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A DIOPHANTINE INEQAULITY (II)***

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Abstract

This paper proves the existence of infinitely many integer solutions to a Diophantine inequality.

§ 1. Introduction

The object of this paper is to prove the following

Theorem. *Let λ_1, λ_2, λ_3, λ_4, λ_5 be non-zero real numbers not all in rational ratios and not all of the same sign. Then, given real number η, the inequality*

$$|\lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2+\lambda_4 x_4^2+\lambda_5 x_5^2+\eta|<(\max_{1\le i\le 5}|x_i|)^{-1/2+\varepsilon} \tag{1}$$

has infinitely many solutions in positive integers x_1, x_2, x_3, x_4, x_5 for any $\varepsilon>0$.

This is an improvement of the result in [6], where the exponent of the right side of inequality (1) is $-1/3+\varepsilon$.

§ 2. The Estimation of I_1, I_2, I_4

Without loss of generality, we may assume that λ_1/λ_2 is negative and irrational. Suppose that q is the denominator of a convergent fraction of λ_1/λ_2.

Let

$$N=q,\ Y=N^{1/2-\varepsilon},\ 0<v<1/3,\ 0<\delta<\varepsilon/5$$

and write

$$f(\alpha)=\sum_{x=1}^{N}e(\alpha x^2),$$

$$f_j(\alpha)=f(\lambda_j\alpha),\ j=1,\ 2,\ 3,\ 4,\ 5.$$

Similar to [6], we can show that $R(N)$, the number of the solutions of inequality

$$|\lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2+\lambda_4 x_4^2+\lambda_5 x_5^2+\eta|<Y^{-1}$$

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in integers satisfying

$$\max_{1\leqslant i\leqslant 5}|x_i|<N,$$

satisfies

$$R(N)\gg Y^{-1}\int_{-\infty}^{+\infty}e(\eta\alpha)\Big(\prod_{j=1}^{5}f_j(\alpha)\Big)K_n\Big(\frac{\alpha}{Y}\Big)d\alpha=Y^{-1}I$$
$$=Y^{-1}(I_1+I_2+I_3+I_4),$$

where I_1, I_2, I_3, I_4 are the integrals over the intervals

$$E_1=\{|\alpha|\leqslant N^{-2+v}\},$$
$$E_2=\{N^{-2+v}<|\alpha\leqslant 1|\},$$
$$E_3=\{1<|\alpha|\leqslant YN^{\delta}\},$$
$$E_4=\{|\alpha|>YN^{\delta}\},$$

respectively.

It is almost the same of [6] (we take $v=\left[\frac{1}{2\delta}\right]+1$ now) that

$$I_4=o(N^3), \tag{2}$$
$$I_2=o(N^3) \tag{3}$$

and

$$I_1\gg N^3. \tag{4}$$

§ 3. The Estimation of I_3

Let $M=\{\alpha:\ |f_j(\alpha)|\geqslant N^{1/2+\delta},\ j=1,\ 2\}\cap E_3$. Then

$$\int_{E_3\setminus M}e(\eta\alpha)\Big(\prod_{j=1}^{5}f_j(\alpha)\Big)K_n\Big(\frac{\alpha}{Y}\Big)d\alpha\ll N^{1/2+\delta}\sum_{j=1}^{5}\int_1^{YN^{\delta}}|f_j|^4d\alpha$$
$$\ll N^{1/2+\delta}\cdot N^2\cdot YN^{\delta}\ll N^{1/2+1/2+2+2\delta-\varepsilon}=o(N^3).$$

In order to estimate the integral I_3, we need a lemma below.

Lemma. *Let $Y\geqslant 1$, $A\geqslant N^{1/2+\delta}$, $B\geqslant N^{1/2+\delta}$ and the total length of the set*

$$\mathscr{E}=\{\alpha\ |\leqslant\alpha\leqslant Y,\ A\leqslant|f_1(\alpha)|<2A,\ B\leqslant|f_2(\alpha)|<2B\}$$

be $m(\mathscr{E})$. Then there is a positive constant C shch that

$$m(\mathscr{E})\ll YN^{2+\delta}(AB)^{-2}q^{-1},\ \text{if}\ AB>2Cq^{1/2}N$$

and

$$m(\mathscr{E})\ll YN^{4+\delta}(AB)^{-4}+Y^{1/4}q^{3/4}N^{1/2+\delta}(AB)^{-2},\ \text{if}\ AB\leqslant 2Cq^{1/2}N.$$

Proof This is essentially the Lemma 1 of [3], only N and η being replaced by N^2 and q.

Now, we subdivide M into several sets

$$\mathscr{E}=M(A,\ B)=\{\alpha:\ A\leqslant|f_1(\alpha)|<2A,\ B\leqslant|f_2(\alpha)|<2B\}\cap E_3.$$

Obviously the number of $M(A,\ B)$ is $\ll(\log N)^2\ll N^{\delta/2}$.

If $AB>2Cq^{1/2}N$, then

$$\int_{M(A,B)} e_{(\eta\alpha)}\Big(\prod_{j=1}^{5} f_{j(\alpha)}\Big)K_n\Big(\frac{\alpha}{Y}\Big)\alpha d$$
$$\ll\Big(\int_{M(A,B)}|f_1f_2|^4d\alpha\Big)^{1/4}\Big(\int_{M(A,B)}|f_3|^4d\alpha\Big)^{1/4}\Big(\int_1^{YN^\delta}|f_4|^4d\alpha\Big)^{1/4}\Big(\int_1^{YN^\delta}|f_5|^4d\alpha\Big)^{1/4}$$
$$\ll(AB(m(\mathscr{E}))^{1/4})((m(\mathscr{E})^{1/4}N)NY^{1/2}N^{\delta/2}\ll AB(m(\mathscr{E}))^{1/2}N^{2+\delta/2}Y^{1/2}$$
$$\ll AB(YN^\delta N^{2+\delta}(AB)^{-2}q^{-1})^{1/2}N^{2+\delta/2}Y^{1/2}\ll YN^{3+3\delta/2-1/2}\ll N^{3-\delta}.$$

If $AB\leqslant 2Cq^{1/2}N$, then

$$\int_{M(A,B)} e(\eta\alpha)\Big(\prod_{j=1}^{5} f_j(\alpha)\Big)K_n\Big(\frac{\alpha}{Y}\Big)d\alpha$$
$$\ll\Big(\int_{M(A,B)}|f_1f_2|^4d\alpha\Big)^{1/4}\Big(\int_1^{YN^\delta}|f_3|d\alpha\Big)^{1/4}\Big(\int_1^{NY^\delta}|f_4|^4d\alpha\Big)^{1/4}\Big(\int_1^{YN^\delta}|f_5|^4d\alpha\Big)^{1/4}$$
$$\ll(AB(m(\mathscr{E}))^{1/4})(YN^\delta N^2)^{3/4}$$
$$\ll AB(YN^\delta N^{4+\delta}(AB)^{-4}+Y^{1/4}N^{\delta/4}q^{3/4}N^{1/2+\delta}(AB)^{-2})^{1/4}Y^{3/4}N^{3/2+3\delta/4}$$
$$\ll YN^{5/2+5\delta/4}+(AB)^{1/2}Y^{1/16+3/4}q^{3/16}N^{13/8+5\delta/4}$$
$$\ll N^{3-\delta}+Y^{13/16}q^{7/16}N^{17/8+5\delta/4}$$
$$\ll N^{3-\delta}+N^{95/32+5\delta/4-13\varepsilon/16}\ll N^{3-\delta}.$$

Hence

$$I_3=o(N^3). \tag{5}$$

From (2), (3), (4), (5) we have

$$R(N)\gg Y^{-1}N^3.$$

The theorem follows immediately.

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