geqslant 3$.

It is not hard to see that $\psi(x)$ is a harmonic map from R^n into S^{n-1}, i. e., ψ satisfies the equation $\Delta\psi+\psi|\nabla\psi|^2=0$. W. Jäger and H. Kaul have shown that ψ is an energy minimizing harmonic map if $n\geqslant 7$ [2]. H. Brezis, J. M. Coron and E. Lieb have proved that ψ: $R^3\to S^2$ is the unique minimizing harmonic map [3]. Recently, Fan-Hua Lin has proved that ψ is an energy minimizing harmonic map for all $n\geqslant 3$ [1]. The questions we shall discuss are whether ψ is the unique energy minimizing harmonic map or not and whether ψ is the unique harmonic map or not for all $n\geqslant 3$.

Let B be the unit ball in R^n, and consider the class

$$C=\{u\in H^1(B,\ S^{n-1})\,|\,u(x)=x \text{ on } \partial B\}.$$

Then we set

$$E=\inf_{u\in C}\int_B|\nabla u|^2dx.$$

At first, we prove the following

Theorem 1. *The map $\psi(x)$ is the unique minimizer of E for all $n\geqslant 3$.*

The technique we use in the proof of the above theorem is so called "adding a null Lagrangian". This is from Lin's paper[1] and has been used already in [1] in proving that ψ is a minimizer for E for $n\geqslant 3$.

Lemma 1. *Let u: $R^n\to S^{n-1}(n\geqslant 4)$ be a C^1 map in a neighborhood of $x_0\in R^n$ which*

Manuscript received February 10, 1989.
* Department of Mathematics, Zhejiang university, Hangzhou, Zhejiang, China.
** Supported by the National Natural Science Foundation of China.

satisfies

$$P(\nabla u)(x_0)=|\nabla u|^2(x_0)+\frac{1}{n-2}[\operatorname{tr}(\nabla u)^2-(\operatorname{div} u)^2](x_0)=0. \tag{1}$$

Then we have

$$\partial u_i/\partial x_j(x_0)=a[\delta_{ij}-u_i(x_0)u_j(u_0)],\ i,\ j=1,\ 2,\ \cdots,\ n, \tag{2}$$

where $a=\frac{1}{n-1}\operatorname{div} u(x_0)$.

Proof Let us choose a rotation $R=(r_{ij})$ in R^n which satisfies

$$R(e_n)=u(x_0),$$

where e_n is n-th unit coordinate vector. Then, denote by y_0 the point $R^t x_0$ and define

$$v(x)=R^t\cdot u\cdot R(x) \text{ for } x\in R^n.$$

It is easy to verify that

$$v(y_0)=R^t\circ u(x_0)=e_n, \tag{3}$$

$$\nabla v(x)=R^t\nabla u(Rx)R \text{ for all } x\in R^n, \tag{4}$$

$$P(\nabla v)(x)=P(\nabla u)(Rx) \text{ for all } x\in R^n. \tag{5}$$

In particular we have $P(\nabla v)(y_0)=0$.

Setting $a_{ij}=\frac{\partial v_i}{\partial x_j}(y_0)$ and applying the relation (3), we obtain $a_{nj}=0$ for $j=1$, $2,\cdots,v$. So we have

$$(\operatorname{div} v(y_0))^2=\Big(\sum_{i=1}^{n-1}a_{ii}\Big)^2=(n-1)\sum_{i=1}^{n-1}a_{ii}^2-\frac{1}{2}\sum_{i,j=1}^{n-1}(a_{ii}-a_{jj})^2, \tag{6}$$

$$\operatorname{tr}(\nabla v)^2(y_0)=\sum_{i,j=1}^{n}a_{ij}a_{ji}=\sum_{i=1}^{n-1}a_{ii}^2-\sum_{i\neq j}a_{ij}^2+\frac{1}{2}\sum_{i\neq j}(a_{ij}+a_{ji})^2, \tag{7}$$

$$|\nabla v|^2(y_0)=\sum_{i,j=1}^{n}a_{ij}^2=\sum_{i=1}^{n-1}a_{ii}^2+\sum_{i\neq j}a_{ij}^2. \tag{8}$$

Hence we get

$$P(\nabla v)(y_0)=\frac{n-3}{n-2}\sum_{i\neq j}a_{ij}^2+\frac{1}{2(n-2)}\Big[\sum_{i\neq j}(a_{ij}+a_{ji})^2+\sum_{i,j=1}^{n-1}(a_{ii}-a_{jj})^2\Big]. \tag{9}$$

Consequently one has

$$a_{ij}=0\ i\neq j,\ a_{11}=\cdots=a_{n-1n-1}.$$

By a simple computation we obtain from formula (4)

$$\partial u_i/\partial x_j(x_0)=a(\delta_{ij}-r_{in}r_{jn}),$$

where $a=a_{11}=\frac{1}{n-1}\operatorname{div} v(y_0)=\frac{1}{n-1}\operatorname{div} u(x_0)$. Recalling the choice of rotation R, we have

$$r_{in}=u_i(x_0),\ i=1,\ 2,\ \cdots,\ n.$$

The required equality (2) then follows.

Lemma 2. *Let* u: R^3–S^2 *be a* C^1 *map in a neighborhood of* $x_0\in R^3$ *which satisfies*

$$P(\nabla u)(x_0)=|\nabla u|^2(x_0)+[\operatorname{tr}(\nabla u)^2-(\operatorname{div} u)](x_0)=0.$$

Then there exist unit vectors r_1, r_2 *in* R^3 *and a real number* b *such that* $(r_1, r_2)=0$, $(r_k,$

$u(x_0))=0$ *for* $k=1, 2$ *and*

$$\partial u_i/\partial x_j(x_0)=a[\delta_{ij}-u_i(x_0)u_j(x_0)]+b(r_{i1}r_{j2}-r_{i2}r_{j1}),$$

$i, j=1, 2, 3$, *where* $a=\frac{1}{2}\operatorname{div}u(x_0)$.

Proof Follow the same argument as in the proof of Lemma 1.

Proof of the Theorem 1. Let u be a minimizer for E. We claim that u equals ψ.

From the Schoen-Uhlenbeck result[4], we know that the singular set of map u is a closed set with Hausdorff dimension at most $n-3$. In particular, u is a C^1 map in some neighborhood of $x\in B$ for a. e. $x\in B$.

From Lin's paper[1] we see that

$$P(\nabla u)\geqslant 0,\ E(u)=E=\inf_{u\in C}\int_B|\nabla u|^2dx=\frac{n-1}{n-2}\operatorname{meas}(S^{n-1}),$$

$$\int_B[(\operatorname{div}u)^2-\operatorname{tr}(\nabla u)^2]dx=(n-1)\operatorname{meas}(S^{n-1}).$$

So we have by the definition of $P(\nabla u)$

$$\begin{aligned}\frac{n-1}{n-2}\operatorname{meas}(S^{n-1})&=\int_B|\nabla u|^2dx\\&=\int_B P(\nabla u)dx+\frac{1}{n-2}\int_B[(\operatorname{div}u)^2-\operatorname{tr}(\nabla u)^2]dx\\&=\int_B P(\nabla u)dx+\frac{n-1}{n-2}\operatorname{meas}(S^{n-1}).\end{aligned}$$

Hence $P(\nabla u)(x)=0$ for a. e. $x\in B$. Then form Lemma 1 and Lemma 2 we obtain for all $n\geqslant 3$

$$\partial u_i/\partial x_j(x)=a(x)(\delta_{ij}-u_iu_j)+b(x)\delta_{3n}(r_{i1}r_{j2}-r_{i2}r_{j1}).$$

By a simple computation we find that

$$a^2=\frac{1}{n-1}|\nabla u|^2\quad(n\geqslant 4),\ a^2+b^2=\frac{1}{2}|\nabla u|^2\quad(n=3).$$

So we have

$$\int_B a^2(x)dx\leqslant\frac{1}{n-2}\operatorname{meas}(S^{n-1})\text{ for all }n\geqslant 3.$$

Next set $f(x)=(\psi(x), u(x))$. We want to prove that $f=1$ for a. e. $x\in B$. By a simple computation we get

$$\begin{aligned}\partial f/\partial x_i(x)=&a[\psi_i-f(x)u_i]+[u_i-f(x)\psi_i]\frac{1}{|x|}\\&+\delta_{3n}b(x)[(r_1,\psi)r_{i2}-(r_2,\psi)r_{i1}]\quad\text{a. e. }x\in B,\text{ all }i.\end{aligned}\tag{10}$$

According to Fubini theorem we conclude that there exists a set $F\subset S^{n-1}$, meas $(S^{n-1}\backslash F)=0$, such that for any $\sigma\in F$ the formula(10) holds at the point $t\sigma$ for a. e $t\in[0, 1]$ and

$$N(\sigma)=\int_0^1 r^{n-1}a^2(r\sigma)dr<\infty.$$

Now fix $\sigma\in F$ and let $g(t)$ denote the function $1-f(t\sigma)$. We obtain by formula (10) and boundary condition

$$g'(t)=-\frac{\partial f}{\partial x_i}(t\sigma)\sigma_i=-a(t\sigma)[1+f(t\sigma)]g(t) \text{ a. e. } t\in[0,1], \tag{11}$$

$$g(1)=0. \tag{12}$$

For any given $d\in(0,1]$ we get

$$M=\int_d^1 a^2(r\sigma)dr\leqslant d^{1-n}N(\sigma)<\infty.$$

Consequently we obtain by Hölder inequality

$$g(t)=\int_t^1 a(s\sigma)[1+f(s\sigma)]g(s)ds$$

$$\leqslant 2\sqrt{M}\left[\int_t^1 |g(s)|^2 ds\right]^{1/2} \text{for all } t\in[d,1].$$

Hence $g\in C[d,1]$ and

$$|g(t)|^2\leqslant 4M\int_t^1 |g(s)|^2 ds, \text{ for all } t\in[d,1].$$

Using the Bellman inequality we immediatly get

$$g(t)=0 \text{ for all } t\in[d,1].$$

So, $g(t)=0$ for all $t\in(0,1]$. Recalling the choice of the set F, it follows that $f(x)=1$ for a. e. $x\in B$.

This comnletes the proof of Theorem 1.

Now we prove the following

Theorem 2. *The map $\psi(x)$ is the unique harmonic map with identity boundary value from R^n into S^{n-1} for all $n\geqslant 3$.*

Proof Let $u\in H^1(B^n)$ be a harmonic map from R^n to S^{n-1}. Then we have identity

$$\operatorname{div}((x\cdot Du_i)Du_i)-\frac{1}{2}x\cdot D(|Du_i|^2)=-(x\cdot Du_i)u_i|\nabla u|^2+|Du_i|^2,$$

$$i=1,2,\cdots,n. \tag{13}$$

Summing over i we then obtain

$$\sum_i \operatorname{div}((x\cdot Du_i)Du_i)-\frac{1}{2}x\cdot D(|\nabla u|^2)=|\nabla u|^2. \tag{14}$$

By above identity and divergence theorem one has

$$\left(\frac{n}{2}-1\right)\int_B |\nabla u|^2 dx=\frac{1}{2}\int_{S^{n-1}}|\nabla u|^2 d\sigma-\sum_i\int_{S^{n-1}}(x\cdot Du_i)^2 d\sigma. \tag{15}$$

If we denote the tangential gradient of u on S^{n-1} by $\nabla_T u$, then it is obtained that

$$\int_B |\nabla u|^2 dx=\frac{1}{n-2}\int_{S^{n-1}}|\nabla_T u|^2 d\sigma-\frac{1}{n-2}\int_{S^{n-1}}|\nabla_r u|^2 d\sigma, \tag{16}$$

where

$$|\nabla_r u|^2=|\nabla u|^2-|\nabla_T u|^2\geqslant 0.$$

Since $u(x)=x$ on S^{n-1}, we have

$$\int_{S^{n-1}}|\nabla_T u|^2 d\sigma=(n-1)|S^{n-1}|.$$

So

$$\int_B|\nabla u|^2 dx\leqslant E.$$

It follows that u is a minimizer of E and $u\equiv\psi$ by Theorem 1.

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