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RELAXATION AND EXISTENCE IN NONISOTROPICAL PLASTICITY PROBLEMS

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Abstract

Giving a nonisotropic plasticity problem on a Sobolev Space, the author tries to relax it to a new problem which is posed on a subset of BD(Ω). It is proved that this relaxation does not change the equilibrium energy and that the solutions exist in such settings under not too restricted conditions.

§ 1. Introduction

The theory of plasticity is attracting more and more people's attention. In the study of fracture, dislocation and buckling, a lot of efforts are made in considering plastic behavior of the material.

In the later 70's and ealier 80's, many mathematicians concentrated on making and solving a reasonable mathematical model for isotropic, homogeneous plasticity problem which obeys the Von Mises yielding criterion. The reason is that if a material obeys this criterion, its deformations are only permanent elastic with respect to hydrostatic pressures, while for other directions, it will be plastified when the force is sufficiently strong. This theory has some relations with the Navier-Stokes equations. It is proved that the displacement solutions for these problems lies in the space

$$U(\Omega)=\{u\in L^1(\Omega)^3,\ e(u)=(\nabla u+\nabla u^T)/2\in M(\Omega;E),\ \operatorname{div} u\in L^2(\Omega)\}$$

which is nonlocal in the sense that if $\varphi\in C^1(\overline{\Omega})$, $u\in U(\Omega)$, u is not sure to be an element of $U(\Omega)$ due to the effect of $\operatorname{div}\varphi u$. But the study of Navier-Stokes equations has accumulated a lot of experience in dealing with the divergence operator. We can avoid the difficulty and get appropriate approximation results.

When we deal with nonisotropic plasticity, some materials behave in an irregular way. For example the reinforced concrete, its yielding criterion makes the admissible elastic stress set a general unbounded convex set and we have to face the displacement function space

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$$U^{\psi}(\Omega)=\{u\in L^1(\Omega)^3,\ e(u)\in M(\Omega) \text{ such that } \psi(e(u))\in M(\Omega)\}.$$

In this case, we have no more desired approximation results. This would cause a lot of trouble in computation because the discretised problems are often posed on a subset of Sobolev spaces. If we can not prove, by putting the problem on Sobolev spaces and BD(Ω) as what we should do in proving the relaxation, that the equilibrium energy remains invariant, we shall probably meet the so called Lavrentiev phenomenon (cf. [1]), which infer that the finite element method or finite difference method is difficult to apply.

This paper, however, gives an approximation result for the latter case when Ω is star-shaped. We prove that under not too restrict boundary conditiont, the two problems

$$\text{P;}\ \operatorname*{Inf}_{\substack{u\in W^{1,p}\\ u|_{\Gamma}=u_0}}\left\{\int_\Omega \psi(eu))dx-\int_\Omega f(x)u(x)dx\right\},$$

$$\text{Q:}\ \operatorname*{Inf}_{u\in BD(\Omega)}\left\{\int_\Omega \psi(e(u))dx+\int_{\partial\Omega}\psi_\infty(\mathscr{T}(u_0-u))d\Gamma-\int_\Omega f(x)u(x)dx\right\}$$

have the same infinimum where Problem P is on Sobolev space and respects the boundary condition while Q neither is on Sobolev spaces nor respects the boundary conditions totally, where $BD(\Omega)=\{u\in L^1(\Omega)^3,\ e(u)\in M(\Omega;E)\}$. It is proved also that under limit charge hypothesis, Q admits at least one solution in $U_\psi(\Omega)=\{u\in BD(\Omega),\ {}_\Omega\psi(e(u))<+\infty\}$.

§ 2. Notations

We use some standard convex analysis notations to write down our problem. In the theory of quasi-static plasticity that we are interested in, it is always supposed that there exists a potential of energy $\psi(e(u))$ such that

$$\sigma(x)\in\partial\psi(e(u))(x),\ \forall \text{a. e. } x\in\Omega. \tag{2.1}$$

Here we suppose that the mechanical object is occupying an open star-shaped C^2 region in R^3 (We suppose that it is star-shaped with respect to the origin.) and, of course, it is supposed to be bounded. u: $\Omega\to R^3$ is the displacement function. $e(u)=(\nabla u+\nabla u^T)/2$ is the symmetric part of ∇u and is called the strain tensor of u. $\sigma(x)$ represents the Caucey stress tensor. ∂ is the sub-differential notation: let ψ: $X\to R$ be a convev function where X is a reflexive Banach space, then for any $x\in X$, $\sigma\in X'\cap\partial\psi(x)$ if and only if

$$\psi(y)\geqslant\psi(x)+\langle\sigma,\ y-x\rangle_{X',X}.$$

If we denote by ψ^* the Legendre-Lagrange transform of ψ:

$$\psi^*(\sigma)=\operatorname*{Sup}_{x\in X}\{\langle\sigma,\ x\rangle-\psi(x)\},\ \forall\sigma\in X',$$

then (2.1) is equivalent to saying that

$$e(u)\in-\partial\psi^*(-\sigma) \text{ a. e. } x\in\Omega. \tag{2.2}$$

We suppose that ψ^* satisfies

$$\psi^*(\eta)>C|\eta|^{p'},\ p'>1,\ \forall\eta\in E=R^{3\times3}_{sym}, \psi^*(0)=0; \tag{2.3}$$

dom $\psi^*=\{\eta\in E,\ \psi^*(\eta)<+\infty\}$ contains a nonvoid ball in E with center 0, radius r and is a closed set; (2.4)

$$\psi^*(\eta)\leqslant\widetilde{C}|\eta|^{p'} \text{ on dom } \psi^*; \tag{2.5}$$

ψ^* is a strict convex function on dom ψ^* (2.6)

We denote further by $f\in L^3(\Omega)^3$ the density of body force, by $u_0\in W^{1,p}(\Gamma)^3$ the displacement boundary condition. Then the problem can be posed as follows:

Find u, σ in an appropriate function space such that

$$\begin{cases} e(u)\in-\partial\psi^*(-\sigma) & \text{in } \Omega,\\ \operatorname{div}\sigma+f=0 & \text{in } \Omega,\\ u|_\Gamma=u_0 & \text{on } \Gamma,\end{cases} \tag{2.7}$$

with $\Gamma=\partial\Omega$.

It is well known that, in genearl, the sub-differential of a convex function is not a single valued mapping. Therefore (2.7) can hardly get a satisfied interpretation in form of equations. So the mathematical explaination that we try to find is a variational one. By the standard arguments employed in convex analysis and variational analysis, it follows that (2.7) is equivalent to the following two minimization problems:

$$P:\ \operatorname*{Inf}_{\substack{u\in W^{1,p}\\ u|_{\Gamma_0}=u_0}}\left\{\int_{(\Omega)^3}\psi(e(u))dx-\int_\Omega f(x)u(x)dx\right\}, \tag{2.8}$$

$$P^*\ \operatorname*{Sup}_{\substack{\sigma\in L^{p'}(\Omega,E)\\ \operatorname{div}\sigma+f=0}}\left\{-\int_\Omega\psi^*(-\sigma)dx+\int_{\Gamma_0}(\sigma.\,n)u_0d\Gamma\right\} \tag{2.9}$$

where $p>6$, $p'>1$ such that $1/p+1/p'=1$ ψ is the Legendre-Lagrange transform of ψ^*:

$$\psi(\xi)=\operatorname*{Sup}_{\eta\in E}\{\xi:\eta-\psi^*(\eta)\},\forall\xi\in E, \tag{2.10}$$

where $\xi:\eta=\xi_{ij}\eta_{ij}$, the convention of summation over repeated indices are supposed here and in the following. By direct estimation, we get

$$C(|\xi|-1)\leqslant\psi(\xi)\leqslant c(|\xi|^p+1),$$
$$\psi(0)=0;\ \psi(\xi)>0, \text{ for } \xi\neq0. \tag{2.11}$$

It follows from Ekland-Temam[3] that

$$\operatorname{Inf} P=\operatorname{Sup} P^*. \tag{2.12}$$

Finally, I would like to say one word more about dom ψ^*. If the stress soution σ satisfies, for any $x\in\Omega$, $\sigma(x)\in\operatorname{Int}$ dom ψ^* then $\sigma(x)$ causes only elastic deformation, while if $\sigma(x)\in\partial$ dom ψ^* then on the point x plastic (irreversible) deformations happen.

§ 3. Limit Analysis Hypothesis

In the previous section, we transformed the mechanical problem to a pair of minimization problems. Due to the fact that ψ^* is given in an abstract form, we can not get more informations on ψ but (2.11). However, (2.11) can not in general guarantee that Inf $P \neq \infty$. This can also be explained in the dual form: if f is suchthat there exists no $\sigma \in L^{p'}(\Omega; E)$, div $\sigma + f = 0$ such that $\psi^*(-\sigma(x)) \in L^1(\Omega)$, then Sup $P^* = -\infty$ by definition. The mechanical explaination for this phenomenon is that the force f is so strong that it destroys the structure of the body considered.

Evidently we need criteria to characterize the above mentioned phenomenon. This leads to the idea of introdncing the so called limit analysis problems.

Definitions 3.1. *The limit analysis problems of P are defined as*

$$PLA: \inf_{\substack{u \in W^{1,p}(\Omega)^3 \\ u|_\Gamma = 0 \\ \int_\Omega f(x)u(x)dx = 1}} \left\{ \int_\Omega \psi_\infty(e(u))dx \right\} \tag{3.1}$$

where $\psi_\infty(\xi) = \lim_{t \to \infty} \psi(t\xi)/t, \forall \xi \in E$, and

$$PLA^*: \sup_{\substack{\sigma \in L^{p'}(\Omega, \mathrm{dom}\,\psi^*) \\ \mathrm{div}\sigma + \lambda f = 0}} \{\lambda\}. \tag{3.2}$$

Remark 3.2. If $\psi(\xi) = |\xi|^p$ for some $p > 1$, it is easy to see that $\psi_\infty(\xi) = +\infty$ for any $\xi \neq 0$ which implies Inf $PLA = +\infty$. From the following proposition, it is easy to see that the nature of plasticity problems determines that we can not have the form $\psi(\xi) = |\xi|^p$ because, when the force increases in certain directions, the object is expected to be destroyed, that is to say, for some force density f, we should have Inf $P = -\infty$, or it is equivalent to say that for some ξ we have $\psi_\infty(\xi) < +\infty$, otherwise, it is easy to verify that Sup $PLA^* = +\infty$.

Remark 3.3. It is easy to verify that for any $\alpha \leqslant 0$, $\psi_\infty(\alpha\xi) = \alpha\psi_\infty(\xi)$.

Remark 3.4. In general we have (cf. Ekland-Temam[3])

$$+\infty > \text{Inf PLA} \geqslant \text{Sup PLA}^* \geqslant -\infty. \tag{3.3}$$

Proposition 3.5. *The infimum of Problem P is finite if and only if* Inf PLA^* $\geqslant 1$. *In this case, Problem P^* admits a unique solution σ.*

Proof When $\lambda = $ Sup $PLA^* > 1$, there exists a σ which is PLA^* admissible, e. g., $\sigma \in L^{p'}(\Omega;$ dom $\psi^*)$, div $\sigma + \lambda f = 0$. These imply that $\sigma/\lambda \in L^{p'}(\Omega;$ dom $\psi^*)$, div $\sigma/\lambda + f = 0$ so that

$$-\int_{\Omega}\psi^*(-\sigma/\lambda)dx+\lambda^{-1}\int_{\Gamma_0}(\sigma.\,n)u_0 d\Gamma \qquad (3.4)$$

exists and is finite, which implies that Inf $P=$ Sup $P^*>-\infty$. That the prsblem P^* admits a unique solution follows from the classical convex analysis theory.

Proposition 3.6. *If* Sup $PLA^*>1$, *any minimizing sequence of* P *is bounded in* $LD(\Omega)=\{u\in L^1(\Omega),\ e(u)\in L^1(\Omega;\ E)\}$.

Proof We take the following as notation at first:

$$P_{\bar\lambda}:\ \operatorname*{Inf}_{\substack{u\in W^{1,p(\Omega)}\\ u|_\Gamma=u_0}}\int_\Omega \psi(u))dx-\bar\lambda\int_\Omega f(x)u(x)dx,\Big\},\ \forall 0\leqslant\bar\lambda\leqslant\lambda.$$

As Sup $P^*=\lambda>1$, Inf $P_\lambda+C$ is a finite constant. Let u_m be a minimizing sequence for P. We have then

$$\int_\Omega\psi(e(u_m))dx-\int_\Omega f(x)u_m(x)dx=\frac{1}{\lambda}\Big[\Big[\int_\Omega\psi(e(u_m))dx-\lambda\int_\Omega f(x)u_m(x)dx$$

$$+(1-1/\lambda)\int_\Omega\psi(e(u_m))dx\geqslant(1/\lambda)\operatorname{Inf}P_\lambda+(1-1/\lambda)\int_\Omega\psi(e(u_m))dx$$

$$=C_1\int_\Omega\psi(e(u_m))dx+C_2.$$

Thus the conclusion is clear.

Definition 3.7. *The limit analysis hypithesis of* P *is given as*

$$\text{Sup } PLA^*>1. \qquad (3.5)$$

Remark 3.8. Contrary to [8], we do not give the limit analysis hypothesis as Inf $PLA>1$ because we have not proved the fact that Inf $PLA=$ Sup PLA^* which the auther believes to be true for a big class of potentials ψ.

§ 4. Relaxation of Problem P

In the following, we always suppose that Sup $PLA^*>1$. As is well known now that the solutions to plasticity problems are not in usual Sobolev spaces. So before solving the problem of existence, we have to put the problem on an appropriate function set and prove that all the related conditions are properly posed.

Following the general idea in studying variational problems, we put the problem on the function set $U_\psi(\Omega)$:

Definition 4.1.

$$U_\psi(\Omega)=\Big\{u\in L^1(\Omega)^3,\ e(u)\in M(\Omega;\ E),\int_\Omega\psi(e(u))<\infty\Big\}, \qquad (4.1)$$

where M denotes the space of bounded Radon measures.

Remark 4.2. The definition of $\psi(e(u))$ when $e(u)\in M(\Omega;\ E)$ is as gollows: for any $\varphi\in C_0(\Omega)$, $\varphi(x)\geqslant 0$, we have

$$\langle\psi(e(u)),\ \varphi\rangle=\sup_{\sigma\in C_0^0(\Omega,E)}\int_\Omega[\sigma:e(u)-\psi^*(\sigma)]\varphi. \tag{4.2}$$

The details can be found in [2].

Remark 4.3. In the practice, if there exists a convex function ψ_0 which is convenient for the discussion andthere exits C, $\tilde{C}>0$ such that

$$C(\psi_0(\xi)-1)\leqslant\psi(\xi)<C(\psi_0(\xi)+1), \tag{4.3}$$

then we can work in $U_{\psi_0}(\Omega)$.

Remark 4.4. If we note $BD(\Omega)=\{u\in L^1(\Omega)^3,\ e(u)\in M(\Omega;\ E)\}$ with its natural norm

$$\|u\|_{BD(\Omega)}=|u|_{L^1(\Omega)^3}+|e(u)|_{M(\Omega,E)}, \tag{4.4}$$

then BD is a Banach space and $U_\psi(\Omega)$ is a nonempty convex subset of $BD(\Omega)$. Also if $\{u_n\}\subset U_\psi(\Omega)$, $u_n\to u$ in $L^1(\Omega)^3$, $e(u_n)\overset{*}{\to}e(u)$ in $M(\Omega;\ E)$ weak-star, $|\psi(e(u_n))|_{M(\Omega)}\leqslant$conct., then $u\in U_\psi(\Omega)$ and by (4.2) we have

$$\int_\Omega\psi(e(u))\leqslant\varliminf_{u\to\infty}\int_\Omega\psi(e(u_n)). \tag{4.5}$$

We mention here that the study of $BD(\Omega)$ has been done in [6]and [8]. It is well known also that in solving plasticity problems, the boundary conditions are not respected in general, this seems to be the case here too. The central point is to introduce the generalized problem and prove that the new problem achives the same energy level as the original one.

First, we give a relaxation of the problem P:

$$PR:\quad \operatorname*{Inf}_{u\in W^{1,p}(\Omega)^3}\left\{\int_\Omega\psi(e(u))dx+\int_{\partial\Omega}\psi_\infty(\mathscr{T}(v_0-u))d\Gamma-\int_\Omega f(x)u(x)dx\right\}, \tag{4.6}$$

where $\mathscr{T}_{ij}(p)=(p_i n_j+p_j n_i)/2$ for all $p\in R^3$, n denotes the outward unit normal of $\partial\Omega$. By dual calculation, we can show (cf. [9]) that PR^* and P^* are exactly the same. So

$$\operatorname{Inf} PR=\operatorname{Inf} P. \tag{4.7}$$

Remark 4.5. PR does not mean that the solution does not respect the boundary conditions totally. As a matter of fact, on all directions ξ where $\psi_\infty(\xi)=\infty$, we have $\mathscr{T}(u_0-u):\xi=0$.

The first step of relaxation goes well. The second step is to put the problem on right function set: let

$$Q:\ \operatorname*{Inf}_{u\in U_\chi(\Omega)}\left\{\int_\Omega\psi(e(u))+\int_{\partial\Omega}\psi_\infty(\mathscr{T}(u_0-u))d\Gamma-\int_\Omega f(x)u(x)dx\right\}. \tag{4.8}$$

Now we hope to prove that

$$\operatorname{Inf} P=\operatorname{Inf} Q, \tag{4.9}$$

which is much less evident. We specify again the data f and u_0: let $f\in L^\infty(\Omega)$, $u_0\in W^{1,p}(\Gamma)^3$. As Γ is the C^2 boundary of a star-shapedregion, there exists $s_0>0$ such that for any t: $|t-1|<s_0$, the region $R_{s_0}=\{x:\ x\in t^{-1}\Gamma,\ |t-1|<s_0\}$ is a ring-like

region such that for any $y\in R_{\varepsilon_0}$, there exists a unique $x\in\Gamma$ and t: $|t-1|<\varepsilon_0$ such that $y=x/t$. We define U_0: $R_{\varepsilon_0}\to R^3$ such that

$$U_0(y)=u_0(x), \tag{4.10}$$

where y and x are linked by the relation $y=x/t$, $x\in\Gamma$. Then it isclear that $U_0\in W^{1,p}(R_{\varepsilon_0})$.

We define $\Omega_\varepsilon=\Omega/(1+\varepsilon)=\{y\in R^3,\ y=x/(1+\varepsilon)$ for some $x\in\Omega\}$ and

$$P_\varepsilon:\ \operatorname*{Inf}_{\substack{u\in W^{1,p}(\Omega)^3\\ u|\Omega\backslash\Omega_{\varepsilon_0}=U_0}}\left\{\int_\Omega \psi(e(u))dx-\int_\Omega f(x)u(x)dx\right\}. \tag{4.11}$$

It is olear that

$$\operatorname{Inf} P_\varepsilon\geqslant \operatorname{Inf} PR\geqslant \operatorname{Inf} Q. \tag{4.12}$$

So to prove (4.9), it is enough to prove that $\lim\limits_{\varepsilon\to 0}\operatorname{Inf} P_\varepsilon=\operatorname{Inf} Q$. Let u_δ be a Q admissible function satisfying

$$\int_\Omega \psi(e(u_\delta))+\int_{\partial\Omega}\psi_\infty(\mathscr{T}(u_0-u))d\Gamma-\int_\Omega f(x)u_\delta(x)dx<\operatorname{Inf} Q+\delta. \tag{4.13}$$

We extend u_δ to $\tilde u$ such that

$$\tilde u=\begin{cases}u_\delta, & x\in\Omega,\\ U_0, & x\in\Omega_{-\varepsilon_0}\backslash\Omega,\end{cases} \tag{4.14}$$

and define $\lambda(\varepsilon',\varepsilon)=\operatorname{dist}(\partial\Omega_{\varepsilon'},\partial\Omega_\varepsilon)$, $0<1(\varepsilon)<\min\{\lambda(i\varepsilon/3,\ (i+1)\varepsilon/3);\ i=-3,-2,\cdots,2\}$. Let θ_1 be the standard mollifying sequence. We define

$$V_{\varepsilon 1}=\theta_{1(-\varepsilon)}*\tilde u(((1-\varepsilon/(1+\varepsilon))x). \tag{4.15}$$

It is clear that $V_{\varepsilon 1}$ is not P_ε admissible. We have to make a truncation near the boundary: let φ_ε be a truncation function such that

$$\begin{cases}\varphi_\varepsilon(x)\in C^1(R^3),\\ 0\leqslant\varphi_\varepsilon(x)\leqslant 1,\\ \varphi_\varepsilon(x)=0 \text{ in } \Omega_{\varepsilon/3},\\ \varphi_\varepsilon(x)=1 \text{ in } R^3/\Omega_{2\varepsilon/3},\\ \varphi_\varepsilon(x)=\text{constant on each } \partial\Omega_\delta \text{ when } \varepsilon/3\leqslant\delta\leqslant 2\varepsilon/3.\end{cases} \tag{4.16}$$

Let

$$u_{\varepsilon 1}=(1-\varphi_\varepsilon)V_{\varepsilon 1}+\varphi_\varepsilon U_0, \tag{4.17}$$

which is $P_{2\varepsilon/3}$ admissible independent of $1(\varepsilon)$ by our choice. Therefore, for any $0<s<1$,

$$\begin{aligned}\operatorname{Inf} P_{2\varepsilon/3}\leqslant&\int_\Omega[s(1-\varphi_\varepsilon)\psi(e(V_{\varepsilon 1}\nu s)+s\varphi_\varepsilon\psi(e(U_0)/s)]dx\\ &+(1-t)\int_\Omega\psi(\nabla\varphi_\varepsilon\otimes(U_0-V_{\varepsilon 1})_s/(1-s))dx+\int_\Omega f(x)u_{\varepsilon 1}(x)dx.\end{aligned} \tag{4.18}$$

Then as $1\to 0$ first, from the generalized Jensen's inequality (cf. [7])and by an an alysis of the support of φ_ε, $\nabla\varphi_\varepsilon$ and the definition of U_0 (cf. (4.10)), we get

$$\lim_{1\to 0}\int_\Omega\psi(\nabla\varphi_\varepsilon\otimes(U_0-V_{\varepsilon 1})_s/(1-s))\leqslant\int_\Omega\psi(\nabla\varphi_\varepsilon\otimes(U_0-U_0(((1-\varepsilon)/(1+\varepsilon))x))_s/(1-s)=$$

$$\lim_{1\to 0}\int_{\Omega} s(1-\varphi_\varepsilon)\psi(e(V_{\varepsilon 1})/s\leqslant\int_{\Omega} s(1-\varphi_\varepsilon)\psi(e(\tilde{u})(((1-\varepsilon)/x1+\varepsilon))x)/s)$$

$$=\int_{\Omega_\varepsilon} s(1-\varphi_\varepsilon)\psi(e(u)(((1-\varepsilon)/(1+\varepsilon))x)/s)$$

$$+\int_{\partial\Omega_\varepsilon} s(1-\varphi_\varepsilon)\psi_\infty(\mathscr{T}(U_0-u)(((1-\varepsilon)/(1+\varepsilon))x)/s)d\Gamma$$

$$+\int_{\Omega/\tilde{\Omega}_\varepsilon} s(1-\varphi_\varepsilon)\psi(e(U_0)(((1-\varepsilon)/(1+\varepsilon))x)/s)dx,$$

$$\lim_{1\to 0}\int_{\Omega} f(x)u_{\varepsilon 1}(x)dx=\int_{\Omega} f(x)[(1-\varphi_\varepsilon)V_\varepsilon+\varphi_\varepsilon U_0]dx.$$

where $V_\varepsilon=\tilde{U}(((1-\varepsilon)/(1+\varepsilon))x)$.

Thus it follows from (4.18) that

$$\text{Inf } P_{2\varepsilon/3}\leqslant\int_{\Omega_\varepsilon} s(1-\varphi_\varepsilon)\psi(e(u)(((1-\varepsilon)/(1+\varepsilon))x)/s)$$

$$+\int_{\partial\Omega_\varepsilon} s(1-\varphi_\varepsilon)\psi(\mathscr{T}(U_0-u)(((1-\varepsilon)/(1+\varepsilon))x/s)d\Gamma$$

$$+\int_{\Omega/\tilde{\Omega}_\varepsilon} s(1-\varphi_\varepsilon)\psi(e(U_0)(((1-\varepsilon)/(1+\varepsilon))x))dx$$

$$-\int_{\Omega} f(x)((1-\varphi_\varepsilon)V_\varepsilon+\varphi_\varepsilon U_0)dx.$$

By letting $\varepsilon\to 0$, it is easy to see that

$$\lim_{\varepsilon\to 0}\text{Inf } P_{2\varepsilon/3}\leqslant\int_{\Omega} s\psi(e(u)(x)/s)+\int_{\partial\Omega} s\psi_\infty(\mathscr{T}(U_0-u)(x)/s)d\Gamma-\int_{\Omega} f(x)u(x).$$

Then tending s to 1 we get

$$\lim_{\varepsilon\to 0}\text{Inf } P_{2\varepsilon/3}<\text{Inf } Q+\delta, \tag{4.19}$$

which gives us (4.9).

§ 5. Existence of Solutions

From our arguments given in § 4, we know now that it is reasonable to investigate Problem Q when we deal with the existence of solutions. If u is a solution to Problem Q, it is called a generalized solution of P.

Theorem 5.1. *Under all the hypotheses made before and*

$$\text{Sup } PLA^*>1,$$

Problem Q admits at least one solution.

Proof Let u_m be a sequence of Q admissible functions such that

$$\lim_{m\to\infty}\int_{\Omega}\psi(e(u_m))+\int_{\partial\Omega}\psi_\infty(\mathscr{T}(U_0-u_m))d\Gamma-\int_{\Omega} f(x)u_m(x)dx=\text{Inf } Q.$$

Then we know that

$$\{u_m\} \text{ is bounded in } BD(\Omega).$$

Up to choosing a subsequence, we may suppose that $u_m\overset{*}{\rightharpoonup}u_0$ in $BD(\Omega)$. Then $u_m(x)\to$

$u_0(x)$ in $L^{2/3}(\Omega)$, which implies $\int_\Omega f(x)u_m(x)dx \to \int_\Omega f(x)u_0(x)dx$. We define

$$\bar{u}_m(x)=\begin{cases}u_m(x), & x\in\Omega,\\ U_0(x), & x\in\Omega_{-\varepsilon_0}/\Omega.\end{cases}$$

Then $\{\bar{u}_m(x)\}$ is bounded in $BD(\Omega_{\varepsilon_0})$ and it is easy to verify that

$$\bar{u}_m(x)\rightharpoonup\bar{u}_0(x) \text{ in } BD(\Omega) \text{ weak-star},$$

where $\bar{u}_0=u_0(x)$ for $x\in\Omega$, $=U_0(x)$ for $x\in\Omega_{-\varepsilon_0}/\Omega$. By Remark 4.4, we have

$$\lim_{m\to\infty}\int_{\Omega_{-\varepsilon_0}}\psi(e(\bar{u}_m))\geqslant\int_{\Omega_{-\varepsilon_0}}\psi(e(\bar{u}_0)).$$

Thus

$$\lim_{m\to\infty}\int_{\Omega_{-\varepsilon}}(e(\bar{u}_m))=\lim_{m\to\infty}\left\{\int_\Omega\psi(e(u_m))+\int_\Gamma\psi_\infty(\mathscr{T}(U_0-u_m))+\int_{\Omega_{-\varepsilon}/\bar{\Omega}}\psi(e(U_0))\right.$$

$$\geqslant\int_\Omega\psi(e(u_0))+\int_\Gamma\psi_\infty(\mathscr{T}(U_0-u_0))+\int_{\Omega_{-\varepsilon}/\bar{\Omega}}\psi(e(U_0)).$$

Thus we have

$$\operatorname{Inf} Q\geqslant\int_\Omega\psi(e(u_0))+\int_\Gamma\psi_\infty(\mathscr{T}(U_0-u_0))-\int_\Omega f(x)u_0(x)dx$$

and the theorem is proved.

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