Chin. Ann. of Math.
12B: 3(1991), 318—334.

LINEAR TRANSPORT EQUATION WITH INDEFINITE COLLISION OPERATOR

Chu Yi (褚 轶)*

Abstract

The operator theory on indefinite inner product spaces is used to discuss the half-range problem of linear transport equation with indefinite collision operator. A counter-example to [1] is given and a relation between measures of nonuniqueness and noncompleteness is established.

§ 0. Introduction

In this paper we consider the following linear transpoort equation with partial range boundary value condition

$$\begin{cases} T\dfrac{df}{dt}+Af=0, \\ Q_+f(0)=f_+, \\ \|f(t)\|=O(1), \quad (t\to+\infty). \end{cases} \tag{0.1}$$

Here the coordinate operator T and collision operator A are selfadjoint operatorsion on a separable complex Hilbert space H. T is assumed to be injective and bounded and Q_+ is the spectral projection of T with respect to $(0, +\infty)$. A is Fredholm with finite-dimensional negative part.

Since R. Beals' pioneering work[1], which in the situation when A is positive definite proved the existence and uniqueness of equation (0, 1), many efforts have been gone to the study of the nonuniqueness of equation (0, 1) when A is not assumed to be positive. [3] considers the case when A is semi-positive and Fredholm. [2] is one of the few papers which drop the positivity condition on the collision operator A. In [2], the measures of nonumiqueness, and noncompleteness are studied under the assumption that A is T-regular. Unfortunately, we find that the main results of [2] are incorrect. A counter-example will be given in § 3.

In this paper, we drop the regularity condition on A, only assuming that the negative part of A is finite-dimensional. Full-range theory is developed and the meesures of nonuniqueness and noncompleteness are studied and a relation of these

Manuscript received June 16, 1989.

* Department of Applied Mathematics, Shanghai Jiao Dong University, Shanghai, China.

two measures is established. As an application, the measures of nonuniqueness and noncompleteness of symmetric multigroup transport equation in isotropic media are fully characterized.

In the case that A is indefinite, it is natural to introduce an indefinite metric structure into the problem. When A is assumed to be T-regular, as is required by [2], the analysis can be done separably on a Hilbert space and on a finite-dimensional space The analysis is much complex in the absence of regularity condition on A and we find that this generalization is by no means trivial.

Note. That a subspace of a Krein space is said to be positive/negative definite means in this paper that it is positive/negative and nondegenerate.

§ 1. Full-Rang Theory

To deal with the nontrivial kernel of A, we first need a lemma. Let us recall our assumptions on the operator pair $\{T, A\}$. T and A are selfadjoint. T is bounded and injective. A is Fredholm with finite-dimensional negative part.

Lemma 1.1. *A is T-regular at zero, i.e., the root space $Z_0(K)$ of $K=T^{-1}A$ at zero is finite-dimnsional and nondegenerate under the indefinite inner product*

$$(\cdot, \cdot)_T=(T\cdot, \cdot). \tag{1.1}$$

The proof of this lemma is elementary and we shall omit it here. Since A is T-regular at zeao we have the following decomposition

$$H=Z_0(K)\dot{+}(TZ_0(K))^{\perp}. \tag{1.2}$$

Since $Z_0(K)\subset D(A)$ and $Z_0(K)$ is finite-dimensional, the space $(TZ_0(K))^{\perp}\cap D(|A|^{1/2})$ is a Π_k space under the indefinite inner product

$$(\cdot, \cdot)_A=(A\cdot, \cdot). \tag{1.3}$$

Let

$$\Pi_A=\{(TZ_0(K))^{\perp}\cap D(|A|^{1/2}), (\cdot, \cdot)_A\}. \tag{1.4}$$

Then $K_1=K|_{\Pi_A}$ is selfadjoint on Π_A. Let E be the spectral resolution[4] of K_1 on the real line.

Lemma 1.2. *The solution of the equation*

$$\begin{cases}\dfrac{df}{dt}+K_1f=0, & f(t)\in\Pi_A,\ 0<t<+\infty,\\ \|f(t)\|_{\Pi_A}=O(1). & (t\to+\infty),\end{cases}$$

is

$$f(t)=\exp(-tK_1)f_0.$$

Here

$$f_0\in E(0, +\infty)+\sum_{\xi\in\sigma_p(K),\mathrm{Re}\xi>0}Z_\xi(K)+\sum_{\lambda\in\sigma_p(K),\mathrm{Re}\lambda=0}\ker(K-\lambda).$$

Proof Let $a, b>0$ and a, b are not critical points[4] of K_1. Clearly, we only need to prove that

$$\|\exp(-tK_1)E(a, b)f_0\|\to 0, \quad t+\to\infty.$$

We assume without loss of generality that

$$\sigma(K_1)\subset[a, b].$$

Let the triangular model[4] of K_1 be

$$K_1=\begin{bmatrix} S & F & G & Q \\ & A_N & & -F^* \\ & & A_P & G^* \\ & & & S^* \end{bmatrix}\begin{matrix} Z \\ N \\ P \\ Z^* \end{matrix}$$

and

$$\Pi_A=N\oplus_A\{Z\times Z^*\}\oplus_A P.$$

Moreover, we choose a space decomposition such that $\exp(-tK_1)$ also has triangular form (cf. [4])

$$\exp(-tK_1)=\begin{bmatrix} S(t) & F(t) & G(t) & Q(t) \\ & A_N(t) & & -F(t)^* \\ & & A_P(t) & G(t)^* \\ & & & S(t)^* \end{bmatrix}.$$

From the relation $\exp(-2tK_1)=(\exp(-tK_1))^2$, it can be calculated that

$$\begin{cases} F(2t)=S(t)F(t)+F(t)A_N(t), \\ G(2t)=S(t)G(t)+G(t)A_P(t), \\ Q(2t)=S(t)Q(t)+Q(t)S^*(t)-F(t)F(t)^*+G(t)G(t)^*. \end{cases} \tag{1.5}$$

It is obvious that $F(t)$, $G(t)$, $Q(t)$, $S(t)$, $A_P(t)$ and $A_N(t)$ are all continuous in t in uniform topology and

$$\begin{cases} S(t)=e^{-tS}, \\ A_P(t)=e^{-tA_N}, \\ A_N(t)=e^{-tA_N}. \end{cases}$$

Since

$$\sigma(S)\cup\sigma(A_P)\cup\sigma(A_N)\subset\sigma(K_1)\subset[a, b],$$

we have

$$\begin{cases} \|e^{-tA_P}\|\leqslant e^{-ta}, \\ \|e^{-tA_N}\|\leqslant e^{-ta}, \\ \|e^{-tS}\|\leqslant e^{-ta/2}. \end{cases}$$

Thus from (1.5) we see that as $t\to+\infty$,

$$\begin{cases} \|F(t)\|\to 0, \\ \|G(t)\|\to 0, \\ \|Q(t)\|\to 0. \end{cases}$$

Therefore

$$\|\exp(-tK_1)E(a, b)f_0\| \to 0, \ t \to +\infty.$$

Hence we see that if

$$f_0 \in E(0, +\infty) + \sum_{\xi \in \sigma_p(K), \mathrm{Re}\xi > 0} Z_\xi(K) + \sum_{\lambda \in \sigma_p(K), \mathrm{Re}\lambda = 0} \ker(K - \lambda),$$

then $f(t) = \exp(-tK_1)f_0$ is bounded as $t \to +\infty$.

On the other hand, if $f(t) = \exp(-tK_1)f_0$ is bounded as $t \to +\infty$, then we see that

$$f_0 \in E(-\infty, +\infty) + \sum_{\in \sigma_p(\xi K), \mathrm{Re}\xi > 0} Z_\xi(K) + \sum_{\lambda \in \sigma_p(K), \mathrm{Re}\lambda = 0} \ker(K - \lambda).$$

We need to preve that $E(0, +\infty)f_0 = E(-\infty, +\infty)f_0$. Let $b > a > 0$ be arbitrary positive real numbers such that a and b are not critical points of K_1. From the above paragraph we see that

$$\|\exp(tK_1)E(-b, -a)\| \to 0 \quad (t \to +\infty).$$

Let

$$g(t) = \exp(tK_1)E(-b, -a)f(t).$$

Then

$$\frac{dg(t)}{dt} = K_1 g(t) - K_1 g(t) = 0.$$

Hence $g(t)$ is constant. Now

$$\|g(t)\| \leqslant \|\exp(tK_1)E(-b, -a)\| \|f(t)\| \to 0, \ t \to +\infty.$$

Therefore $g(t) \equiv 0$, i.e., $E(-b, -a)f_0 = 0$. Thus

$$E(-\infty, +\infty)f_0 = E(0, +\infty)f_0.$$

Now we fix $b > a > 0$ such that

$$C(K_1) \subset (-b, -a) \cup (a, b), \tag{1.6}$$

where $C(K_1)$ is the set of critical points of K_1. Then we have the following lemma.

Lemma 1.3. *$E(a, b)$ is a Π_k space under $(\cdot, \cdot)_T$. The dimension of maximal negative subspaces is the same as that of maximal negative subspaces under $(\cdot, \cdot)_A$. Furthermore, $E(a, b)$ is closed in H and the induced topologies by $(\cdot, \cdot)_A$, $(\cdot, \cdot)_T$ and $(\cdot, \cdot)$ are all equivalent.*

Proof Let

$$W = (K_1|_{E(a,b)})^{-1}. \tag{1.7}$$

Then for $x, y \in E(a, b)$,

$$(x, y)_T = (Wx, y)_A. \tag{1.8}$$

Let W have triangular model[4]

$$W = \{S, W_N, W_P, F, G, Q\}. \tag{1.9}$$

Now

$$\sigma(S) \cup \sigma(W_N) \cup \sigma(W_P) \subset \sigma(W) \subset \left[\frac{1}{b}, \frac{1}{a}\right].$$

The space $E(a, b)$ has the following decomposition

$$E(a, b)=N\oplus_A\{Z\dot{+}Z^*\}\oplus_A P$$
$$=N\oplus_T P\oplus_T\{z+\{z^*+W_N^{-1}F^*z^*-W_P^{-1}G^*z^*\,|\,z^*\in Z^*\}\}. \qquad (1.10)$$

Under $(\cdot, \cdot)_T=(W\cdot, \cdot)_A$, P is positive definite and on P

$$(W\cdot, \cdot)_A=(W_P\cdot, \cdot)_A.$$

Hence on the subspace P, $(\cdot, \cdot)_T$ is equivalent to $(\cdot, \cdot)_A$ since $\sigma(W_P)\subset\left[\frac{1}{b}, \frac{1}{a}\right]$ and $\{P, (\cdot, \cdot)_T\}$ is a Hilbert space.

Similarly, $(\cdot, \cdot)_T$ is equivalent to $(\cdot, \cdot)_A$ on N and $\{N, -(\cdot, \cdot)_T\}$ is a Hilbert space of finite dimension.

Now we shall prove that the third term in (1.10) is nondegenerate under $(\cdot, \cdot)_T$. If

$$x=z+z^*+W_N^{-1}F^*z^*-W_P^{-1}G^*z^*,\ z\in Z,\ z^*\in Z^*,$$

such that $(x, z_1)_T=0$ for any $z_1\in Z$, then

$$0=(Wz_1, z+z^*+W_N^{-1}F^*z^*-W_P^{-1}G^*z^*)_A=(Sz_1, z^*)_A.$$

Therefore $z^*=0$ since S is surjective. Now if x is $(\cdot, \cdot)_T$ orthogonal to $\{z_0^*+W_N^{-1}F^*z_0^*-W_P^{-1}G^*z_0^*\,|\,z_0\in Z^*\}$, then $z=0$. Hence $x=0$. Thus the third term in (1.10) is nondegenerate under $(\cdot, \cdot)_T$. Therefore we have proved that $E(a, b)$ is Π_k space under $(\cdot, \cdot)_T$. Now since the third term in (1.10) has dimension 2dim$Z$ and Z is a null space under $(\cdot, \cdot)_T$, we see that the maximal negative subspace of $E(a, b)$ under $(\cdot, \cdot)_T$ has dimension dim$N$+dim$Z$, which is equal to the dimension of maximal negative subspace of $E(a, b)$ under $(\cdot, \cdot)_A$.

Next we shall prove that $E(a, b)$ is closed in H. Consider the identity map

$$i: \{E(a, b), (\cdot, \cdot)\}\to\{E(a, b), (\cdot, \cdot)_A\}.$$

We shall prove that i is bicontinuous. Let $x_n\in E(a, b)$ and $x_n\to 0$ in the topology induced by $(\cdot, \cdot)_A$. Then for any $y\in\Pi_A$, $(x_n, y)_A\to 0$. We see that for any $z\in(TZ_0(K))^\perp\cap D(A)$, $(x_n\, Az)\to 0$. Now

$$H=(A(D(A)\cap(TZ_0(K))^\perp)+TZ_0(K)$$

and $x_n\perp TZ_0(K)$. Hence $x_n\to 0$ weakly in H. Thenefore i^{-1} is continuous.

Conversely, if $x_n\in E(a, b)$, $x_n\to 0$ in H, then for any $y\in E(a, b)$, $(x_n, y)_T=(x_n, Ty)\to 0$, i.e., $(x_n, W_y)_A\to 0$. Hence $x_n\to 0$ weakly in Π_A since W is invertible in Π_A. Therefore i is continuous. Hence $E(a, b)$ is a closed subspace of H, and the topologies on $E(a, b)$ induced by $(\cdot, \cdot)$ and $(\cdot, \cdot)_A$ are equivalent. The equivalence of topologies on $E(a, b)$ induced by $(\cdot, \cdot)_A$ and $(\cdot, \cdot)_T$ is obvious since their weak topologies are equivalent.

Lemma 1.4. $E(a, b)\subset D(A)$.

Proof $E(a, b)$ reduces K_1 and $K_1|_{E(a,b)}$ is continuous. Hence $E(a, b)\subset D(K_1)\subset D(K)\subset D(A)$.

Now let

$$\{a_1, \bar{a}_1, a_2, \bar{a}_2, \cdots, a_p, \bar{a}_p\}=\{\xi\in\sigma_p(K)\,|\,\mathrm{Im}\xi\neq 0\}$$

and denote

$$X=\sum_{i=1}^{d}\{Z_{\alpha_i}(K)+Z_{\bar{\alpha}_i}(K)\}+E(a,\ b)+E(-b,\ -a). \tag{1.11}$$

We know from Lemma 1.3 and Lemma 1.4 that X is a Krein space under $(\cdot,\ \cdot)_T$ and $X\subset D(A)$. Let η and β be metric operators[4] on the Krein spaces $\{X,\ (\cdot,\ \cdot)_T\}$ and $\{Z_0(T),(\cdot,\ \cdot)_T\}$, respectively. Let P_0 be the $(\cdot,\ \cdot)_A$ orthogonal projection of Π_A onto $\Pi_A\ominus_A X$. Let P be the parallel projection of H onto $(TZ_0(K))^\perp$ along $Z_0(K)$. Then we define an auxiliary operator on H

$$A_{\beta,\eta}=AP_0+T\eta(P-P_0)|_X+T\beta(1-P)|_{Z_0(K)}. \tag{1.12}$$

Lemma 1.5. *$A_{\beta,\eta}$ is a positive operator on H with bounded inverse and*

$$D(A_{\beta,\eta})=D(A).$$

Proof Since $Z_0(K)\subset D(A)$ and $X\subset D(A)$, we see that $D(A_{\beta,\eta})=D(A)$. Clearly $\ker A_{\beta,\eta}=\{0\}$. Now we prove that $A_{\beta,\eta}$ is positive. For $x\in D(A)\cap \mathrm{ran}P_0$, $y\in X$, $z\in Z_0(K)$,

$$(A_{\beta,r}(x+y+z),\ x+y+z)=(Ax+T\eta y+T\beta z,\ x+y+z)$$
$$=(x,\ x)_A+(\eta y,\ y)_T+(\beta z,\ z)_T\geqslant 0$$

(since $(T\eta y,\ x)=(K_1^{-1}\eta y,\ x)_A=0$).

Therefore $A_{\beta,\eta}$ is positive. Next we shall show that ran $A_{\beta,\eta}=H$. From (1.12) we see that

$$A_{\beta,\eta}=A(P_0+K_1^{-1}\eta|_X)+T\beta|_{Z_0(K)}.$$

Hence $\mathrm{ran}A_{\beta,\eta}\supset A(D(A)\cap(TZ_0(K))^\perp)+TZ_0(K)=H$.

Now we denote

$$K_{\beta,\eta}=T^{-1}A_{\beta,\eta} \tag{1.13}$$

and define two auxiliary Hilbert spaces

$$H_{A_{\beta,\eta}}=\{D(A),\ (\cdot,\ \cdot)_{A_{\beta,\eta}}\}^- \tag{1.14}$$

and

$$H_{K_{\beta,\eta}}=\{H_{A_{\beta,\eta}},\ (|K_{\beta,\eta}^{-1}|\cdot,\ \cdot)_{A_{\beta,\eta}}\}^-. \tag{1.15}$$

We see that $K_{\beta,\eta}$ is selfadjoint on $H_{A_{\beta,\eta}}$ Let $Q_\pm$ be the spectral projections of $K_{\beta,\eta}$ with respect to $(0,\ +\infty)/(-\infty,\ 0)$. $Q_\pm$ can be extended to a pair of complementary projections on $H_{K_{\beta,\eta}}$.

Remark. $H_{A_{\beta,\eta}}$ and $H_{K_{\beta,\eta}}$ are irrelevant to different choices of metric operators η and β. They are also irrelevant to the choice of the real numbers a and b in the definition of X. Thus in the sequel, we suppress β and η in $H_{A_{\beta,\eta}}$ and $H_{K_{\beta,\eta}}$.

Lemma 1.6.[8] *There exists a unique bounded injective albedo operator E with dense range such that*

$$E:\ H_T\to H_K\cap H_T,$$

where $H_T=\{H,\ (|T|\cdot,\ \cdot)\}^-$, and

1) $Q_\pm E Q_\pm f = Q_\pm f$,

2) $\hat{P}_\pm E Q_\pm f = 0$.

Here $Q_\pm$ is the spectral projection of T with respect to $(0, +\infty)/(-\infty, 0)$.

Lemma 1.7.[3] *Equation*

$$\begin{cases} T\dfrac{df}{dt} + A_{\beta,\eta} f = 0, \\ Q_+ f(0) = f_+, \\ \|f(t)\| = O(1), \quad t \to +\infty \end{cases}$$

has a unique solution for every $f_+ \in Q_+(H_T)$ such that

$$f: (0, +\infty) \to H_K$$

is differentiable and $f(0) \in H_T \cap H_K$. The solution is

$$f(t) = \exp(-tK_{\beta,\eta}) E f_+.$$

The space $H_0 = \{H_A \ominus_A X, (\cdot, \cdot)_A\}$ is a Hilbert space and K restricted to H_0 is selfadjoint and injective. Let

$$H_{0,K} = \{H_0, (|K^{-1}|\cdot, \cdot)_A\}^-. \tag{1.16}$$

Then we have

Lemma 1.8. *$H_K = H_{0,K} \oplus X \oplus Z_0(K)$. Here on X and $Z_0(K)$, the inner products are $(\eta\cdot, \cdot)_T$ and $(\beta\cdot, \cdot)_T$, respectively. The spectral resolution $E(\cdot)$ can be extended to H_K for every Borel set in $\mathbf{R}\backslash\{(a, b) \cup (-b, -a)\}$ and the extended projections coincide with the spectral resolution of $K|_{H_0,K}$.*

§ 2. Half-Range Theory

Now let us consider the following equation

$$\begin{cases} \dfrac{df_0}{dt} + Kf_0 = 0, \text{ on } Z_0(K), \\ \dfrac{df_1}{dt} + Kf_1 = 0, \quad \text{on } X, \\ \dfrac{df_2}{dt} + Kf_2 = 0, \quad \text{on } H_{0,K}, \\ Q_+ f(0) = f_+ \in Q_+ H_T, \\ \|f(t)\|_{H_K} = O(1) \quad (t \to +\infty). \end{cases} \tag{2.1}$$

Here

$$f: [0, +\infty) \to H_K$$

is continuous with $f(0) \in H_T \cap H_K$ and f is differentiable on $(0, +\infty)$ and

$$f(t) = f_0(t) + f_1(t) + f_2(t),$$

where $f_0(t) \in Z_0(K)$, $f_1(t) \in X$ and $f_2(t) \in H_{0,K}$.

We introduce measures of nonuniqueness δ^+ and noncompleteness γ^+ by the following definition

$$\delta^+ = \dim\{\text{solutions of equation (2.1) with } Q_+f(0)=0\}, \tag{2.2}$$

$$\gamma^+ = \dim Q_+H_T/\{f_+\in Q_+H_T \mid \text{for } f_+, \text{ equation (2.1) has a solution}\}^-. \tag{2.3}$$

Denote

$$M_r = \sum_{\mathrm{Re}a_i=0}(\ker(K-a_i)+\ker(K-\bar{a}_i)) + \sum_{\mathrm{Re}a_i>0}(Z_{a_i}(K)+Z_{\bar{a}_i}(K)) + \ker A + E(a,\ b). \tag{2.4}$$

We have

Lemma 2.1. $\gamma^+ = \dim Q_+H_T/\{Q_+[(M_r+E(0,\ a)+E(b,\ +\infty))\cap H_T]\}^-$ *and* $\delta^+ = \dim\ker\{Q_+: (M_r+E(0,\ a)+E(b,\ +\infty))\cap H_T \to Q_+H_T\}$.

Proof For $f(t)$ to satisfy Equation (2.1), we see that $f_0(0)\in\ker A$, $f_1(0)\in E(a,\ b)+\sum_{\mathrm{Re}\xi=0}(\ker(K-a_i)+\ker(K-\bar{a}_i))+\sum_{\mathrm{Re}a_i>0}(Z_{a_i}(K)+Z_{\bar{a}_i}(K))$ and $f_2(0)\in E(0,\ a)+E(b,\ +\infty)$.

Hence $f(0)\in M_r+E(0,\ a)+E(b,\ +\infty)$.

Therefore our lemma follows immediately.

For $\lambda\in\sigma_p(K)$ and $\mathrm{Re}\lambda=0$, $\lambda\neq 0$, we have the following decomposition

$$\begin{cases}\ker(K-\lambda)=k_\lambda(K)\dot{+}k_\lambda(K)',\\ Z_\lambda(K)=\ker(K-\lambda)\dot{+}Z_\lambda(K)'+Z_\lambda(K)''\end{cases} \tag{2.5}$$

such that $\{k_\lambda(K),\ k_{\bar{\lambda}}(K)\}$, $\{k_\lambda(K)',\ Z_{\bar{\lambda}}(K)'\}$ and $\{Z_\lambda(K)'',\ Z_{\bar{\lambda}}(K)''\}$ are Hilbert pairs[4] under $(\cdot,\ \cdot)_T$ and these pairs are mutually orthogonal. Moreover,

$$(K-\lambda)Z_\lambda(K)=Z_\lambda(K)''+k_\lambda(K)'. \tag{2.6}$$

Similarly we have

$$\begin{cases}\ker A=k_0(K)\dot{+}k_0(K)',\\ Z_0(K)=\ker K\dot{+}Z_0(K)'+Z_0(K)''\end{cases} \tag{2.7}$$

such that $\{k_0(K)',\ Z_0(K)'\}$ is a Hilbert pair under $(\cdot,\ \cdot)_T$ and

$$Z_0(K)=k_0(K)\oplus_T\{k_0(K)'+Z_0(K)'\}\oplus_T Z_0(K)''. \tag{2.8}$$

Moreover, $KZ_0(K)=k_0(K)'+Z_0(K)''$.

Denote

$$M_s=\sum_{\lambda\in\sigma_p(K),\,\mathrm{Re}\lambda=0}(Z_\lambda(K)'+Z_\lambda(K)'')+\sum_{\xi\in\sigma_p(K),\,\mathrm{Re}\xi<0}Z_\xi(K)+E(-b,\ -a) \tag{2.9}$$

and

$$M=M_r+M_s=\sum_{\lambda\in\sigma_p(K),\,\mathrm{Im}\lambda\neq 0}Z_\lambda(K)+E(a,\ b)+E(-b,\ -a). \tag{2.10}$$

Now

$$M_r=(M_r)_0+(M_r)_n, \tag{2.11}$$

where

$$(M_r)_0=\sum_{\lambda\in\sigma_p(K),\,\mathrm{Re}\lambda=0}k_\lambda(K)' \tag{2.12}$$

and

$$(M_r)_n=\sum_{\lambda\in\sigma_p(K),\,\mathrm{Re}\lambda=0}k_\lambda(K)+\sum_{\xi\in\sigma_p(K),\,\mathrm{Re}\xi>0}Z_\xi(K)+E(a,\ b), \tag{2.13}$$

and

$$M_s=(M_s)_0+(M_s)_n,$$

where

$$(M_s)_0=\sum_{\lambda\in\sigma_p(K),\mathrm{Re}\lambda=0} Z_\lambda(K)', \tag{2.15}$$

$$(M_s)_n=\sum_{\lambda\in\sigma_p(K),\mathrm{Re}\lambda=0} Z_\lambda(K)''+\sum_{\xi\in\sigma_p(K),\mathrm{Re}\xi<0} Z_\xi(K)+E(-b,\ -a). \tag{2.16}$$

We see that $(M_r)_0$ and $(M_s)_0$ are neutral under $(\cdot,\ \cdot)_T$ and $\{(M_r)_0,\ (M_s)_0\}$ is a Hilbert pair. Moreover, M is a Krein space under $(\cdot,\ \cdot)_T$ and

$$M=(M_r)_n\oplus_T(M_s)_n\oplus_T\{(M_r)_0+(M_s)_0\}=X\oplus_T Z_0(K). \tag{2.17}$$

Now choose metric operator $\theta=\eta+\beta$ such that

$$\begin{cases}\theta((M_r)_0+(M_s)_0)=(M_r)_0+(M_s)_0,\\ \theta((M_r)_n)=(M_r)_n,\\ \theta((M_s)_n)=(M_s)_n,\end{cases} \tag{2.18}$$

and the corresponding canonical decompositions[4] are

$$\begin{cases}(M_r)_0+(M_s)_0=((M(_{r0}+(M_s)_0)_-\oplus_T((M_r)_0+(M_s)_0)_+,\\ (M_r)_n=(M_r)_-\otimes_T(M_r)_+,\\ (M_s)_n=(M_n)_-\oplus_T(M_s)_+.\end{cases} \tag{2.19}$$

From (1.13), we see that

$$\mathrm{ram}\hat{P}_+=E(0,\ a)+E(b,\ +\infty)+((M_r)_0+(M_s)_0)_++(M_r)_++(M_s)_+ \tag{2.20}$$

and from Lemma 1.6 we see that

$$Q_+:\ \mathrm{ran}\hat{P}_+\cap H_T(\subset H_K\cap H_T)\to Q_+H_T$$

is bijective and continuous. We also see that ran $\hat{P}_+\cap H_T=EQ_+(H_T)$. Thus $Q_+[(E(0,a)+E(b,\ +\infty))\cap H_T]$ is closed in Q_+H_T. Denote $F=Q+[(E(0,\ a)+E(b,\ +\infty))\cap H_T]$. We have

$$\begin{aligned}\gamma^+&=\dim Q_+H_T/\{Q_+[(M_r+E(0,\ a)+E(b,\ +\infty))\cap H_T]\}^-\\ &=\dim Q_+H_T/[Q_+M_r+F]\\ &=\dim Q_+H_T/[Q_+(M_r)_++F]-\dim\frac{Q_+M_r+F}{Q_+(M_r)+F}\end{aligned}$$

Now the first term can be calculated as the following

$$\begin{aligned}\dim\frac{Q_+H_T}{Q_+(M_r)_++F}&=\dim\frac{Q_+(\mathrm{ran}\hat{P}_+\cap H_T)}{Q_+(M_r)_++F}\\ &=\dim(M_s)_++\dim(M_r)_0+(M_s)_0)_+,\end{aligned}$$

and the second term is

$$\begin{aligned}\dim\frac{Q_+M_r+F}{Q_+(M_r)_++F}&=\dim\frac{Q_+(M_r)_-+Q_+((M_r)_0)+Q_+(M_r)_++F}{R_+(M_r)_++F}\\ &=\dim(M_r)_-+\dim(M_r)_0-\delta^+.\end{aligned}$$

Thus

$$\begin{aligned}\gamma^+&=\dim(M_s)_++\dim((M_r)_0+(M_s)_0)_+-\dim(M_r)_--\dim(M_r)_0+\delta^+\\ &=\dim(M_s)_+-\dim(M_r)_-+\delta^+.\end{aligned}$$

Therefore we have

Lemma 2.2. $\dim(M_s)_+ - \gamma^+ = \dim(M_r)_- - \delta^+.$

Now denote

$$\begin{cases} M_R = M_r, \\ M_L = \sum_{\lambda\in\sigma_p(K), \mathrm{Re}\lambda=0} (K-\lambda)Z_\lambda(K) + \sum_{\xi\in\sigma_p(K), \mathrm{Re}\xi<0} Z_\xi(K) + E(-b, -a). \end{cases} \tag{2.21}$$

We have

Lemma 2.3. $M_L = (M_L)_0 \oplus_T (M_L)_n$,

where

$$(M_L)_0 = \sum_{\lambda\in\sigma_p(K), \mathrm{Re}\lambda=0} k_\lambda(K)'$$

is neutral under $(\cdot, \cdot)_T$ *and*

$$(M_L)_n = \sum_{\lambda\in\sigma_p(K), \mathrm{Re}\lambda=0} Z_r(K)'' + \sum_{\xi\in\sigma_p(K), \mathrm{Re}\xi<0} Z_s(K) + E(-b, -a)$$

is a Π_k *space under* $-(\cdot, \cdot)_T$, *and if*

$$(M_L)_n = (M_L)_- \oplus_T (M_L)_+$$

is a canonical decomposition, then

$$\dim(M_L)_\pm = \dim(M_s)_\pm.$$

Combining Lemma 2.2 and Lemma 2.3 we obtain a relation between measures of nonuniqueness δ^+ and noncompleteness γ^+.

Theorem 2.4. $\dim(M_L)_+ - \gamma^+ = \dim(M_R)_- - \delta^+.$

Next we shall characterize the geometric feature of δ^+ and γ^+. Let

$$M_+ = \{Q_+H_T + E(-a, 0) + E(-\infty, -b)\} \cap M, \tag{2.22}$$

$$M_- = \{Q_-H_T + E(0, a) + E(b, +\infty)\} \cap M. \tag{2.23}$$

We have the following

Lemma 2.5. $M_\pm$ *is positive/negative definite under* $(\cdot, \cdot)_T$ *and*

$$M = M_+ \oplus_T M_-$$

is a canonical decomposition of M.

Proof Let $m = q + e \in M_+$, $q \in Q_+H_T$ and $e \in E(-a, 0) + E(-\infty, -b)$. Then

$$0 \leqslant (q, q)_T = (m, m)_T + (K^{-T}e, e)_A$$

and $(K^{-1}e, e)_T \leqslant 0$. Thus we see that $(m, m)_T \geqslant 0$ and $(m, m)_T = 0$ iff $e = 0$, $q = 0$, i.e., $m = 0$. Therefore M_+ is positive and nondegenerate. Similarly M_- is negative and nondegenerate. Let $M_+^{[1]}$ denote the $(\cdot, \cdot)_T$ orthogonal complement of M_+ in M. Then

$$\begin{aligned} M_+^{[1]} &= \{E(-\infty, -b) + E(-a, 0) + E(0, a) + E(b, +\infty) \\ &\quad + Q_-H_T \cap [M + E(0, a) + E(b, +\infty)]\} \cap M \\ &= \{Q_-H_T + E(0, a) + E(b, +\infty)\} \cap M = M_-. \end{aligned}$$

Hence $M_+^{[1]} = M_-$. Similarly $M_-^{[1]} = M_+$. So M_+ and M_- are closed in $\{M, (\cdot, \cdot)_T\}$. Now it suffices to prove that $\{M_+, (\cdot, \cdot)_T\}$ is a Hilbert space. Let $x_n \in M_+$ such that $(x_n, x_n)_T \to 0$. Write $x_n = t_n + f_n$, $t_n \in Q_+H_T$ and $f_n \in E(-\infty, -b) + E(-a,$

0). Then

$$(t_n, t_n)_T=(x_n, x_n)_T+(f_n, f_n)_T\to 0.$$

Hence $(t_n, t_n)_T\to 0$ and $\|f_n\|_{H_K}\to 0$. Therefore for any $m\in M$,

$$(m, x_n)_T=(m, t_n)_T+(m, f_n)_T$$
$$=(Q_+m, t_n)_T-(|K_{\beta,\eta}^{-1}|m, f_n)_A\to 0.$$

Hence $x_n\to 0$ in M weakly. Therefore $\{x_n\}$ is bounded in M. Thus $\{M_+, (\cdot, \cdot)_T\}$ is a Hilbert space. Now by [4], we see that $M=M_+\oplus_T M_-$ is a canonical decomposition.

Theorem 2.6. $\delta^+=\dim M_+\cap M_R$. *and* $\gamma^+=\dim M_+\cap M_L$.

Proof By Lemma 2.1 and since $(E(0, a)+E(b, +\infty))\cap H_T$ is $(\cdot, \cdot)_T$ positive definite, we have

$$\delta^+=\dim\{(M_R+E(0, a)+E(b, +\infty))\cap Q_-H_T\}$$
$$=\dim\{(Q_-H_T+E(0, a)+E(b, +\infty))\cap M_R\}=\dim M_-\cap M_R.$$

As for γ^+, by Theorem 2.4, it suffices to prove that

$$\dim(M_R)_- -\dim M_-\cap M_R=\dim(M_L)_+ -\dim M_+\cap M_L.$$

From Lemma 2.5 we know that $M=M_-\oplus_T M_+$ is a canonical decomposition. By Lemma 1.3 we know that M_R is a direct sum of a Π_k space and a neutral space. Thus by [4],

$$M_R=\{(0, y)\,|\,y\in L\subset M_-\}\oplus_T\{(u, Bu)\,|\,u\in D(B)\subset M_+\}.$$

Here B is a continuous linear operator from $D(B)$ to $\overline{\operatorname{ran}B}$, and L and $D(B)$ are closed subspaces of M_- and M_+, respectively. Let $B=U|B|$ be the polar decomposition. Then U is a partial isometry from $D(B)$ onto $\overline{\operatorname{ran}B}$. Let $\sigma(\cdot, \cdot)$ be the spectral resolution of $|B|$ in $D(B)$. Then we know that $\sigma((1, +\infty))$ is finite-dimensional. Now it can be easily calculated that

$$M_L=M_R^{[\perp]}=\{(x, 0)\,|\,x\perp D(B) \text{ in } M_+\}\oplus_T\{(B^*v, v)\,|\,v\in\overline{\operatorname{ran}B}\}$$
$$\oplus_T\{(0, y)\,|\,y\perp L+\overline{\operatorname{ran}B} \text{ in } M_-\}.$$

Since

$$M_R=\{(0, y)\,|\,y\in L\}\oplus_T\{(u, Bu)\,|\,u\in\sigma((1, +\infty))\}$$
$$\oplus_T\{(u, Bu)\,|\,u\in\sigma([0, 1])\}\oplus_T\{(u, Bu)\,|\,u\in\sigma(\{1\})\}$$

and

$$\dim(N_R)_-=\dim\{\{(u, Bu)\,|\,u\in\sigma((1, +\infty))\}\oplus_T\{(0, y)\,|\,y\in L\}\}$$

and $M_R\cap M_-=\{(0, y)\,|\,y\in L\}$, we have

$$\dim(M_R)_- -\dim M_R\cap M_-=\dim\sigma((1, +\infty)).$$

Similarly,

$$M_L=\{(0, y)\,|\,y\perp L+\overline{\operatorname{ran}B}\}\oplus_T\{(|B|v, Uv)\,|\,v\in((0, 1))\}$$
$$\oplus_T\{(x, 0)\,|\,x\perp D(B)\}\oplus\{(|B|v, Uv)\,|\,v\in\sigma((1, +\infty))\}$$
$$\oplus_T\{(|B|v, Uv)\,|\,v\in\sigma(\{1\})\},$$

$\dim(M_L)_+ = \dim\{\{(x,\ 0)\,|\,x\perp D(B)\}+\dim\{|B|v,\ Uv)\,|\,v\in\sigma((1,\ +\infty))\}$, $M_L\cap M_+ = i\{(x,\ 0)\,|\,x\perp D(B)\}$.

Thus $\dim(M_L)_+ - \dim M_L\cap M_+ = \dim\sigma((1,\ +\infty))$. Therefore we have proved our theorem.

Corollary 1. $\delta^+ \leqslant \dim(M_R)_-$ *and* $\gamma^+ \leqslant \dim(M_L)_+$.

Corollary 2. *If A is semi-positive and Fredholm, then* $\delta^+ = \dim(\ker A)_-$ *and* $\gamma^+ = 0$.

Proof When A is semi-positive, every Jordan chain of K restricted to $Z_0(K)$ is at most of length two[3]. Hence $\dim(M_L)_+ = 0$. Therefore by Corollary 1, $\gamma^+ = 0$. Then by Theorem 2.4 we see that $\delta^+ = \dim(M_R)_- = \dim(\ker A)_-$.

Generally, δ^+ need not equal $\dim(M_R)_-$. The following example is to illustrate this.

Example. Let $H = \mathrm{span}\{e_1,\ e_2,\ e_3,\ e_4\} = \mathbf{C}^4$. $\{e_1,\ e_2,\ e_3,\ e_4\}$ is an orthonormal base. Under this base

$$T = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix},\quad S = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix},$$

$$A^{-1} = -\frac{1}{6}\begin{bmatrix} 5 & 4 & 15 & 12 \\ 4 & 5 & 12 & 15 \\ 15 & 12 & 5 & 4 \\ 12 & 15 & 4 & 5 \end{bmatrix}.$$

It is easily calculated that $ST = -TS$ and $SA = AS$. Moreover

$$\begin{cases} A^{-1}T(2e_1+e_2+2e_3+e_4) = -2(2e_1+e_2+2e_3+e_4) \\ A^{-1}T(e_1+2e_2-e_3-2e_4) = -(e_1+2e_2-e_3-2e_4), \\ A^{-1}T(2e_1+e_2-2e_3-e_4) = (2e_1+e_2-2e_3-e_4), \\ A^{-1}T(e_1+2e_2+e_3+2e_4) = 2(e_1+2e_2+e_3+2e_4). \end{cases}$$

Hence

$$Z_1(A^{-1}T) = \mathrm{Span}\{2e_1+e_2-2e_3-e_4\}$$

is $(\cdot,\ \cdot)_T$ positive definite and

$$Z_2(A^{-1}T) = \mathrm{Span}\{e_1+2e_2+e_3+2e_4\}$$

is $(\cdot,\ \cdot)_T$ negative definite. Furthermore

$$Q_+M_R = Q_+\{Z_1(A^{-1}T)+Z_2(A^{-1}T)\} = \mathrm{Span}\{e_1-e_3,\ e_1+e_3\} = Q_+H.$$

Hence the half-range problem has a unique solution for every $f_+\in Q_+H$ and therefore $\delta^+ = 0$. But $\dim(M_R)_+ = 1$.

Remark. This example shows that the half-range problem may have uniqueness even though the spectral subspace of $A^{-1}T$ with respect to the right half plan is indefinite under $(\cdot, \cdot)_T$. Therefore this is a counter-example to the main result of [2] (Theorem 1 and Theorem 2).

For the application in the next section, we impose more restrictions on the operator pair $\{T, A\}$ such that under these restrictions the equality $\delta^+=\dim(M_R)_-$ holds.

Assumption 1. All kernels of $K=T^{-1}A$ on the negative real line $(-\infty, 0)$ are $(\cdot, \cdot)_T$ negative definite.

Assumption 2. Every Jordan chain of K in $Z_0(K)$ is at most of length two.

Assumption 3. T and A are real, i.e., there exists a conjugation operator C (antilinear isometric with $C^2=Id$) that commutes both T and A.

Now b yAssumption 1, in the definitions of X and M ((1.11) and (2.10)) we do not include the term $E(-b, -a)$. Denote

$$Y=\sum_{\lambda\in\sigma_p(K),\,\mathrm{Re}\lambda=0}(k_\lambda(K)+Z_\lambda(K)'')+\sum_{\xi\in\sigma_p(K),\,\mathrm{Im}\xi\neq0,\,\mathrm{Re}\xi\neq0}Z_\xi(K)+E(a, b) \quad (2.24)$$

and

$$Y_R=\sum_{\lambda\in\sigma_p(K),\,\mathrm{Re}\lambda=0}k_\lambda(K)+\sum_{\xi\in\sigma_p(K),\,\mathrm{Im}\xi\neq0,\,\mathrm{Re}\xi>0}Z_\xi)K)+E(a, b). \quad (2.25)$$

Lemma 2.7. *We have a canonical decomposition for Y:*

$$Y=Y_+\oplus_T Y_-,$$

where

$$Y_-=\{Q_-H_T+E(0, a)+E(b, +\infty)+(M_r)_0\}\cap Y \quad (2.26)$$

is $(\cdot, \cdot)_T$ negative definite and

$$Y_+=\{Q_+H_T+E(-\infty, 0)+(M_r)_0\}\cap Y \quad (2.27)$$

is $(\cdot, \cdot)_T$ pcsitive definite.

Proof Since
$$M=Y+(M_r)_0+(M_s)_0$$
and $(M_r)_0$ is neutral under $(\cdot, \cdot)_T$, as in the proof of Lemma 2.5, $Y_\pm$ is positive/negative and nondegenerate and $Y=M\ominus_T((M_r)_0+(M_s)_0)$ is a Krein space under $(\cdot, \cdot)_T$. The orthogonal complement $Y_+^{[\perp]}$ of Y_+ in Y with respect to $(\cdot, \cdot)_T$ can be calculated as we have done for $M_+^{[\perp]}$ in Lemma 2.5.

$$\begin{aligned}Y_+^{[\perp]}=&Y\cap[\{(M_r)_0+(M_s)_0+E(-\infty, 0)+E(0, a)+E(b, +\infty)\\&+Q_-H_T\cap\{(M_r)_0+(M_s)_0+Y+E(-\infty, 0)\}\\&\cap\{(M_r)_0+Y+E(-\infty, +\infty)\}]\\=&Y\cap\{Q_-H_T+E(-\infty, 0)+(M_r)_0\}=Y_-.\end{aligned}$$

Now it remains to prove that $\{Y_+, (\cdot, \cdot)_T\}$ is a Hilbert space, which can be done as we did for $\{M_+, (\cdot, \cdot)_T\}$ in Lemma 2.5.

Lemma 2.8. $\dim Y_R\cap Y_-=\dim M_R\cap M_-$.

Proof Since $M_R=Y_R+(M_r)_0$, we can write $x=y+z$, $y\in(M_r)_0$ and $z\in Y_R$ if $x\in M_R\cap M_-$. Then $z\in Y_R\cap Y_-$. Conversely, if $z\in Y_R\cap Y_-$, then there exists a y in

$(M_r)_0$ such that $x=y+z\in M_R\cap M_-$. Since $(M_r)_0$ is neutral under $(\cdot,\cdot)_T$, the corresponding of x to z is one to one and onto. Hence we have proved our lemma.

Since T and A are real, $Y_\pm$, $M_\pm$, M_s, M_r, $(M_r)_0$, $(M_s)_0$ and $\{Z_\lambda(K)+Z_{\bar\lambda}(K)\}$ are all real subspaces (a subspace S is real if $CS=S$). Now

$$\dim Y_-=\dim(M_s)_-+\dim(M_r)_-,$$

and since $CZ_\lambda(K)=Z_{\bar\lambda}(K)$ and $C\ker(K-\lambda)=\ker(K-\bar\lambda)$ for $\lambda\in\sigma_p(K)$, $\operatorname{Im}\lambda\neq 0$, we see that

$$\dim Y_-\cap Y_R=\dim(M_r)_-=\dim(M_R)_-.$$

Thus we have the following

Theorem 2.9. *When the operator pair* $\{T, A\}$ *satisfies Asumptions* 1, 2 *and* 3,

$$\delta^+=\dim(M_R)_- \text{ and } \gamma^+=\dim(M_L)_+.$$

Proof Since $\delta^+=\dim M_R\cap M_-$ by Theorem 2.6 and $\dim M_R\cap M_-=\dim Y_R\cap Y_-$ by Lemma 2.8, we have $\delta^+=\dim(M_R)_-$. Now by Theorem 2.4 we see that $\gamma^+=\dim(M_L)_+$.

§ 3. Symmetric Multigroup Transport Equation

The symmetric multigroup approximation in neutron transport with isotropic scattering leads to the coupled set of N equations

$$u\frac{\partial f_i}{\partial t}(t, u)+\sigma_i f_i(t, u)=\frac{1}{2}\sum_{j=1}^{N}C_{ij}\int_{-1}^{1}f_j(t, u')du',$$
$$i=1, 2, \cdots, N, \tag{3.1}$$

where $u\in[-1, 1]$ and $C=(C_{ij})_{N\times N}$ is a real symmetric matrix. We take $\sigma_1\geqslant\sigma_2\geqslant\cdots\geqslant\sigma_N=1$ and denote by Γ the diagonal matrix with diagonal entries $\sigma_1, \cdots, \sigma_N$. We do not consider the condition $C_{ij}\geqslant 0$ required for physics reasons, neither the physics necessity that $f_i(t, u)$ be nonnegative ($i=1, 2, \cdots, N$).

Let $H=\bigoplus_1^N L^2(-1, 1)$. Define for f in H,

$$\begin{cases} Tf(u)=uf(u), \\ (Bf)_i(u)=\dfrac{1}{2}\sum_{j=1}^{N}C_{ij}\int_{-1}^{1}f_j(u')du', \\ A=\Gamma-B. \end{cases} \tag{3.2}$$

As above, $Q_\pm$ is the orthogonal projection onto the maximal T-positive/T-negative T-invariant subspace of H. Now we come to consider equation(0, 1). From [2] we know that the operator pair $\{T, A\}$ satisfies our requirment tnat T and A are selfadjoint and T is injective and bounded, and A is Fredholm with finite-dimensional negative part. Moreover, A is T-regular.

Lemma 3.1. *All Jordan chains of $T^{-1}A$ restricted to $Z_0(T^{-1}A)$ have length two.*

Proof For x in $\ker A$, $x=\Gamma^{-1}Bx$ is constant, and hence $Tx\perp\ker A$. Thus every Jordan chain is at least of length two. If x_1, x_2 and $x_3\in Z_0(T^{-1}A)$ and $T^{-1}Ax_3=x_2$, $T^{-1}Ax_2=x_1$ and $x_1\in\ker A$, we shall prove that $x_1=0$. First we see that

$$(Ax_2,\ x_2)=(Tx_1,\ x_2)=(x_1,\ Ax_3)=(Ax_1,\ x_3)=0,$$

and $\Gamma x_2-Bx_2=ux_1$, i.e., $x_2=\Gamma^{-1}(Bx_2+ux_1)$. Hence

$$\Gamma(\Gamma^{-1}(Bx_2+ux_1))-B\Gamma^{-1}(Bx_2+ux_1)=ux_1.$$

Since $B\Gamma^{-1}ux_1=0$, we see that $A\Gamma^{-1}Bx_2=0$, i.e., $\Gamma^{-1}Bx_2\in\ker A$. Now

$$0=(Ax_2,\ x_2)=(A\Gamma^{-1}ux_1,\ \Gamma^{-1}ux_1)=(ux_1,\ \Gamma^{-1}ux_1)-(B\Gamma^{-1}ux_1,\ \Gamma^{-1}ux_1)$$
$$=(\Gamma^{-1}ux_1,\ ux_1).$$

Hence $ux_1=0$, i.e., $x_1=0$.

From the proof of the above lemma, we also have the following corollary.

Corollary 3.2. For every $x\in Z_0(T^{-1}A)$, $(Ax,\ x)\geqslant 0$.

Lemma 3.3. *For $\lambda\in(\mathbf{R}\setminus\{0\})\cap\sigma_p(T^{-1}A)$, the root space is $(\cdot,\ \cdot)_T$ positive/negative definite for $\lambda\gtrless 0$. Furthermore, λ is not a critical point and $Z_\lambda(T^{-1}A)=\ker(T^{-1}A-\lambda)$.*

Proof Let $h\in\ker(T^{-1}A-\lambda)$ and $h\neq 0$. We see that

$$h(u)=(\Gamma-\lambda u)^{-1}Bh.$$

Let $Bh=\mathrm{col}[f_1,\ \cdots,f_N]$, f_i being constant. Suppose $\lambda>0$. For $i\in\{1,\ \cdots,\ N\}$, if $\lambda\geqslant\sigma_i$, then $f_i=0$. Denote

$$h=\min\{i\,|\,\lambda\geqslant\sigma_i\}-1.$$

Since $h\neq 0$, we see that $h\geqslant 1$. For $1\leqslant j\leqslant n$,

$$\frac{1}{(\sigma_j-\lambda u)^2}\geqslant\frac{1}{(\sigma_j+\lambda u)^2},\ \text{for}\ u\in(0,\ 1).$$

Thus

$$(Th,\ h)=\sum_{k=1}^{h}\int_{-1}^{1}\frac{u|f_k|^2}{(\sigma_k-\lambda u)^2}du>0.$$

Therefore, $\ker(T^{-1}A-\lambda)$ is $(\cdot,\ \cdot)_T$ and hence $(\cdot,\ \cdot)_A$ positive definite. We know from the triangular model of self-adjoint operators on Π_k spaces[4] that this implies $Z_\lambda(T^{-1}A)=\ker(T^{-1}A-\lambda)$ and λ is not a critical point.

Lemma 3.4. *The eigenvalues of $T^{-1}A$ outside the real line are contained in the imaginary axis. Furthermore, if $\lambda\in\sigma_p(T^{-1}A)\cap\{z\in\mathbf{C}\,|\,\mathrm{Re}z=0,\ z\neq 0\}$, then $Z_\lambda(T^{-1}A)=\ker(T^{-1}A-\lambda)$.*

Proof Let $\lambda\in\sigma_p(T^{-1}A)$, $\mathrm{Im}\lambda\neq 0$. From the proof of the above lemma we see that for $h\neq 0$ in $\ker(T^{-1}A-\lambda)$,

$$h(u)=(\Gamma-\lambda u)^{-1}Bh,$$

and $Bh=\mathrm{col}[f_1,\ \cdots,\ f_N]$, f_i being constant. Since

$$(\sigma_j-u\mathrm{Re}\lambda)^2\gtrless(\sigma_j+u\mathrm{Re}\lambda)^2(\mathrm{Re}\lambda\gtrless 0)$$

for all $u\in(0,1)$, we have for all $u\in(0,1)$,

$$\frac{1}{|\sigma_j-\lambda u|^2}=\frac{1}{|\sigma_j+\lambda u|^2}\quad(\mathrm{Re}\lambda\gtrless 0).$$

Therefore

$$(Th,h)=\sum_{j=1}^{N}\int_{-1}^{1}\frac{u|f_j|^2}{|\sigma_j-\lambda u|^2}du\gtrless 0\quad(\mathrm{Re}\lambda\gtrless 0).$$

But we know from the operator theory on Π_k spaces (cf. [4]) that $(Ah,h)=0$. Thus

$$0=(Ah,h)=\lambda(Th,h)\neq 0.$$

This contradiction provesthat $\sigma_p(T^{-1}A)\cap\{z\in\mathbf{C}|\mathrm{Im}\,\lambda\neq 0\}$ is contained in the imaginary axis.

Now let $\lambda\in\sigma_p(T^{-1}A)\cap\{z\in\mathbf{C}|\mathrm{Re}z=0,\ z\neq 0\}$ and $0\neq h\in\ker(T^{-1}A-\lambda)$. We see that $h=(\Gamma-\lambda u)^{-1}Bh$ and $Bh=f=\mathrm{col}[f_1,\cdots,f_N]$, f_i being constant and

$$\begin{aligned}f_i=(Bh)_i&=\frac{1}{2}\sum_{j=1}^{N}C_{ij}\int_{-1}^{1}h_j(u)du\\&=\frac{1}{2}\sum_{j=1}^{N}C_{ij}\int_{-1}^{1}\frac{f_j}{\sigma_j-\lambda u}du\\&=\frac{1}{2}\sum_{j=1}^{N}C_{ij}\int_{-1}^{1}\frac{\sigma_j f_j}{|\sigma_j|^2+|\lambda u|^2}du.\end{aligned}$$

Let Λ be the diagonal matrix with diagonal entries

$$\Lambda_j=\frac{1}{2}\int_{-1}^{1}\frac{\sigma_j}{|\sigma_j|^2+|\lambda u|^2}du.$$

Then

$$\ker(T^{-1}A-\lambda)=\{(\Gamma-\lambda u)^{-1}f'\,|\,f'\in\mathbf{C}^N,\ f'\in\ker(1-C\Lambda)\}$$

and

$$\begin{aligned}\ker(AT^{-1}+\lambda)&=\ker(T^{-1}A-\lambda)^*=T\ker(T^{-1}A+\lambda)\\&=\{u(\Gamma+\lambda u)^{-1}f'\,|\,f'\in\mathbf{C}^N,\ f'\in\ker(1-C\Lambda)\}.\end{aligned}$$

Now if there is $g\neq 0$ such that $(T^{-1}A-\lambda)g=h$, then

$$h\perp\ker(T^{-1}A-\lambda)^*=\ker(AT^{-1}+\lambda).$$

Since $u(\Gamma+\lambda u)^{-1}f$ is in $\ker(AT^{-1}+\lambda)$, we have

$$\begin{aligned}0&=\sum_{j=1}^{N}\int_{-1}^{1}\frac{u|f_j|^2}{(\sigma_j-\lambda u)^2}du\\&=\sum_{j=1}^{N}\int_{-1}^{1}\frac{|f_j|^2u(\sigma_j+\lambda u)^2}{(|\sigma_j|^2+|\lambda u|^2)^2}du\\&=\sum_{j=1}^{N}\int_{-1}^{1}\frac{2\lambda u^2\sigma_j|f_j|^2}{(|\sigma_j|^2+|\lambda u|^2)^2}du\neq 0.\end{aligned}$$

This contradiction proves that $Z_\lambda(T^{-1}A)=\ker(Te^{-1}A-\lambda)$.

From Corollary 3.2 we see that the negative index of the Π_k space $\{(TZ_0(T^{-1}A))^\perp,\ (\cdot,\cdot)_A\}$ equals the dimension of A-invariant A-negative subspace in H. Hence by Theorem 2.9 we have

Proposition 3.5. For symmetric multigroup transport equation with isotropic scattering, we have the following description of the measures of nonuniqueness and noncompleteness:

δ^{+} = The number of negative eigenvalues (counted by multiplicity) of the $N\times N$ matrix $\Gamma-C$;

$\gamma^{+}=0$.

References

[1] Beals, R., On an abstract treatment of some forward-backward problems of transport and scattering, *J. Functional Analysis*, **34**(1979), 1—20.

[2] Greenberg, W. & van der Mee, C. V. M., Abstract Kinetic equations relevant to supercritical media, *J. Functional Analysis*, **57**(1984), 111—142.

[3] Greenberg, W., van der Mee, C. V. M. & Zweifel, P. F., Generaliznd Kinetic equations, *Integral Equations and Operator Theory*, **7**(1984), 60—95.

[4] Daoxing Xia & Shaozong Yan, Spectral Theory of Linear Operatos II: Operator Theory on Indefinite Inner Product Spaces, Chinese Academic Press, 1987 (in Chinese).