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TIME-DELAY AND SPECTRAL DENSITY FOR STARK HAMILTONIANS (II) ——ASYMPTOTICS OF TRACE FORMULAE***

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Abstract

This paper studies the Schrodinger operator with a homogeneous electric field of the form $-\Delta+x_1+V(x)$, where $x=(x_1, \cdots, x_n)\in R^n$. It is proved that in the specctral representation of the free Stark Hamiltonian, the time-delay operator in scattering theory can be expressed in trems of scattering matrix and under reasonable assumptions on the decay of the potential V, the on-shell time-delay operator is of trace class and its trace is related to the local spectral density via an explicit integral formula. Some asymptotics for the trace are established when the energy tends to infinity.

§0. Introduction

This paper is a continuation of our work [14], in which we proved the existence of a global time-delay operator in scattering theory for Stark Hamiltonians and established the relation between time-delay operator and scattering operator.

In this work, we want to study the relationship between time-delay and local spectral density and to give asymptotics for the scattering phase and the trace of on-shell time-delay.

Let $H_0=-\Delta+x_1$ be the free Stark Hamiltonian and $H=H_0+V(x)$, where $V\in L^2_{Loc}$ and satisfies that for some $\varepsilon_0>0$

$$|V(x)|\leqslant C\langle x_1\rangle^{1-\varepsilon_0}, \text{ fox } x=(x_1, x')\in \mathbf{R}\times\mathbf{R}^{n-1}, x_1>0,$$

and

$$|V(x)|\leqslant C\langle x_1\rangle^{-\frac{1}{2}-\varepsilon_0}, \text{ otherwise.}$$

Then it is known that the scattering operator S for the scattering process of (H_0, H) exists and is unitary on $L^2(\mathbf{R}^n)$. Let $\mathscr{F}_A$: $L^2(\mathbf{R}^n)\to L^2(\mathbf{R}: L^2(\mathbf{R}^{n-1}))$ be the unitary operator defined by

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$$(\mathscr{F}_A u)(\lambda, y) = (\gamma(\lambda) e^{-iG(D)} u)(y), \quad y \in \mathbf{R}^{n-1},$$

where $G(D) = (D_1^3/3) - D_1 \Delta_{x'}$, $D_1 = i^{-1}(\partial/\partial x_1)$, $\Delta_{x'} = \sum_{j=2}^{n} (\partial^2/\partial x_j^2)$, and $\gamma(\lambda)$ is the trace operator on the hyperplane $x_1 = \lambda$. $\mathscr{F}_A$ gives us a spectral representation for H_0

$$(\mathscr{F}_A H_0 \mathscr{F}_A^{-1} \psi)(\lambda, y) = \lambda \psi(\lambda, y).$$

Since S commutes with H_0, by a theorem of von Neumann, S is decomposable in the spectral representation of H_0:

$$(\mathscr{F}_A S \mathscr{F}_A^{-1} \psi)(\lambda, y) = (S(\lambda) \psi(\lambda, \cdot))(y), \quad \psi \in L^2(\mathbf{R}: L^2(\mathbf{R}^{n-1})),$$

where $S(\lambda): L^2(\mathbf{R}^{n-1}) - L^2(\mathbf{R}^{n-1})$ is called the scattering matrix. By Kato-Kuroda theory [8], we have

$$S(\lambda) = 1 - 2\pi i T(\lambda), \tag{0.1}$$

where

$$T(\lambda) = \gamma(\lambda) e^{-iG(D)} (V - V(H - \lambda - i0)^{-1}. V) e^{iG(D)} \gamma(\lambda)^*, \ \lambda \notin \sigma_{pp}(H) \tag{0.2}$$

whenever $(H - \lambda \mp i0)^{-1} = \lim_{\varepsilon \downarrow 0} (H - \lambda \mp i\varepsilon)^{-1}$ exists. In [14], we have also seen that time-delay operator t_D for the scattering process of (H_0, H) satisfies

$$t_D = S^* [D_1, S].$$

By studying the smoothness of $S(\lambda)$ on λ, we shall derive from (0.2) that in the spectral representation of H_0, t_D is given by a family of operators $\{t_D(\lambda), \lambda \in \mathbf{R}\}$, where

$$t_D(\lambda) = -i S^*(\lambda) \frac{d}{d\lambda} S(\lambda). \tag{0.3}$$

(0.3) is Eisenbud-Wigner formula of time-delay for Stark Hamiltonians. With suitable assumptions on V, we shall prove that $t_D(\lambda)$ is of trace class on $L^2(\mathbf{R}^{n-1})$ and its trace, called mean time-delay, is related to the local spectral density by

$$\langle t_D \rangle(\lambda) \equiv \operatorname{tr}(t_D(\lambda)) = -2\pi \int_{\mathbf{R}^n} \partial x_1 V(x) \frac{\partial e}{\partial \lambda}(x, x; \lambda) dx, \tag{0.4}$$

where $\frac{\partial e}{\partial \lambda}(\cdot, \cdot; \lambda)$ is the kernel of $\frac{\partial}{\partial \lambda} E_H(\lambda)$, $E_H(\lambda)$ being the spectral resolution of H. (0.4) is basic for studying asymptotics of trace of time-delay on energy-shell λ. The organization of this work is as follows. In Section 1, we study the smoothness of the boundary values of resolvents and prove in particular (0.3). We give also several estimates for the norm of $T(\lambda)$ in Schatten classes, which guarantees $t_D(\lambda)$ to be of trace class. In Section 2, we establish (0.4). Section 3 is devoted to studying the asymptotics of some traces related to scattering phase. In Section 4, we give asymptotics for mean time-delay $\langle t_D \rangle(\lambda)$, as $\lambda \to -\infty$. The same problem for $\lambda \to +\infty$ is more subtle and is left as an open question here; it will be studied elsewhere.

§ 1. Smoothness of Resolvents and Scattering Matrix

We begin with giving some estimates over the resolvents $R(\lambda\pm i\varepsilon)=(H-\lambda\mp i\varepsilon)^{-1}$ as $\varepsilon\to 0_+$. The following results have certainly something to do with the recent works of Jensen [5,6] and Wang[17] (Theorem 5.1). But the conditions here are different from those in the above-mentioned works. At first, we assume that

$$V\in C^2(\mathbf{R}^n),\ \frac{\partial^k}{\partial x_1^k}V\in L^\infty(\mathbf{R}^n),\ \text{for } k=1,2, \tag{1.1}$$

$$\lim_{x_1\to-\infty}\ \inf_{x'\in\mathbf{R}^{n-1}} \partial_{x_1}V(x_1,x')>-1. \tag{1.2}$$

There exist ε_1, $\varepsilon'>0$ such that for any $R>0$, one has

$$\langle x_1\rangle^{\varepsilon_1-1}|V(x)|+|\partial_{x_1}V(x)|\leqslant C\langle x'\rangle^{-\varepsilon'} \text{ for every } (x_1,x'),\ x_1>-R,\ x'\in\mathbf{R}^{n-1}. \tag{1.3}$$

Here $C>0$ may depend on R.

From (1.1)—(1.3), it is known (see [2]) that $H=-\Delta+x_1+V(x)$, defined on $D(-\Delta)\cap D(x_1)$, is essentially selfadjoint and H has no eigenvalues. In the following, we denote still by H the selfadjoint realization of $-\Delta+x_1+V(x)$. By Mourre's method, we can easily prove the following

Proposition 1.1. *For every $s<1/2$, $\lambda\in\mathbf{R}$, the limits:*

$$\langle D_1\rangle^{-s}R(\lambda\pm i0)\langle D_1\rangle^{-s}\equiv\lim_{\varepsilon\downarrow 0}\langle D_1\rangle^{-s}R(\lambda\pm i\varepsilon)\langle D_1\rangle^{-s}$$

exist in norm of operators on $L^2(\mathbf{R}^n)$. Moreover the map $\lambda\to\langle D_1\rangle^{-s}R(\lambda\pm i0)\langle D_1\rangle^{-s}$ is locally Holder continuous on $\mathbf{R}$ (Here $\langle D_1\rangle=(1+|D_1|^2)^{1/2}$).

Proof We verify that $A=D_1$ is a conjugate operator for H in any bounded interval $I\subset\mathbf{R}$. Clearly $i[H,A]=I+\partial_{x_1}V$ and $i^2[[H,A],A]=(\partial^2/\partial x_1^2)V$. So by (1.1), these commutators are bounded on $L^2(\mathbf{R}^n)$. Let $g\in C^\infty(\mathbf{R})$ such that

$$g(x_1)=1 \text{ for } x_1\geqslant -1,\ g(x_1)=0 \text{ for } x_1\leqslant -2.$$

By (1.2) and (1.3), for $R>0$ large enough, there exists $\alpha>0$ such that

$$E_H(I)i[H,A]E_H(I)\geqslant\alpha E_H(I)+E_H(I)g(x_1/R)\partial_{x_1}VE_H(I). \tag{1.4}$$

It remains to check that $E_H(I)g(x_1/R)\partial_{x_1}VE_H(I)$ is compact. Then we can apply Mourre's theory which gives the desired results. But $(H_0+i)^{-1}g(x_1/R)\partial_{x_1}V$ is clearly compact. Since

$$[(H+i)^{-1}-(H_0+i)^{-1}]g(x_1/R)\partial_{x_1}V=-(H+i)^{-1}V(H_0+i)^{-1}g(x_1/R)\partial_{x_1}V$$

and by (1.2) and (1.3) $V(H+i)^{-1}$ is bounded, we see that $(H+i)^{-1}g(x_1/R)\partial_{x_1}V$ is compact, so is $E_H(I)g(x_1/R)\partial_{x_1}V$.

Suppose in addition that

$$(\partial_k/\partial_1^k)V\in L^\infty(\mathbf{R}^n),\ k\leqslant j,\ j\geqslant 2. \tag{1.1$_j$}$$

We have the following result on smoothness of $\lambda\to\langle D_1\rangle^{-s}R(\lambda\pm i0)\langle D_1\rangle^{-s}$ defined on $\mathbf{R}$ with values in the Banach space of bounded operators on $L^2(\mathbf{R}^n)$.

Proposition 1.1' *Assume* (1.1)$_j$, (1.2) *and* (1.3). *Let* $s>j+1/2$. *Then*

$$\lambda \to \langle D_1\rangle^{-s} R(\lambda \pm io) \langle D_1\rangle^{-s}$$

is $(j-1)$ *times continuously differentiable and*

$$\frac{d^j}{d\lambda^j}\langle D_1\rangle^{-s} R(\lambda \pm io)\langle D_1\rangle^{-s} = jl\langle D_1\rangle^{-s} \lim_{\varepsilon\to 0} R(\lambda \pm i\varepsilon)^{j+1}\langle D_1\rangle^{-s}, \ \forall \lambda \in \mathbf{R}.$$

Proof It is an easy consequence of multiple commutators techniques [7].

Remarks 1.2. Arguing as in our previous work [14] we can replace in propositions 1.1 and 1.1' the weight $\langle D_1\rangle^{-1}$ by $\rho^{1/2}$ where $\rho \in C^\infty(\mathbf{R})$, $\rho(x_1)=1$ for $x_1 \geqslant 0$, $\rho(x_1)=\langle x_1\rangle^{-1}$ for $x^1 \leqslant -1$.

Proposition 1.3. (i) *Assume on* V (1.1), (1.2), (1.3) *and*

$$|V(x_1, x')| \leqslant C.\ \rho(x_1)^s \ \textit{for}\ s>1/2. \tag{1.5$_0$}$$

Then $\lambda \to S(\lambda)$ *is locally Holder continuous on* $\mathbf{R}$ *with values in* $\mathscr{L}(L^2(\mathbf{R}^{n-1}))$.

(ii) *Assume on* V (1.1), (1.2), (1.3), (1.5)$_0$ *and*

$$|\partial_{x_1} V(x_1, x')| \leqslant C.\ \rho(x_1)^s. \tag{1.5$_1$}$$

Then $\lambda \to S(\lambda)$ *is continuously differentiable from* $\mathbf{R}$ *into* $\mathscr{L}(L^2(\mathbf{R}^{n-1}))$ *and* $\frac{ds}{d\lambda}$ *is locally Holder from* $\mathbf{R}$ *into* $\mathscr{L}(L^2(\mathbf{R}^{n-1}))$.

Proof (i) Let us denote $\mathscr{F}_A(\lambda) = \gamma(\lambda).e^{-iG(D)}$.

It is sufficient to prove continuity for $\lambda \to \mathscr{F}_A(\lambda)\rho^s$ with values in $\mathscr{L}(L^2(\mathbf{R}^n))$, $L^2(\mathbf{R}^{n-1})$ for $s>1/4$.

For that we have from abstract scattering theory [8]

$$\rho^s \mathscr{F}_A(\lambda)^* \cdot \mathscr{F}_A(\lambda)\cdot \rho^s = \frac{1}{2i\pi}\rho^s((H_0-\lambda-io)^{-1} - (H_0-\lambda+io)^{-1})\rho^s.$$

So we have clearly the result.

(ii) Using translation invariance we have $T(\lambda+\varepsilon, V) = T(\lambda, V_\varepsilon)$ with $V_\varepsilon(x_1, x') = V(x_1+\varepsilon, x')$.

Then (ii) results easily from (0.1). In particular we have

$$\frac{dT}{d\lambda}(\lambda) = \mathscr{F}_A(\lambda)[\partial_{x_1}V - (\partial_{x_1}V)(H-\lambda-io)^{-1}V - V(H-\lambda-io)^{-1}(\partial_{x_1}V)$$
$$- V(H-\lambda-io)^{-1}(\partial_{x_1}V)(H-\lambda-io)^{-1}V]\mathscr{F}_A(\lambda)^*.$$

Remark 1.4. Clearly if we have $|\partial_{x_1}^k V(x_1, x')| \leqslant C\rho(x_1)^s$ for $0 \leqslant k \leqslant j$ then one can prove that $\lambda \to S(\lambda)$ is of class C^1 on $\mathbf{R}$ with values in $\mathscr{L}(L^2(\mathbf{R}^{n-1}))$.

Corollary 1.5. *Let* t_D *denote the time-delay operator for the scattering process of* (H_0, H). *Then* t_D *is decomposable in the spectral representation for* H_0, *and we have*

$$(\mathscr{F}_A t_D \mathscr{F}_A^{-1}\psi)(\lambda, y) = (t_D(\lambda)\psi(\lambda, \cdot))(y) \tag{1.6}$$

for any $\psi \in C_0^1(\mathbf{R}; L^2(\mathbf{R}^{n-1}))$. *In addition*

$$(0.3)\ t_D(\lambda) = -S^*(\lambda)\frac{d}{d\lambda}S(\lambda), \ \forall \lambda \in \mathbf{R}.$$

Proof Notice that $(\mathscr{F}_A D_1 \mathscr{F}_A^{-1}\psi)(\lambda, y) = -i\dfrac{d}{d\lambda}\psi(\lambda, y)$, for any $\psi \in C_0^1(\mathbf{R}; L^2(\mathbf{R}^{n-1}))$. Proposition 1.3 implies that $C_0^1(\mathbf{R}; L^2(\mathbf{R}^{n-1}))$ is invariant by $\mathscr{F}_A S \mathscr{F}_A^{-1}$. So

$$(\mathscr{F}_A S^* D_1 S \mathscr{F}_A^{-1}\psi)(\lambda, y) = -i\Big[S(\lambda)^* \frac{d}{d\lambda}(S(\lambda)\psi(\lambda, \cdot))\Big](y).$$

Now it follows from (0.2) that t_D is decomposable and

$$t_D(\lambda) = S(\lambda)^*\Big[i^{-1}\frac{d}{d\lambda}, S(\lambda)\Big] = -iS(\lambda)^* \frac{d}{d\lambda}S(\lambda).$$

By an argument of density, we can prove that (1.6) is in fact valid for any $\psi \in L^2(\mathbf{R}; L^2(\mathbf{R}^{n-1}))$ with $\mathscr{F}_A^{-1}\psi \in D(t_D)$. Notice that (0.3) has the same form as the Eisenbud-Wigner formula for time-delay in scattering theory of Schrodinger operators without Stark effect (see for example [16]). Since both time-delay operator and scattering operator may be defined in an abstract setting, we wonder if such formula is universal.

To be able to define scattering cross sections and scattering phases we study now compacity classes for $T(\lambda)$ and $\dfrac{dT}{d\lambda}(\lambda)$ (for theses notions see [4]).

The following lemma will be usefull:

Lemma 1.6. *Consider W a measurable function on $\mathbf{R}_x^n \times \mathbf{R}_\xi^n$ satisfying*

$(+)$ $|W(x_1, x')| \leqslant C\langle x_1\rangle^{\alpha_1}\cdot\langle x'\rangle^{-\delta'}$ *for every* $x_1 \geqslant 0$, $x' \in \mathbf{R}^{n-1}$,

$(-)$ $|W(x_1, x')| \leqslant C\langle x_1\rangle^{-\delta_1}\cdot\langle x'\rangle^{-\delta'}$ *for every* $x_1 < 0$, $x' \in \mathbf{R}^{n-1}$.

Suppose $\alpha_1 \in \mathbf{R}$ arbitrary, $\delta_1 > 0$, $\delta' > 0$. Then the operator $\mathscr{F}_A(\lambda)\cdot W$ is compact. Moreover if $p > 0$ is such that $\delta_1 > ((n+1)/2p) + 1/2$ and $\delta' > (n-1)/p$ then $\mathscr{F}_A(\lambda)\cdot W$ is in the Schatten class C^p (see [4] for definition).

Proof It is sufficient to study compacity of $W\cdot\mathscr{F}_A(\lambda)^*\cdot\mathscr{F}_A(\lambda)\cdot W$.

We write down

$$W\cdot\mathscr{F}_A(\lambda)^*\cdot\mathscr{F}_A(\lambda)W = W\cdot\chi(H_0-\lambda)\cdot(dE_{H_0}(\lambda)/d\lambda)\cdot\chi(H_0-\lambda)\cdot W,$$

where $\chi \in C_0^\infty(\mathbf{R})$, $\chi(u) = 1$ for $|u| \leqslant \dfrac{1}{2}$ and write for $\dfrac{1}{4} < s < 1$

$$W\mathscr{F}_A(\lambda)^*\mathscr{F}_A(\lambda)\cdot W = W\chi(H_0-\lambda)\rho^{-s}\cdot(\rho^s(dE_{H_0}(\lambda)/d\lambda)\rho^s)\rho^{-s}\chi(H_0-\lambda)W.$$

So we are reduced to studying compacity classes for

$$A = W\cdot\chi(H_0-\lambda)\rho^{-s}.$$

Write it as $A = W\rho^{-s}\chi(H_0-\lambda) + W[\chi(H_0-\lambda), \rho^{-s}] \equiv B + E$.

At first we study B. We have clearly, for the p-norms in the Schatten class C^p, the inequality

$$\|B\|_p \leqslant c(\|B_1\|_p + \|B_2\|_p + \|B_3\|_p),$$

where

$$B_1 = \langle x'\rangle^{-\delta'}\langle x_1\rangle^{\alpha_1+s}g(x_1)(H_0+i-\lambda)^{-N},$$

$$B_2=\langle x'\rangle^{-\delta}\langle x_1\rangle^{-\delta_1+s}(1-g(x_1))\gamma_a(x, D)(H_0+i-\lambda)^{-N},\ N\gg 0,$$
$$B_3=\langle x'\rangle^{-\delta'}\langle x_1\rangle^{-\delta_1+s}(1-g(x_1))(1-\gamma_a(x, D))(H_0+i-\lambda)^{-N}.$$

Using estimates of p–norms for pseudodifferential operator[15] we have

$$\|B_1\|_p^p\leqslant C\int[\langle x'\rangle^{-\delta'}\langle x_1\rangle^{\alpha_1+s}(1+|x_1|+|\xi|^2)^{-N}]^p dx d\xi<+\infty \text{ if } \delta'>\frac{n-1}{p}.$$

The cut-off symbol $\gamma_a(x, \xi)$ is defined as

$$\gamma_a(x, \xi)=1-\theta\left(\frac{x_1+|\xi|^2}{\langle\xi\rangle^a}\right),\ 0<a\leqslant 2,$$

$\theta\in C_0^\infty\left]-\frac{1}{2}, \frac{1}{2}\right[$, $\theta=1$ on $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

We have

$$\|B_2\|_p^p\leqslant C\int\langle x'\rangle^{-\delta'}\langle x_1\rangle^{\delta_1+s}\langle\xi\rangle^{-2Na})^p dx d\xi.$$

So $\|B_2\|_p^p< +\infty$ if $\delta'>\frac{n-1}{p}$ and $\delta_1>\frac{1}{p}+s$.

For B_3 remark that on supp $(1-\gamma_a(x, \xi))$ we have $|x_1+|\xi|^2|\leqslant\frac{1}{2}\langle\xi\rangle^a$. So

$$\|B_3\|_p^p\leqslant C\left(\int_{\mathbf{R}^{n-1}}\langle x'\rangle^{-p\delta'}dx'\right)\cdot\int_{|x_1+|\xi|^2|\leqslant\frac{\langle\xi\rangle^a}{2}}\langle x_1\rangle^{p(s-\delta_1)}dx_1 d\xi<+\infty$$

if $\delta'>\frac{n-1}{p}$, $\delta_1>\frac{n+a}{2p}+s$.

To study the operator E we have to control the commutator $[\chi(H_0-\lambda), f(x_1)]$, where $f(x_1)=\rho^{-s}(x_1)$. We can write

$$[\chi(H_0-\lambda),\ f]=\frac{1}{2\pi}\int\hat{\chi}(t-\lambda)e^{itH_0}(f(x_1)-f(x_1-2tD_1-t^2))dt \tag{1.7}$$

and

$$f(x_1)-f(x_1-2tD_1-t^2)=(2tD_1+t^2)\cdot\int_0^1 f'(x_1+s(2tD_1+t^2))ds$$
$$-2t\cdot i^{-1}\int_0^1 f''(x_1+s(2tD_1+t^2))ds. \tag{1.8}$$

Here we have used a specific property for Weyl quantization:for a first order operator $\alpha\cdot x+\beta\cdot D_x$ and f a smooth function, the symbol of $f(\alpha x+\beta D_x)$ (defined by the spectral theorem) is exactly $f(\alpha\cdot x+\beta\cdot\xi)$.

From (1.7), (1.8) and (2.2) of [14], we have

$$[\chi(H_0-\lambda), f]=\int(2tD_1-t^2)\hat{\chi}(t-\lambda)e^{itH_0}\left(\int_0^1 f'(x_1+s(2tD_1+t^2))ds\right)$$
$$-2i^{-1}\int t\cdot\hat{\chi}(t-\lambda)\cdot e^{itH_0}\left(\int_0^1 f''(x_1+s(2tD_1+t^2))ds\right)dt \tag{1.9}$$

and for every $k\geqslant 1$

$$H_0^k\cdot[\chi(H_0-\lambda), tf]=\sum_{0\leqslant j\leqslant k+1}C_j D_1^{j+1}\int\hat{\chi}(t-\lambda)t^{\alpha_j}e^{itH_0}$$
$$\cdot\left(\int_0^1 f^{(\beta_j)}(x_1+s(2tD_1+^2))ds\right)dt, \tag{1.10}$$

where $\beta_j \geq 1$.

As for B we decompose $E = E_1 + E_2 + E_3$.

$$E_1 \equiv W g[\chi(H_0 - \lambda), \rho^{-s}].$$

Using (1.10) with k big enough we see that

$$\|E_1\|_p \leq C \cdot \|B_1\|_p.$$

In the same manner we have for $E_2 \equiv W(1-g) \cdot \gamma_a(x_1 D)[\chi(H_0 - \lambda) \cdot \rho^s]$,

$$\|E_2\|_p \leq C\|B_2\|_p, \text{ choosing } a > 1.$$

For $E_3 \equiv W(1-g)(1-\gamma_a(x, D))[(H_0 - \lambda), \rho^{-s}]$ we have

$$\|E_3\|_p \leq C\|\langle x'\rangle^{-\delta'}(1-\gamma_a(x, D))\langle x_1\rangle^{\frac{1}{2}-\delta_1}\|_p.$$

So $\|E_3\|_p < +\infty$ if $\delta' > \dfrac{n-1}{p}$ and $\delta_1 > \dfrac{n+a}{2p} + \dfrac{1}{2}$.

Summing up the three above estimates we have proved Lemma 1.6.

Remark 1.7. If we replace the weight function $\rho(x_1)$ by $\langle D_1\rangle^{-1}$, Lemma 1.6 can be improved in the sense that we have the same conclusion for $\delta' > \dfrac{n-1}{p}$ and $\delta_1 > \dfrac{n}{2p} + \dfrac{1}{4}$. This is easy to see: we work as in the proof of Lemma 1.6 and we remark that with the new weight the commutation formula is simpler

$$[\chi(H_0 - \lambda), f(D_1)] = \frac{1}{2\pi} \cdot \int t \cdot \hat{\chi}(t - \lambda) e^{itH_0} \left(\int_0^1 f'(D_1 + st) ds \right) dt.$$

Moreover in this proof we can take any $a > 0$.

Proposition 1.8. *Suppose $n \geq 2$ and V satisfies*

$$\text{(i)} \quad \begin{cases} |V(x_1, x')| \leq C\langle x_1\rangle^{1-\varepsilon_1}\langle x'\rangle^{-\delta'} \text{ for } x_1 \geq 0, \\ |V(x_1, x')| \leq C\langle x_1\rangle^{-\delta_1}\langle x'\rangle^{-\delta'} \text{ for } x_1 \leq 0 \end{cases}$$

with $\varepsilon_1 > 0$, $\delta' > \dfrac{n-1}{p}$, $\delta_1 > \dfrac{n}{2p} + \dfrac{1}{2}$, then $T(\lambda) \in C^p(L^2(\mathbf{R}^{n-1}))$.

(ii) *If both V and $\partial_{x_1} V$ satisfy assumption (i) then $\dfrac{dT}{d\lambda}(\lambda)$ and $t_D(\lambda)$ are in $C^p(L^2(\mathbf{R}^{n-1}))$.*

Proof (i) Using (0.1) we have

$$T(\lambda) = \mathscr{F}_A(\lambda) \cdot V \cdot \mathscr{F}_A^*(\lambda) - \mathscr{F}_A(\lambda) \cdot V(H - \lambda - io)^{-1} \cdot V \cdot \mathscr{F}_A(\lambda)^* \equiv T_1(\lambda) - T_2(\lambda).$$

Write $V = V_1 \cdot V_2$ where V_1 and V_2 satisfy

$$|V_i(x_1, x')| \leq C\langle x_1\rangle^{(1-\varepsilon_1)/2}\langle x'\rangle^{-\delta'/2} \quad \text{for } x_1 \geq 0;$$

$$|V_i(x_1, x')| \leq C\langle x_1\rangle^{-\delta_1/2}\langle x'\rangle^{-\delta'/2} \text{ for } x_1 \leq 0,$$

for $i = 1, 2$.

We can compute the compacity class of $T_1(\lambda)$ using Lemma 1.6. For $T_2(\lambda)$ we write down

$$T_2(\lambda) = \mathscr{F}_A(\lambda) V \cdot \rho^{-s} \cdot \rho^s \cdot (H - \lambda - io)^{-1} \rho^s \cdot \rho^{-s} \mathscr{F}_A(\lambda)^*$$

with $s>\frac{1}{4}$; we apply Proposition 1.1′ (with Remark 1.2) and Lemma 1.6.

Now we want to consider p-norm estimates for $T(\lambda)$, $\frac{dT}{d\lambda}(\lambda)$ as $\lambda\to\pm\infty$.

At first we need some uniform energy estimates for boundary values of resolvents.

Lemma 1.9. *For every $s>1/2$, there exists $C>0$ such that*

$$\|\langle D_1\rangle^{-s}(H_0-\lambda\pm io)^{-1}\langle D_1\rangle^{-s}\|\leqslant C,\ \forall\lambda\in\mathbf{R}.$$

Proof By translation in x_1 variable, one has

$$\|\langle D_1\rangle^{-s}(H_0-\lambda\pm io)^{-1}\langle D_1\rangle^{-s}\|=\|\langle D_1\rangle^{-s}(H_0\pm io)^{-1}\langle D_1\rangle^{-s}\|,\ \forall\lambda\in\mathbf{R}.$$

Lemma 1.10. *Let V satisfy* (1.1)—(1.3). *Assume in addition that $V\in L^\infty(\mathbf{R}^n)$. Then for every $s>1/2$, there exists $C>0$ such that*

(i) $\|\langle D_1\rangle^{-s}R(\lambda\pm io)\langle D_1\rangle^{-s}\|\leqslant C,\ \forall\lambda\leqslant 0,$

(ii) $\|\rho(x_1-\lambda)^{s/2}R(\lambda\pm io)\rho(x_1-\lambda)^{s/2}\|\leqslant C,\ \forall\lambda\leqslant 0.$

Proof (i) Let $I_\lambda=[\lambda-1,\lambda+1]$. We claim that there are $\alpha>0$, $R>0$ such that

(*) $$E_H(I_\lambda)[H, iD_1]E_H(I_\lambda)\geqslant\alpha,\text{ uniformly in }\lambda<-R.$$

Then (i) follows from Mourre's commutator method. Remark that under the assumptions (1.2) and (1.3), we have

$$E_H(I_\lambda)[H, iD_1]E_H(I_\lambda)\geqslant\alpha_0E_H(I_\lambda)+E_H(I_\lambda)g(x_1)(\partial_{x_1}V)E_H(I_\lambda),$$

where $\alpha_0>0$ and $g\in C^\infty(\mathbf{R})$ with $g(x_1)=0$, for $x_1<-R_0$ and $g(x_1)=1$, for $x_1>-R_0+1$. Take $\varphi\in C_0^\infty(]-2,2[)$ such that $\varphi\equiv1$ on $[-1,1]$. Then

$$\|\varphi(H-\lambda)g(x_1)\partial_{x_1}V\|\leqslant C\|\varphi(H_\lambda)\langle x'\rangle^{-s'}g(x_1+\lambda)\|\leqslant C'\|(H_\lambda+i)^{-1}\langle x'\rangle^{-s'}g(x_1+\lambda)\|$$
$$\leqslant C''\|(H_0+i)^{-1}g_1(x_1)\langle x'\rangle^{-s}g(x_1+\lambda)\|,\ \lambda\leqslant 0.$$

Here $g_1(x_1)=0$, if $x_1<-R_0-1$ and $g_1(x_1)=1$ if $x_1\geqslant -R_0$. It is clear that $(H_0+i)^{-1}g_1(x_1)\langle x'\rangle^{-s}$ is compact and the multiplication by $g(x_1+\lambda)$ converges strongly to 0 in $L^2(\mathbf{R}^n)$, as $\lambda\to-\infty$. Consequently

$$\|(H_0+i)^{-1}g_1(x_1)\langle x'\rangle^{-s}g(x_1+\lambda)\|\to0\text{ as }\lambda\to-\infty.\ (*)$$

is proved.

(ii) Since $\|\rho(x_1-\lambda)^{s/2}\varphi(H-\lambda)\langle D_1\rangle^{s}\|$ is uniformly bounded, (cf. Lemma 2.9 in [14]), (ii) follows easily.

Lemma 1.11. *Suppose V satisfies* (1.1), (1.3) *and*

$$\|\partial_{x_1}V(x_1,x')\|_{L^\infty(\mathbf{R}_{x_1}\times\mathbf{R}^{n-1}_{x'})}<1. \tag{1.2'}$$

Then the estimates (i) *and* (ii) *of Lemma* 1.10 *hold for $\lambda\in\mathbf{R}$ (in particular for $\lambda\to+\infty$).*

Proof It is the same as for Lemma 1.10: assumption (1.2)′ gives a uniform Mourre estimate.

Lemma 1.12. *Consider W a measurable function satisfying* (+) *and* (−) *of Lemma* 1.6. *Assume $\delta'>\frac{n-1}{p}$, $\delta_1>\frac{n}{2p}+\frac{1}{4}$.*

For every $\varepsilon>0$, *there exists* $C>0$ *such that*

(i) $\|\mathscr{F}_A\cdot W\|_p\leqslant C(\langle\lambda\rangle^{\alpha_1}+\langle\lambda\rangle^{n/\varepsilon p-\delta_1})$ *for* $\lambda\leqslant 0$;

(ii) $\|\mathscr{F}_A\cdot W\|_p\leqslant C(1+\langle\lambda^{\alpha_1}\rangle+\langle\lambda\rangle^{(n/2-1)/p+\varepsilon})$ *for* $\lambda\geqslant 0$.

Proof We use the same method as in the proof of Lemma 1.6 (with Remark 1.7). Then $\|\mathscr{F}_A(\lambda)\cdot W\|_p\leqslant\|W\langle D_1\rangle^{-1}\chi(H_0-\lambda)\|_p+\|W[\chi(H_0-\lambda),\langle D_1\rangle^{-s}]\|_p$.

We use translation in λ.

As in Lemma 1.6 we have six terms to estimate.

1) $\|B_1(\lambda)\|_p^p\leqslant C_N\int_0^{+\infty}\langle x_1+\lambda\rangle^{\alpha_1 p}\langle x_1\rangle^{-N}dx_1,\ \forall N\gg 0$.

So $\|B_1(\lambda)\|_p=O(|\lambda|^{\alpha_1}),\ |\lambda|\to+\infty$.

2) $\|B_2(\lambda)\|_p^p\leqslant C\int_{-\infty}^0\langle x_1+\lambda\rangle^{-\delta_1 p}dx_1$.

So $\|B_2(\lambda)\|_p=O(|\lambda|^{-\delta_1})$ for $\lambda\to-\infty$,

$=O(1)$ for $\lambda\to+\infty$.

3) $\|B_3(\lambda)\|_p^p\leqslant C\iint\langle x_1+\lambda\rangle^{-\delta_1 p}\langle x_1\rangle^{sp/2}dx_1d\xi$,

the integral is taken over the set $\{-|\xi|^2-\langle\xi\rangle^a/2\leqslant x_1\leqslant-|\xi|^2+\langle\xi\rangle^a/2\}$.
Compute the last integral in polar coordonates in ξ we obtain

$$\|B_3(\lambda)\|_p=\begin{cases}O(|\lambda|^{n/2p-\delta_1}), & \lambda\to-\infty,\\ O(\lambda|^{n/}|^{2-1+2a}), & \lambda\to+\infty.\end{cases}$$

We estimate the other three corresponding terms in the same way. So we obtain Lemma 1.12.

Proposition 1.13. *Suppose that* V *satisfies the assumption* (1) *with* $\delta_1=\varepsilon_1-1$ *of Proposition* 1.8. *Then we have*

$$\|T(\lambda)\|_p=O(|\lambda|^{n/2p-\delta_1})\ for\ \lambda\to-\infty;$$

$$\forall\varepsilon>0,\ \|T(\lambda)\|_p=O(|\lambda|^{n/2p-1/p+1/2+\varepsilon})\ for\ \lambda\to+\infty.$$

Furthermore if $\partial_{x_\cdot}V$ *satisfies the same assumption as* V *then the above estimates hold also for* $t_D(\lambda)$. *For the estimate* $\lambda\to+\infty$ *we assume* (1.2)$'$ *of Lemma* 1.11.

Proof As in Proposition 1.8 we write $T(\lambda)=T_1(\lambda)-T_2(\lambda)$. We have

$$\|T_1(\lambda)\|_p=O(\langle\lambda\rangle^{n/2p-\delta_1})\ \text{for}\ \lambda\leqslant 0$$

and

$$\|T_1(\lambda)\|_p=O(\langle\lambda\rangle^{-\delta_1/2}+\langle\lambda\rangle^{(n/2-1)/2p+\varepsilon})^2\ \text{for}\ \lambda\geqslant 0.$$

By applying Lemma 1.12, one has

$$T_2(\lambda)=\mathscr{F}_A(\lambda)V\rho^{-s}(x_1-\lambda)\cdot\rho^{-s}(x_1-\lambda)\cdot\rho^{s}(x_1-\lambda)(H-\lambda-io)^{-1}\cdot\rho^{s}(\chi_1-\lambda)\cdot\rho^{-s}(x_1-\lambda)\mathscr{F}_A(\lambda)^*.$$

Using Lemma 1.10 for $\lambda\leqslant 0$ and Lemma 1.11 for $\lambda\geqslant 0$ we get

$$\|T_2(\lambda)\|_p\leqslant C\cdot\|\mathscr{F}_A(\lambda)\cdot V\cdot\rho^{-s}(x_1-\lambda)\|_{2p}^2,$$

with C independent of λ.

Then for every $s>1/4$ we have

$$\|T_2(\lambda)\|_p = O(\langle\lambda\rangle^{n/2p+2s-2\delta_1}) \text{ for } \lambda\leqslant 0;$$
$$\|T_2(\lambda)\|_p = O((1+\langle\lambda\rangle^{\alpha_1}+\langle\lambda\rangle^{(n/2-1)/2p+s})^2\langle\lambda\rangle^{2s}) \text{ for } \lambda>0.$$

Putting together these estimates we prove Proposition 1.13. The corresponding estimates for $t_D(\lambda)$ are obtained via $\frac{dT}{d\lambda}$ and formulae (0.2), (1.5) by using Lemma 1.12.

Remark 1.14. If $\delta_1>\frac{n}{2}+\frac{1}{2}$ and V, $\partial_{x_1}V$ satisfy assumption (i) of Proposition 1.8 then Proposition 1.13 shows that $\lim_{\lambda\to-\infty} T(\lambda)=\lim_{\lambda\to-\infty} t_D(\lambda)=0$ in trace norm (see § 4).

§ 2. Mean Time-Delay-Scattering Phase and Local Spectral Density

In § 1 we have studied the time-delay operator and the time delay matrix $t_D(\lambda)$. If V and $\partial_{x_1}V$ satisfy

$$\begin{cases} |V(x_1,x')|+|\partial_{x_1}v(x_1,x')|\leqslant C\langle x_1\rangle^{1-\varepsilon_1}\langle x'\rangle^{-\delta'} \text{ for } x_1\geqslant 0, \\ |V(x_1,x')|+|\partial_{x_1}V(x_1,x')|\leqslant C\langle x_1\rangle^{-\delta_1}\langle x'\rangle^{-\delta'} \text{ for } x_1\leqslant 0 \end{cases} \tag{2.1}$$

with $\delta'>n-1$ and $\delta_1>(n+1)/2$ then for each $\lambda\in\mathbf{R}$, $t_D(\lambda)$ is of trace-class (Proposition 1.8) and we can define the mean time-delay

$$\langle t_D\rangle(\lambda)=\mathrm{tr}(t_D(\lambda))=-i\cdot\mathrm{tr}\Big(S^*(\lambda)\cdot\frac{ds}{d\lambda}\Big).$$

We know also from Proposition 1.8 that $\langle t_D(\lambda)\rangle$ is continuous on $\mathbf{R}$ (locally Holder).

Moreover if for $0\leqslant j\leqslant k+1$ we have

$$\begin{cases} |\partial^j_{x_1}V(x_1,x')|\leqslant C\langle x_1\rangle^{1-\varepsilon_1}\langle x'\rangle^{-\delta'} \text{ for } x_1\geqslant 0, \\ |\partial^j_{x_1}V(x_1,x')|\leqslant C\langle x_1\rangle^{-\delta_1}\langle x'\rangle^{-\delta'} \text{ for } x_1\leqslant 0, \end{cases} \tag{2.1)$_j$}$$

then $\langle t_D\rangle$ is C^k-smooth on $\mathbf{R}$.

To make connexion between $\langle t_D\rangle$ and the spectral shift function of Birman-Krein [B. K] we have to study compacity class for $(H-z)^{-k}-(H_0-z)^{-k}$.

Lemma 2.1. *Suppose* $n\geqslant 2$. *If* V *satisfies* (2.1) *then for real* $p\geqslant 1$, $k\geqslant 1, k\in\mathbf{N}$ *such that* $\delta'>\frac{n-1}{p}$, $\delta_1>\frac{n}{2p}$, $k>\frac{n+2}{2p}+1-\varepsilon_1$ *then* $(H-z)^{-k}-(H_0-z)^{-k}$ *is in the compacity class* C^p, *for every* $z\in\mathbf{C}\backslash\mathbf{R}$.

For proving Lemma 2.1 we use the following

Lemma 2.2. *Let us consider a measurable function* W *on* $\mathbf{R}^n_x\times\mathbf{R}^n_\xi$ *as in Lemma* 1.6. *If* $\delta'>\frac{n-1}{p}$, $\delta_1>\frac{n}{2p}$, $i>\frac{n}{2p}$ *and* $j>\frac{n+2}{2p}+\alpha_1$, *then the operator* $W(H_0-z)^{-j}$ *is in the Schatten class* C^p *for every* $z\in\mathbf{C}\backslash\mathbf{R}$.

Proof of Lemma 2.2 By using the same method as in the proof of Lemma 1.6 the result is easily obtained.

Proof of Lemma 2.1 Iterating the resolvant identity we have

$$(H-z)^{-1}-(H_0-z)^{-1}=(H_0-z)^{-1}\cdot\sum_{j=1}^{N}(-V(H_0-z)^{-1})^j$$
$$+(H-z)^{-1}(-V(H_0-z)^{-1})^{N+1}. \tag{2.2}$$

N is chosen big enough.

Taking $(k-1)$ derivatives in z we get

$$(H-z)^{-k}-(H_0-z)^{-k}$$
$$=\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\Big\{(H_0-z)^{-1}\cdot\sum_{j=1}^{N}(V(H_0-z)^{-1})^j+(H-z)^{-1}(-V(H_0-z)^{-1})^{N+1}\Big\}. \tag{2.3}$$

First we choose N big enough such that the second term on r. h. s. is in the Schatten C^p. The other terms are of the form

$$(H_0-z)^{-m}\prod_{1\leqslant j\leqslant n}(V(H_0-z)^{-q_j}),$$

$m+q_1+q_2+\cdots+q_r\geqslant k+r$, $1\leqslant q_j$, $1\leqslant m\leqslant k$.

So we have to compute compacity class of products like

$$(H_0-z)^{-m}\cdot V\cdot(H_0-z)^{-q_1}\cdot V(H_0-z)^{-q_2}\cdots V(H_0-z)^{-q_r}.$$

This can be done by using Lemma 2.2 and Holder inequality in Schatten classes. This finishes the proof of Lemma 2.1.

It results now, from Birman-Krein theory([B. K] extended to non semi-bounded operators), that we have

Lemma 2.3. *Suppose V satisfies* (2.1) *with $\delta'>n-1$ and $\delta_1>\frac{n}{2}$. Then*

i) *For every $f\in C_0^\infty(\mathbf{R})$, $f(H)-f(H_0)$ is of trace class;*

ii) *There exists $\xi\in L^1_{loc}(\mathbf{R})$ such that*

$$\mathrm{tr}(f(H)-f(H_0))=\int f'(\lambda)\cdot\xi(\lambda)d\lambda \textit{ for every } f\in C_0^\infty(\mathbf{R}).$$

Suppose furthermore that $\delta_1>\frac{n+1}{2}$. Then

iii) $\det S(\lambda)=e^{-2i\pi\xi(\lambda)}$ *for a. e.* $\lambda\in\mathbf{R}$.

Now we want to connect the spectral shift function ξ to the spectral resolution of H:

$$f(H)=\int f(\lambda)dE_H(\lambda)\ (\text{for every } f\in C_0^\infty(\mathbf{R})).$$

Lemma 2.4. *Under the assumptions* (2.1) *on V, $\left(\delta_1>\frac{n}{2}\text{ suffices here}\right)$ for every $f\in C_0^\infty(\mathbf{R})$, $\partial_{x_1}V\cdot f(H)$ is of trace class and wehave*

$$\mathrm{tr}(f(H)-f(H_0))=-\mathrm{tr}(\partial_{x_1}V\cdot f(H)).$$

Proof At first for k big enough, $\partial_{x_1}V(H_0+i)^{-k}$ is of trace class. Then using the trick of Lemma 2.1 we can prove easily that $\partial_{x_1}V(H+i)^{-k}$ is of trace class. So $\partial_{x_1}V\cdot f(H)$ is of trace class.

The relation we want to prove is clearly equivalent to

$$\operatorname{tr}([D_1,\ H]f(H)-[D_1,\ H_0]f(H_0))=0.$$

For every $\chi\in C_0^\infty(\mathbf{R}_n^x)$, $[\chi D_1,\ H]f(H)$ and $[\chi\cdot D_1,\ H_0]f(H_0)$ are of trace class. Cyclicity property of the trace gives

$$\operatorname{tr}([\chi D_1,\ H_0]f(H_0))=\operatorname{tr}([\chi D_1,\ H_0]f(H_0))=0.$$

Take now $\chi_R(x)=\chi\left(\frac{x}{R}\right)$ where $\chi\in C_0^\infty(\mathbf{R})$, $\chi\equiv1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. We have

$$\begin{aligned}0&=\operatorname{tr}([\chi_R\cdot D_1,\ H]f(H)-[\chi_R\cdot D_1,\ H_0]f(H_0))\\&=\operatorname{tr}(\chi_R\cdot([D_1,\ H]f(H)-[D_1,\ H_0]f(H_0))+\operatorname{tr}\{[\chi_R,\ H]D_1\cdot f(H))\\&\quad-[\chi_R,\ H_0]D_1f(H_0)\}.\end{aligned}$$

Continuity property of the trace gives

$$\lim_{R\to+\infty}\operatorname{tr}(\chi_R[D_1,\ H]f(H)-[D_1,\ H_0]f(H_0))=\operatorname{tr}([D_1,\ H]f(H)-[D_1,\ H_0)f(H_0)).$$

But we have

$$[\chi_R,H]=[\chi_R,\ H_0]=[\Delta,\ \chi_R]+\frac{1}{R^2}\chi_R''+\frac{2}{R}\chi_R'\cdot\frac{\partial}{\partial x_1}.$$

So we have to show that

$$\lim_{R\to+\infty}\operatorname{tr}([\Delta,\ \chi_R]\cdot D_1)(f(H)-f(H_0)))=0.$$

Choose N big enough and write

$$\begin{aligned}f(H)-f(H_0)&=(H+i)^{-N}g(H)-(H_0+i)^{-N}g(H_0)\\&=((H+i)^{-N}-(H_0+i)^{-N})g(H)+(H_0+i)^{-N}(g(H)-g(H_0)),\end{aligned}\tag{2.4}$$

where $g\in C_0^\infty(\mathbf{R})$. Then from (2.3) and (2.4) we obtain

$$f(H)-f(H_0)=(H_0+i)^{-1}\cdot K,\ K \text{ being of trace class.}$$

It results

$$[\Delta,\ \chi_R]D_1(f(H)-f(H_0))=[\Delta,\ \chi_R]D_1(H_0+i)^{-1}\cdot K\equiv L(R)\cdot K.$$

Using microlocal localization as in Lemma 1.6 we see easily that

$$\|L(R)\|\leqslant C\quad\forall R\geqslant1.$$

But $L(R)$ converges strongly on $L^2(\mathbf{R}^n)$ to 0 as $R\to+\infty$. So by a well known theorem about convergence of traces [4] we obtain

$$\lim_{R\to\infty}\operatorname{tr}(L(R).\ K)=0,$$

which ends the proof of lemma 2.4.

Proposition 2.5. (i) *Suppose V satisfies the assumption of Lemma 2.4.*

Then the derivative in the distributional sense of the spectral shift function ξ is given by

$$\xi'(\lambda)=\eta(\lambda)=\operatorname{tr}\left(\partial_xV.\ \frac{\partial E_H}{\partial\lambda}(\lambda)\right);$$

where η is by definition the distribution

$$\langle \eta, \varphi\rangle = \mathrm{tr}(\partial_{x_1} V\cdot \varphi(H)).$$

$$\left(\textit{Notice that a priori } \partial_{x_1}V\cdot \frac{\partial F_H}{\partial\lambda}(\lambda)\textit{ isnot of trace class } 1\right).$$

(ii) *Suppose furthermore $\delta_1>(n+1)/2$. Then, for every $1/2\geqslant s>1/4$, $\rho^{-s}\cdot\partial_{x_1}V\cdot \frac{\partial E_H}{\partial\lambda}\cdot\rho^s$ is of trace-class.*

(iii) *Suppose that $\delta_1>(n+1)/2$, $\delta'>n-1$ and V satisfies (2.1)$_j$ for every $j\geqslant 1$. Then $\frac{\partial E_H}{\partial\lambda}(\lambda,.,.)$ is C^∞ on $\mathbf{R}^n\times\mathbf{R}^n$, $\partial_{x_1}V(x)\cdot\frac{\partial E_H}{\partial\lambda}(\lambda, x, x)$ is integrable on $\mathbf{R}^n$ and we have*

$$\xi'(\lambda)=\int \partial_{x_1}V(x)\cdot\frac{\partial E_H}{\partial\lambda}(\lambda; x, x)dx.$$

In particular the spectral shift function ξ is continuously derivable on $\mathbf{R}$.

Proof At first recall the Stone formula

$$dE_H(\lambda)=(2i\pi)^{-1}(R(\lambda+io)-R(\lambda-io))d\lambda.$$

Write down

$$\rho^{-s}\partial_{x_1}V\frac{\partial E_H}{\partial\lambda}\rho^s=\rho^{-s}\cdot\partial_{x_1}V\cdot(H-\lambda+i)^{-N}\cdot\rho^{-s}\cdot\rho^s(H-\lambda+i)^N\frac{\partial E_H}{\partial\lambda}\rho^s$$

with N big enough. Using Proposition 1.1′, Remark 1.2 and Stone formula we have to prove first that

$$A=\rho^{-s}\cdot\partial_{x_1}V(H-\lambda+i)^{-N}\cdot\rho^{-s} \text{ is of trace class.}$$

We consider separately

$$A_1=\rho^{-s}\partial_{x_1}V(H_0-\lambda+i)^{-N}\rho^{-s},$$
$$A_2=\rho^{-s}(\partial_{x_1}V((H-\lambda+i)^{-N}-(H_0-\lambda+i)^{-N})\rho^{-s}$$

and

$$A_1'=\rho^{-s}\partial_{x_1}V\rho^{-s}(H_0-\lambda+i)^{-N},$$
$$A_1''=\rho^{-s}\partial_{x_1}V[(H_0-\lambda+i)^{-N}, \rho^{-s}].$$

A_1' is easily checked by the method of proof of Lemma 1.6.

For A_1'' we compute the commutator by derivating $(N-1)$ times the identity

$$[(H_0-\lambda+i)^{-1}, f]=(H_0-\lambda+i)^{-1}\left(2f'\frac{d}{dx}+f''\right)(H_0-\lambda+i)^{-1},$$

where $f(x_1)=\rho^{-s}(x_1)$. Then

$$(N-1)!\,[(H_0-\lambda+i)^{-N}, f]=\sum_{\substack{1\leqslant j,k\\ j+k=N+2}} k|j|(H_0-\lambda+i)^{-j}(2f'\partial/\partial_{x_1}+f'')(H_0-\lambda+i)^{-k}.$$

As in the proof of Lemma 1.6 we decompose into nine terms:

$$\begin{aligned}[(H_0-\lambda+i)^{-N}, f]&=g[(H-\lambda+i)^{-N}, f]g+(1-g)\gamma[\cdot,\cdot]g\\&+(1-g)(1-\gamma)[\cdot,\cdot]g+g[\cdot,\cdot](1-g)\gamma+(1-g)\gamma[\cdot,\cdot](1-g)\gamma\\&+(1-g)(1-\gamma)[\cdot,\cdot](1-g)\gamma+g[\cdot,\cdot](1-g)(1-\gamma)\\&+(1-g)g[\cdot,\cdot](1-g)(1-\gamma)+(1-g)(1-\gamma)[\cdot,\cdot](1-g)(1-\gamma),\end{aligned}$$

It is not difficult to prove that $\rho^{-s}\partial_{x_1}V\times$(each of the nine terms) is of trace class. In particular remark that $f'=O(\rho^{1-s})$ et $f''=O(\rho^{2-s})$ and on Supp$(1-\gamma)$ we have

$$f'(x_1)\cdot\xi_1=O(\rho^{1-s})=O(1) \text{ with the choice } \frac{1}{4}<s\leqslant\frac{1}{2}.$$

To check A_2 we use identity (2.3). Remark that each term ends by $V(H_0-\lambda+i)^{-1}$ ρ^{-s} and we write

$$V(H_0-\lambda+i)^{-1}\rho^{-s}=V\rho^{-s}(H_0-\lambda+i)^{-1}+V[(H_0-\lambda+i)^{-1}\rho^{-s}].$$

Each term on r. h. s is bounded. With this remark it is not difficult to see that A_2 is of trace class.

The third part of Proposition 2.5 comes from the following lemma.

Lemma 2.6. *Under the assumption of* (iii) *proposition* (2.5) *we have*

(i) $\dfrac{\partial E_H}{\partial\lambda}(\lambda,\cdot,\cdot)$ *is* C^∞ *on* $\mathbf{R}^n_x\times\mathbf{R}^n_y$.

(ii) *For every bounded interval* $I\subseteq\mathbf{R}$, *every real* $r>(n+1)/4$ *and every real* $k\geqslant 0$ *there exists* $C_{1,k,r}$ *such that*

$$\left|\frac{\partial E_H}{\partial\lambda}(\lambda,x,y)\right|\leqslant C_{1,k,r}\rho(x_1)\cdot\rho(y_1)^{-r}\rho(-x_1)^k\rho(-y_1)^k$$

for every $x=(x_1,x')\to\mathbf{R}_{x_1}\times\mathbf{R}^{n-1}_{x'}$, $y=(y_1,y')\in\mathbf{R}_{y_1}\times\mathbf{R}^{n-1}_{y'}$.

Proof of Lemma 2.6 We have, by the Stone formula,

$$\frac{\partial E_H}{\partial\lambda}=\frac{(-1)^N}{2\pi i}(H-\lambda+i)^{-N}(R_H(\lambda+io)-R_H(\lambda-io))(H-\lambda+i)^{-N}$$

with $R_H(\lambda\pm io)=(H-\lambda\mp io)^{-1}$. We have used $(H-\lambda+i)^N\dfrac{\partial E_H}{\partial\lambda}=(i)^N\dfrac{\partial E_H}{\partial\lambda}$.

Denoting $\rho^v(x_1)=\rho(-x_1)$ we have to estimate in $L^\infty(\mathbf{R}^n_x\times\mathbf{R}^n_y)$ the kernel of $F_\pm(\lambda)=\rho^{v-k}\cdot\rho^r(H-\lambda+i)^{-N}R_H(\lambda\pm io)(H-\lambda-i)^{-N}\rho^r\rho^{v-k}$.

From a theorem on kernel of operators[1] it is sufficient to show that $F_\pm(\lambda)$ is in the space $\mathscr{L}(H^{-s}(\mathbf{R}^n),H^s(\mathbf{R}^n))$ from some $s>n/2$. ($H^s(\mathbf{R}^n)$ is the usual Sobolev space). Clearly it suffices to prove this for $F(\lambda)=F_+(\lambda)$. As usual in this paper we decompose $F(\lambda)$ as

$$F(\lambda)=\rho^{v-k}\rho^r(H-\lambda+i)^{-N}\rho^{-\sigma}\cdot\rho^\sigma R_H(\lambda+io)\rho^\sigma(H-\lambda+i)^{-N}\cdot\rho^{v-k}\rho^r$$

for some $\sigma>1/4$. Then we have to prove that

$$G(\lambda)=\rho^{v-k}\cdot\rho^r(H-\lambda+i)^{-N}\rho^{-\sigma}\in\mathscr{L}(L^2(\mathbf{R}^n),H^s(\mathbf{R}^n))$$

for some N, $\sigma>1/4$, $s>n/2$ if $r>(n+1)/4$. For then $G^*(\lambda)\in\mathscr{L}(H^{-s}(\mathbf{R}^n),L^2(\mathbf{R}^n))$. We decompose $G(\lambda)$ in three terms: $G(\lambda)=G_1(\lambda)+G_2(\lambda)+G_3(\lambda)$.

$$G_1(\lambda)\equiv\rho^{v-k}\cdot\rho^r\cdot g(H-\lambda+i)^{-N}\cdot\rho^{-\sigma}.$$

The assumptions on V guaranties uniform ellipticity of $(H-\lambda+i)$ for $x_1>0$ (with weight $|\xi^2|+|x_1|+1$).

So taking N big enough we have

$$G_1(\lambda)\in\mathscr{L}(L^2(\mathbf{R}^n),H^s(\mathbf{R}^n)) \text{ with } s>n/2.$$

$G_2(\lambda) \equiv \rho^{v-k}\rho^r(1-g)\gamma(x, D)(H-\lambda+i)^{-N}\rho^{-\sigma}$.

For this case, as in [14] Lemma 2.7, we can construct a parametrix for $\gamma(x, D)(H-\lambda+i)^{-N}$. Taking N big enough and $r \geqslant \sigma > \frac{1}{4}$ we have clearly $G_2(\lambda) \in \mathscr{L}(L^2(\mathbf{R}^n), H^s(\mathbf{R}^n))$ with $s > \frac{n}{2}$.

$$\begin{aligned} G_3(\lambda) &\equiv \rho^{v-k}\cdot\rho^r(1-g)\cdot(1-\gamma(x, D))(H-\lambda+i)^{-N}\cdot\rho^{-\sigma} \\ &= \rho\ (1-g)(1-\gamma(x\cdot D))\rho^{-\sigma}(H-\lambda+i)^{-N} + \rho^r(1-g)(1-\gamma(x\cdot D))[(H-\lambda+i)^{-N}, \rho^{-\sigma}] \\ &\equiv G_3'(\lambda) + G_3''. \end{aligned}$$

For $G_3'(\lambda)$ we examine the symbol of $\rho^r(1-g)(1-\gamma(x, D))\rho^{-\sigma}$. On the support of $1-\gamma$ we have $|\xi|^2 \approx -x_1$. So the principal symbol of this operator is of order

$$\langle\xi\rangle^{2(\sigma-r)} \text{ uniformly in } x \in \mathbf{R}^n.$$

So $G_3''(\lambda) \in \mathscr{L}(L^2(\mathbf{R}^n), H^s(\mathbf{R}^n))$ for $s > n/2$ if $2(r-\sigma) > n/2$.

For $G_3''(\lambda)$ we use the commutator formula

$$(n-1)\,|\,[(H-\lambda+i)^{-N}, f] = \sum_{\substack{1 \leqslant j,k \\ j+k=N}} k|j|(H-\lambda+i)^{-j}(2f'\partial/\partial_{x_1} + f'')(H-\lambda+i)^{-k},$$

$f = \rho^{-\sigma}$. So this term is bounded on $L^2(\mathbf{R}^n)$ for $1/4 < \sigma < 1/2$ and we have also $G_3''(\lambda) \in \mathscr{L}(L^2(\mathbf{R}^n), H^s(\mathbf{R}^n))$.

To prove that $\frac{\partial E_H}{\partial\lambda}(\lambda)$ has a kernel of class $C^j(\mathbf{R}^n_x \times \mathbf{R}^n_y)$, applying the same theorem about kernel we have to prove that $G(\lambda) \in \mathscr{L}(L^2(\mathbf{R}^n), H^s(\mathbf{R}^n))$ for some $s > n/2 + j$. Then it is possible if $r > \frac{n+1}{4} + \frac{j}{2}$. We have in this case

$$\left| \frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial y^\beta}\cdot\frac{\partial E_H}{\partial\lambda}(\lambda, x, y) \right| \leqslant C\cdot\rho^{-r}(x_1)\rho^{-r}(y_1)\rho^k(-x_1)\rho^k(-y_1).$$

End of the proof of Proqosition 2.5 We want to prove the equality

$$\xi'(\lambda) = \int \partial_{x_1} V(x)\cdot\frac{\partial E_H}{\partial\lambda}(\lambda, x, x)\,dx$$

in the distributional sense. Denote by $\tilde{\xi}(\lambda)$ the right hand side.

From (ii) Proposition 2.5 we know that, for $\frac{1}{4} < s \leqslant \frac{1}{2}$, $\rho^{-s}\partial_{x_1} V\cdot\frac{\partial E_H}{\partial\lambda}\cdot\rho^s$ is of trace class and has a C^∞ kernel. So we have easily

$$\operatorname{tr}\left(\rho^{-s}\partial_{x_1} V\cdot\frac{\lambda E_H}{\partial\lambda}\rho^s\right) = \tilde{\xi}(\lambda).$$

For $\varphi \in C_0^\infty(\mathbf{R})$ we have by Lemma 2.4

$$\langle\varphi, \xi'\rangle = \operatorname{tr}(\partial_{x_1} V\cdot\varphi(H)) = \operatorname{tr}\left(\int \varphi(\lambda)\cdot\rho^{-s}\cdot\partial_{x_1} V\cdot\int \partial E_H \partial\lambda\cdot\rho^s d\lambda\right).$$

The continuity of trace gives $\langle\xi', \varphi\rangle = \int \varphi(\lambda)\cdot\tilde{\xi}(\lambda)\,d\lambda$,

This proves $\xi' = \tilde{\xi}$ and ends the proof of Proposition 2.5.

Remark 2.6. Using smoothness in the energy λ of boundary values of the resolvents $R_H(\lambda \pm io)$ (Proposition 1.1′ and Remark 1.2), we can prove with the method used in Lemma 2.6 that E_H is also C^∞ in λ with corresponding estimates for $\frac{\partial^j}{\partial\lambda^j}\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^\beta}{\partial x^\beta}E_H(\lambda, x, y)$.

Remark 2.7. From Proposition 2.5 and § 1 we have the following equalities

$$\langle t_D\rangle(\lambda) = -2\pi\cdot\xi'(\lambda) = -2\pi\int \partial_{x_1}V(x)\frac{\partial E_H}{\partial\lambda}(\lambda, x, x)dx.$$

§ 3. Asymptotic Trace Formulas

The spectral shift function is defined modulo a constant. We propose here, given $\lambda_0\in\mathbf{R}$, to compute asymptotic for $\xi(\lambda)-\xi(\lambda_0)$ as $\lambda\to\pm\infty$.

To be simple, tape $\lambda_0=0$. Then we define two functions $\xi_\pm(\lambda)$ by

$$\xi_+(\lambda)=\xi(\lambda)-\xi(0)=\int \partial_{x_1}V(x)E_H[0, \lambda](x; x)dx, \text{ for } \lambda\geqslant 0, \tag{3.1}$$

$$\xi_-(\lambda)=\xi(0)-\xi(-\lambda)=\int \partial_{x_1}V_1(x)\cdot E_H(-\lambda, 0](x, x)dx. \tag{3.2}$$

More generally we are interested in studying

$$\sigma_{W,f}(\lambda)=\operatorname{tr}\left(W, f\left(\frac{H}{\lambda}\right)\right) \quad \text{as } \lambda\to+\infty$$

for suitable $W:\mathbf{R}^n\to\mathbf{R}$ and $f:\mathbf{R}\to\mathbf{C}$, supp f compact or f small enough at $\pm\infty$. In all this paragraph V satisfies $(2.1)_j$ for every $j\geqslant 0$; W satisfies the assumptions $(+)$ and $(-)$ of Lemma 1.6 with: $\delta_1>n/2$, $\delta'>n-1$; W is continuous on $\mathbf{R}^n_x$.

As first we want to give some informations about $f(H)$, for a suitable class of smooth functions f. Then we shall proceed by smoothing and approximation.

Lemma 3.1. *Consider φ a complex function analytic in a conic neighborhood of the semi-real axis $[0, +\infty[$ defined by*

$$\Omega=\{z\in\mathbf{C}: |\operatorname{Im}z|\leqslant\varepsilon_0|\operatorname{Re}z|, \quad \operatorname{Re}z\geqslant 0\}\cup\{|z|\leqslant\varepsilon_1\}$$

with ε_0, $\varepsilon_1>0$. Suppose that thete exist $r>0$, $C>0$ such that

$$|\varphi(z)|\leqslant C\langle z\rangle^{-r} \quad \text{for } z\in\Omega.$$

Then $g\cdot\varphi(H^2)$ and $(1-g)\cdot\gamma_a(x, D)\cdot\varphi(H^2)$ are nice pseudodifferential operators. More precisely

i) *$g\cdot\varphi(H^2)=op(a_{\varphi,q})$ with*
$|\partial_x^\alpha\partial_\xi^\beta a_{\phi,q}(x, \xi)|\leqslant C_{\alpha\beta}(\langle\xi\rangle^2+\langle x_1\rangle)^{-2r}\langle\xi\rangle^{-|\beta|}$ *for all $(x, \xi)\in\mathbf{R}^n_x\times\mathbf{R}^n_\xi$.*

ii) *$(1-g)\cdot\gamma_a(x, D)\cdot\varphi(H^2)=op(a_{\varphi,\gamma})$ with*
$|\partial_x^\alpha\partial_\xi^\beta a_{\varphi,\gamma}(x, \xi)|\leqslant C_{\alpha\beta}\cdot\langle\xi\rangle^{-4r}\langle\xi\rangle^{-|\beta|}$ *for all $(x, \xi)\in\mathbf{R}^n_x\times\mathbf{R}^n_\xi$.*

Proof The method is standard in analytic functional calculus. By the method already used in Lemmas 2.6—2.7 of[14] we prove Lemma 3.1 for $\varphi(u)=(u-z)^{-1}$, $z\in\mathbf{C}/\mathbf{R}$ with control in z for $|z|\to+\infty$ such that $|\operatorname{Im}z|=\varepsilon_0|\operatorname{Re}z|$. Then using the

Cauchy formula we get Lemma 3.1 for the general case

$$\varphi(H^2)=\frac{i}{2\pi}\int_{\partial\Omega}\varphi(z)(H^2-z)^{-1}dz$$

(For details about estimates used, see for example [15] or [3]).

Remark 3.2. As usual the restriction $r>0$ in Lemma 3.1 is not essential; we can drop it by writing $\varphi(H^2)=\langle H\rangle^{2k}\cdot\langle H\rangle^{-2k}\cdot\varphi(H^2)$ with some integer $k>r$.

Lemma 3.3. *Consider a function f which can be written as $f(u)=p(u)\cdot\varphi(u^2)$ where p is a polynomial of degree $m\geqslant 0$, φ an analytic function as in Lemma 3.1, (We shall say then that $f\in\mathscr{K}_m^r$). Suppose $2r-m>(n+2)/2+\alpha_1$ and $2r-m>n/2$. Then $W.f(H)$ is of trace class.*

Proof The proof is easily done by using the partition of unity $1=g+(1-g)\cdot\gamma(x,\xi)+(1-g)(1-\gamma(x,\xi))$ and Lemma 3.1 (see § 1.2).

Lemma 3.4. *Consider $f\in\mathscr{K}_m^r$ with $2r-m>(n+2)/2+\alpha_1$, $2r-m>n/2$. Then we have*

$$\operatorname{tr}\Big(W(1-g)(1-\gamma(x,D))f\Big(\frac{H}{\lambda}\Big)\Big)=O(1)\text{ as }\lambda\to+\infty.$$

Proof

$$\Big|\operatorname{tr}\Big(W(1-g)(1-\gamma(x,D))f\Big(\frac{H}{\lambda}\Big)\Big)\Big|\leqslant\|W(1-g)\cdot(1-\gamma(x,D))\|_{Tr}<+\infty$$

by the proof of Lemma 1.6.

We want now to compute asymptotics for

$$F(\lambda)=\operatorname{tr}\Big(W\cdot g\cdot f\Big(\frac{H}{\lambda}\Big)\Big)+\operatorname{tr}\Big(W(1-g)\cdot\gamma(x\cdot D)f\Big(\frac{H}{\lambda}\Big)\Big)=F_1(\lambda)+F_2(\lambda).$$

It results from [3], § 3 (specially Corollary (3.17)) that we can obtain asymptotic expansions in $\lambda\to+\infty$ for the pseudodifferential operators $g\cdot f\left(\frac{H}{\lambda}\right)$ and $(1-g)\cdot\gamma(x\cdot D)\cdot f\left(\frac{H}{\lambda}\right)$. More precisely on the supports of g and $(1-g)\gamma(x,D)\cdot f\left(\frac{H}{\lambda}\right)$ behaves like a pseudodifferential operator with a symbol of the following form

$$\sum_{k=0}^{N}\lambda^{-k}\cdot f^{(k)}\Big(\frac{x_1+|\xi|^2+V}{\lambda}\Big)\cdot a_k(x,\xi)+\lambda^{-N-1}\cdot r_N(\lambda;x,\xi)\qquad(*)$$

with uniform estimates in λ for $r_N(\lambda)$ in symbol spaces; a_k can be computed (see for example [3]). In particular $a_0\equiv 1$.

So we have to estimate

$$J_1(\lambda)=\iint W(x_1,x')g(x_1)f\Big(\frac{x_1+|\xi|^2+V(x)}{\lambda}\Big)dx\,d\xi,$$

$$J_2(\lambda)=\iint W(x_1,x')(1-g)\cdot\gamma(x,\xi)f\Big(\frac{x_1+|\xi|^2+V(x)}{\lambda}\Big)dx\,d\xi.$$

Lemma 3.5. *Suppose $f\in\mathscr{K}_m^r$ with $2r-m>n/2$. Then*

(1) *If $\alpha_1<-1$, we have for $\lambda\to+\infty$*

$$J_1(\lambda)=\lambda^{n/2}\left(\int W(x)g(x_1)dx\right)\left(\int f(|\eta|^2)d\eta\right)+O(\lambda^{n/2-a}),$$

$$J_2(\lambda)=\lambda^{n/2}\left(\int W(x)(1-g)(x_1)dx\right)\left(\int f(|\eta|^2\right)d\eta+O(\lambda^{n/2-a})$$

for some $a>0$.

If $\alpha_1\geqslant -1$, *we assume furthermore*

$$\lim_{x_1\to+\infty} x_1^{-\alpha} W(x_1, x')=1(x') \text{ in } L^1(\mathbf{R}^{n-1}_{x'}).$$

Then

(2) *If* $\alpha_1=-1$,

$$J_1(\lambda)=\lambda^{n/2}\ln(\lambda)\left(\int f(|\eta|^2)d\eta\right)\left(\int 1(x')dx'\right)+O(\lambda^{n/2}),\quad \lambda\to\infty,$$

(3) *If* $\alpha_1>-1$,

$$J_1(\lambda)=\lambda^{n/2+1+\alpha_1}\left(\iint_{u\geqslant 0} u^{\alpha_1}f(u+|\eta|^2)du\, d\eta\right)\left(\int 1(x')dx\right)+o(\lambda^{n/2+1+\alpha_1}),$$

$\lambda\to+\infty$.

Proof (1) Changing of variables $\xi=\sqrt{\lambda}\,\eta$ gives

$$J_1(\lambda)=\lambda^{n/2}\int W(x_1, x')\cdot g(x_1)f\left(|\xi|^2+\frac{x_1+V(x_1, x')}{\lambda}\right)dx\,d\xi.$$

For every $s\in]0, 1]$, there exists C_s such that

$$\left|f\left(|\xi|^2+\frac{x_1+V(x_1, x')}{\lambda}\right)-f(|\xi|^2)\right|\leqslant C_s\cdot\left|\frac{x_1+V(x_1, x')}{\lambda}\right|^s\cdot\langle\xi\rangle^{2(m-2r)} \quad (3.3)$$

for every $x_1\geqslant -1$, $x'\in\mathbf{R}^{n-1}$, $\xi\in\mathbf{R}^n$.

Now choosing s small enough such that $s>\delta_1-1$ we get the asymptotic for $J_1(\lambda)$. $J_2(\lambda)$ is checked in the same manner, remembering that:

$$1-\gamma(x_1+\xi)=\theta\left(\frac{x_1+|\xi|^2+1}{1+\xi|^2}\right)$$

with $\theta\in C_0^\infty\left(\left]-\frac{1}{2}, \frac{1}{2}\right[\right)$, $\theta\equiv 1$ on $\left[-\frac{1}{3}, \frac{1}{3}\right]$ (see([14]).

(2) and (3): It is not difficult to see that we have to estimate

$$\mathcal{J}(\lambda)=\int_{R^n}\int_{x_1\geqslant A} x_1^{-\alpha_1}f\left(\frac{x_1+|\xi|^2}{\lambda}\right)dx_1\,d\xi$$

for every $A>0$.

We have

$$\mathcal{J}(\lambda)=\lambda^{n/2+1+\alpha_1}\iint_{u\geqslant A/\lambda} u^{\alpha_1}f(u+|\eta|^2)du\,d\eta.$$

If $\alpha_1=-1$, integrating by parts gives

$$\mathcal{J}(\lambda)=\lambda^{n/2}\cdot\ln(\lambda)\int(|\eta|^2)d\eta+O(\lambda^{n/2}).$$

If $\alpha_1>-1$, we have clearly

$$\mathcal{J}(\lambda)=\lambda^{n/2+1+\alpha_1}\iint_{u\geqslant 0} u^{\alpha_1}f(u+|\eta|^2)du\,d\eta.$$

This proves (2) and (3) of Lemma 3.5.

Summing up the results obtained above we have proved

Theorem 3.6. *For every $f\in\mathscr{K}_r^m$ with $2r-m>n/2+1+\alpha_1$, $2r-m>n/2$, we have*

(1) *If $\alpha_1<-1$,*

$$\operatorname{tr}\left(W,\ f\left(\frac{H}{\lambda}\right)\right)=(2\pi)^{-n}\lambda^{n/2}\left(\int W(x)dx\right)\left(\int f(|\eta|^2d\eta\right)+o(\lambda^{n/2}),\quad \lambda\to+\infty.$$

For $\alpha_1\geqslant-1$ suppose furthermore

$$\lim_{x_1\to\infty}x_1^{-\alpha_1}W(x_1,x')=1(x')\text{in } L^1(\mathbf{R}_{x'}^{n-1}).$$

(2) *If $\alpha_1=-1$,*

$$\operatorname{tr}\left(W\cdot f\left(\frac{H}{\lambda}\right)\right)=(2\pi)^{-n}\lambda^{n/2}\cdot\ln(\lambda)\left(\int f(|\eta|^2)d\eta\right)\left(\int 1(x')dx'\right)+o(\lambda^{n/2}\ln\lambda);$$

(3) *If $\alpha_1>-1$,*

$$\operatorname{tr}\left(W\cdot f\left(\frac{H}{\lambda}\right)\right)=(2\pi)^{-n}\cdot\lambda^{n/2+1+\alpha_1}\cdot\iint_{u\geqslant0}u^{\alpha_1}\cdot f(u+|\eta|^2)du\,d\eta$$
$$+o(\lambda^{n/2+1+\alpha_1}),\quad \lambda\to+\infty.$$

Corollary 3.7. *Theorem 3.6 holds for every $f\in\mathscr{S}(\mathbf{R})$ (Schwartz space).*

Proof Denote by p_m the m^{th} Hermite polynomial of degree m. Clearly we have

$$p_m(u)\cdot e^{-u^2/2}\in\bigcap_{r>0}\mathscr{K}_r^m,$$

$\{p_m(u)e^{-u^2/2}\}_{m\geqslant0}$ is a total set in $\mathscr{S}(\mathbf{R}^n)$. In particular fixing $N>0$ big enough there exist $s_j>0$, $\lim\limits_{j\to0}s_j=0$ and polynomials q_j of degree m_j such that

$$|f(u)-q_j(u)\cdot e^{-u^2/2}|\leqslant s_j(1+u^2)^{-N}\text{ for every } u\in\mathbf{R}. \tag{3.4}$$

It is sufficient to prove Corollary (3.7) for $W\geqslant0$.

In this case we can write

$$\operatorname{tr}\left(W\cdot f\left(\frac{H}{\lambda}\right)\right)=\operatorname{tr}\left(\sqrt{W}\cdot f\left(\frac{H}{\lambda}\right)\cdot\sqrt{W}\right)$$

and from (3.4) we have

$$\operatorname{tr}\left(\sqrt{W}f_{j,N}^-\left(\frac{H}{\lambda}\right)\sqrt{W}\right)\leqslant\operatorname{tr}\left(\sqrt{W}\cdot f\left(\frac{H}{\lambda}\right)\sqrt{W}\right)\leqslant\operatorname{tr}\left(\sqrt{W}f_{j,N}^+\left(\frac{H}{\lambda}\right)\cdot\sqrt{W}\right)\quad(3.5)$$

$$\text{with, } f_{j,N}^{\pm}(u)=q_j(u)e^{-u^2/2}\pm\frac{s_j}{(1+u^2)^N}.$$

Consider the case (1): $\alpha_1<-1$.

Multiplying (3.5) by $\lambda^{-n/2}$ and making $\lambda\to+\infty$ we get

$$(2\pi)^{-n}\left(\int W(x)dx\right)\left(\int f_{j_N}^-(|\eta|^2)d\eta\right)\leqslant\lim_{\lambda\to+\infty}\lambda^{-n/2}\operatorname{tr}\left(\sqrt{W}\cdot f\left(\frac{H}{\lambda}\right)\sqrt{W}\right);$$

$$\lim_{\lambda\downarrow+\infty}\lambda^{-n/2}\operatorname{tr}\left(\sqrt{W}f\left(\frac{H}{\lambda}\right)\sqrt{W}\right)\leqslant(2\pi)^{-n}\cdot\left(\int W(x)dx\right)\cdot\int f_{j,N}^+(|\eta|^2)d\eta.$$

But we have

$$0\leqslant\int(f_{j,N}^+(|\eta|^2)-f_{j,N}^-(|\eta|^2))d\eta\leqslant2s_j\int(1+|\eta^4|)^{-N}d\eta.$$

So making $j\to+\infty$ we obtain (1) for f.|

Cases (2) and (3) are checked in the same way.

Corollary 3.8. *Define* $\sigma_W^{+}(\lambda)=\mathrm{tr}(W\cdot E_H[0, \lambda])$, $\lambda>0$. *Then the conclusions of Theorem* (3.6) *hold for* $f(u)=1_{[0,1]}(u)$ *where* $1_{[0,1]}$ *is the characteristic function of the interval* $[0, 1]$.

For example in case $\alpha_1<-1$ *we find*

$$\lim_{\lambda\to+\infty}\lambda^{-n/2}\cdot\sigma_W^{+}(\lambda)=\omega_n\cdot\int W(x)dx,$$

ω_n *being the volume of the unit ball in* $\mathbf{R}^n$.

Proof We approach $1_{[0,1]}$ by smooth functions with compactsupports below and above.

Remark that for $W=\partial_{x_1}V$ we have $\int\partial_{x_1}Vdx=0$ so Corollary 3.8 does not give the leading term for $\xi_+(\lambda)$ as $\lambda\to+\infty$. To solve this problem we have to check the second term in the asymptotic of $\mathrm{tr}\left(W\cdot f\left(\frac{H}{\lambda}\right)\right)$.

Theorem 3.9. *Suppose* W *satisfies*$(\pm)$ *of Lemma* 1.6 *with* $-\alpha_1>k+1$, $\delta_1>k+1$, $k\in\mathbf{N}$, $0\leqslant k<n/2$. *Then we have, for every* $f\in\mathscr{K}_m^r$ *with* $2r-m>n/2$,

$$\mathrm{tr}\left(W\cdot f\left(\frac{H}{\lambda}\right)\right)=\sum_{k=0}^{k}\lambda^{n/2-j}C_j(f, W)+O(\lambda^{n/2-k-\mu_0})\ \ as\ \ \lambda\to+\infty$$

for some $\mu_0>0$. *In particular*

$$C_0(f, W)=(2\pi)^{-n}\cdot\left(\int W(x)dx\right)\left(\int f(|\eta|^2)d\eta\right),$$

$$C_1(f, W)=(2\pi)^{-n}\cdot\left(\int x_1W(x_1, x')dx_1dx'f'\right)\left(\int (|\eta|^2)d\eta\right)$$

$$+\ (2\pi)^{-n}\cdot\left(\int W(x)V(x)dx\cdot\int f(|\eta|^2)d\eta.\right.$$

Proof Under our assumptions we remark that we can use Taylor formula in (*) around the point $|\xi|^2/\lambda$ for $f^{(j)}((x_1+|\xi|^2+V(x))/\lambda)$ until the order k, because we have $\int|x_1^lW(x_1, x')|dx_1dx'<+\infty$ for $0\leqslant l\leqslant k$. With this remark we can prove Theorem 3.9 in the same way as Theorem 3.6, (1), (See also Lemma 3.5).

Theorem 3.10. *Suppose* $n\leqslant3$, V *satisfies* (2.1)$_j$ *for every* j *and*

$$|\partial_{x_1}V(x_1, x')|\leqslant C\langle x'\rangle^{-\delta'}\langle x_1\rangle^{\alpha_1},\ with\ \alpha_1<-2,\ for\ x\geqslant0;$$

$$|\partial_{x_1}V(x_1, x')|\leqslant C\langle x'\rangle^{-\delta'}\langle x_1\rangle^{\delta_1},\ with\ \delta_1>2,\ and\ \delta_1>n/2\ for\ x_1<0,$$

where $\delta'>n-1$. *Then we have*

$$\lim_{\lambda\uparrow+\infty}\lambda^{1-n/2}\xi_+(\lambda)=\frac{(2\pi)^{-n}}{2}\cdot\sigma_n\cdot\int V(x)dx,$$

where σ_n *is the aera of unit sphere* S^{n-1}.

Proof We begin by applying Theorem 3.9 to $W=\partial_{x_1}V$ with $k=1$ and $f\in\mathscr{K}_m^r$. Computing the constant we find

$$\lim_{\lambda\to+\infty}\lambda^{1-n/2}\cdot\operatorname{tr}\Big(\partial_{x_1}V\cdot f\Big(\frac{H}{\lambda}\Big)\Big)=\Big(\frac{n-2}{4}\Big)(2\pi)^{-n}\sigma_n\Big(\int V(x)dx\Big)\Big(\int f(u)u^{n/2-2}du\Big). \quad (3.6)$$

Now with (3.6) we end the proof exactly in the same way as that for the proof of Corollary 3.7.

Now we give a result for $\xi_-(\lambda)$.

Theorem 3.11. *Suppose V and $\partial_{x_1}V$ satisfy $(2.1)_0$ and $(2.1)_1$ with $\delta_1=s_1-1>(n+1)/2$.*

Then

(i) $\lim_{\lambda\to+\infty}\xi_-(\lambda)=\xi_-(+\infty)$ *exists;*

(ii) $\rho^{-s}\cdot\partial_{x_1}V\cdot E_H(]-\infty,\lambda])\rho^s$ *is of trace class for every $\lambda\in\mathbf{R}$ and we have*

$$\int_{-\infty}^{\lambda}\xi'(\mu)d\mu=-\frac{1}{2\pi}\int_{-\infty}^{\lambda}\langle t_D\rangle(\mu)d\mu=\operatorname{tr}(\rho^{-s}\partial_{x_1}V\cdot E_H]-\infty,\lambda]\rho^s)$$
$$=\int_{\mathbf{R}^n}\partial_{x_1}V\cdot E_H(\lambda,x,x)dx.$$

Proof We have seen in § 2 that

$$\xi'(\lambda)=\int\partial_{x_1}V\cdot\frac{\partial E_H}{\partial\lambda}(\lambda,x,x)dx=\operatorname{tr}\Big(\rho^{-s}\cdot\partial_{x_1}V\cdot\frac{\partial E_H}{\partial\lambda}\cdot\rho^s\Big),$$
$$\text{for }\frac{1}{4}<s\leqslant\frac{1}{2},\ \text{(Proposition 2.5)}.$$

Using the ideas of the proof of Proposition 1.13 we can show

$$\|\rho^{-s}\cdot\partial_{x_1}V\cdot\frac{\partial E_H}{\partial\lambda}\cdot\rho^s\|_{Tr}=O(|\lambda|^{-\delta_1}),\ \lambda\to-\infty. \quad (3.7)$$

Then $|\xi'(\partial)|\leqslant C|\partial|^{-\delta_1}$ for every $\lambda\leqslant 0$.

This gives (i) $-\Big(\delta_1>\frac{n+1}{2}\geqslant 1\Big)$.

(3.7) gives also that $\rho^{-s}\partial_{x_1}V\cdot E_H[-\infty,\lambda]\rho^s_\lambda$ is of trace class and

$$\operatorname{tr}\Big\{\int_{-\infty}^{\lambda}\Big(\rho^{-s}\cdot\partial_{x_1}V\cdot\frac{\partial E_H}{\partial\mu}\rho^s\Big)d\mu\Big\}=\int_{-\infty}^{\beta}\operatorname{tr}\Big(\rho^{-s}\partial_{x_1}V\,\frac{\partial E_H}{\partial\mu}\,\rho^s\Big)d\mu$$
$$=\operatorname{tr}(\rho^{-s}\cdot\partial_{x_1}V\cdot E_H]-\infty,\lambda]\cdot\rho^s).$$

This proves (ii) by Remark 2.7.

Remark 3.12. With Remark 1.14 we can see that a natural choice for ξ is to take $\xi(-\infty)=0$ (mod. $\mathbf{Z}$).

§ 4. Asymptotics of Mean Time-Delay

In this section we study the asymptotics for the trace of on-shell time-delay operator $t_D(\lambda)$, which is equal to the first derivative of scattering phase. To simplify notations, we assume that $V\in C^{n+1}(\mathbf{R}^n)$ and

$$|\partial_x^\alpha V(x)|\leqslant C\langle x_1\rangle^{-\delta_1-\alpha_1}\langle x'\rangle^{-\delta'},\ x=(x_1,x')\in\mathbf{R}'\times\mathbf{R}^{n-1},$$
$$\alpha\in\mathbf{N}^n,\ |\alpha|\leqslant n+1 \quad (4.1)$$

with $\delta_1>(n+1)/2$, $\delta'>n-1$. Then it is known (see §§ 1 and 2) that $t_D(\lambda)$ is of trace class on $L^2(\mathbf{R}^{n-1})$, for every $\lambda\in\mathbf{R}$ and we have

$$\langle t_D\rangle(\lambda) = -2\pi\int_{\mathbf{R}^n}\partial_{x_1}V(x)\frac{\partial e}{\partial\lambda}(x,\, x;\, \lambda)dx, \tag{4.2}$$

where $\langle t_D\rangle(\lambda)=\mathrm{tr}\,t_D(\lambda)$ and $\frac{\partial e}{\partial\lambda}$ is the local spectral density for $H=-\Delta+x_1+V(x)$. These high energy asymptotics for the mean time-delay $\langle t_D\rangle(\lambda)$ are closely related to those of $\frac{\partial e}{\partial\lambda}$. In this section, we content ourselfves with establishing some asymptotic formulae for $\langle t_D\rangle(\lambda)$ as $\lambda\to+\infty$.

We begin by giving estimates for $\partial E_H/\partial\lambda$ as $\lambda\to-\infty$.

Proposition 4.1. *Let r_1, $r_2>1/4$. Then under the assumptions (4.1), for any $\varepsilon>0$, there exists $C>0$ such that*

$$\|\langle x_1\rangle^{-r_1}\frac{\partial E}{\partial\lambda}(\lambda)\langle x_1\rangle^{-r_2}\|\leqslant C\langle\lambda\rangle^{1+\varepsilon-r_1-r_2},\ for\ every\ \lambda<0, \tag{4.3}$$

Here $E(\lambda)=E_H(\lambda)$.

Proof By Lemma 1.10, one has for every $s>1/2$,

$$\|\langle D_1\rangle^{-s}R(\lambda\pm io)\rangle D_1\rangle^{-s}\|\leqslant C_s; \tag{4.4}$$

$$\|\rho(x_1-\lambda)^{-s/2}R(\lambda\mp io)\rho(x_1-\lambda)^{-s/2}\|\leqslant C,\quad \text{uniformly for } \lambda<0. \tag{4.5}$$

Now put $R_1(\lambda)=(P-\lambda)^{-1}$, for $\lambda<\lambda_0$, where $P=-\Delta+V(x)$, $\lambda_0=\inf\sigma(P)$.

Using resolvent equations we get easily by induction the formula

$$R=\sum_{0\leqslant l\leqslant j;\,0\leqslant m\leqslant k-1}(R_1x_1)^l\cdot R_1\cdot(x_1R_1)^m+(R_1x_1)^j\cdot R(x_1R_1)^k,$$

where $R=(H-z)^{-1}$, $R_1=(P-z)^{-1}$, $z\in\mathbf{C}/\mathbf{R}$.

From this we get

$$\frac{\partial E}{\partial\lambda}(\lambda)=(-1)^{j+k}(R_1(\lambda)x_1)^j\frac{\partial E}{\partial\lambda}(\lambda)\,(x_1R(\lambda_1))^k$$

$$\text{for any, } j,\ k\in\mathbf{N}. \tag{4.5}$$

From the estimate $\|\langle x_1\rangle^{-j}(R_1(\lambda)x_1)^j\langle D_1\rangle\|\leqslant C_{j,s}\langle\lambda\rangle^{-j+(s/2)}$, $\lambda<\lambda_0$, $j\geqslant1$, we derive easily

$$\left\|\langle x_1\rangle^{-j}\frac{\partial E}{\partial\lambda}(\lambda)\langle x_1\rangle^{-k}\right\|\leqslant C\langle\lambda\rangle^{-j-k+(1/2)+\varepsilon},\ \lambda\leqslant\lambda_0.$$

From (4.4), it follows that for every $s>1/4$,

$$\|\langle x_1\rangle^{-s}R(\lambda\pm io)\langle x_1\rangle^{-s}\|\leqslant C_\varepsilon\langle\lambda\rangle^{(1/2)+\varepsilon},$$

(4.3) can be obtained by interpolations.

Now we want to give an approximation for the local spectral density $\frac{\partial e}{\partial\lambda}$, which is the distributional kernel, restricted to the diagonal, for $\frac{\partial E}{\partial\lambda}$. From now on, denote $R_0(z)=(H_0-z)^{-1}$, $H_0=-\Delta+x_1$, and $\frac{\partial E_0}{\partial\lambda}(\lambda)=\frac{1}{2\pi i}[R_0(\lambda+io)-R^0(\lambda-io)]$. Then one has

$$\frac{\partial E}{\partial \lambda}=\frac{\partial E_0}{\partial \lambda}-\frac{\partial E}{\partial \lambda}VR_0(\lambda+io)+R(\lambda-io)V\frac{\partial E_0}{\partial \lambda}. \tag{4.6}$$

As in § 2, we can show that $\langle x_1\rangle^s\partial_{x_1}V\frac{\partial E}{\partial \lambda}\langle x_1\rangle^{-s}$, $s>\frac{1}{4}$, is of trace class in $L^2(\mathbf{R}^n)$.

From (4.5) it sollows that

$$\begin{aligned}\langle t_D\rangle(\lambda)&=-2\pi\operatorname{tr}\left(\langle x_1\rangle^s\partial_{x_1}V\frac{\partial E}{\partial \lambda}(\lambda)\langle x_1\rangle^{-s}\right)\\&=-2\pi\int_{\mathbf{R}^n}\partial_{x_1}V(x)\frac{\partial E_0}{\partial \lambda}(x,x;\lambda)dx+r(\lambda),\end{aligned} \tag{4.7}$$

where $\frac{\partial E_0}{\partial \lambda}(\cdot,\cdot;\lambda)$ is distributional kernel of $\frac{\partial E_0}{\partial \lambda}(\lambda)$ and

$$\begin{aligned}r(\lambda)=2\pi\Big\{&\operatorname{tr}\langle x_1\rangle^s\partial_{x_1}V\frac{\partial E}{\partial \lambda}(\lambda)VR_0(\lambda+io)\langle x_1\rangle^{-s}\\&-\operatorname{tr}\langle x_1\rangle^s\partial_{x_1}VR(\lambda-io)V\frac{\partial E_0}{\partial \lambda}(x)\langle x_1\rangle^{-s}\Big\}.\end{aligned} \tag{4.8}$$

Lemma 4.2. *Under the assumptions of (4.1), let $1/4<s<\delta_1-n/2-1/4$. Then for any $\varepsilon>0$ one has*

$$\left\|\langle x_1\rangle^s\partial_{x_1}V\frac{\partial E}{\partial \lambda}(\lambda)VR_0(\lambda\pm io)\langle x_1\rangle^{-s}\right\|_{\mathrm{tr}}\leqslant C_c\langle\lambda\rangle^{n/2+2+c-2\delta_1},\quad \lambda<0; \tag{4.9}$$

$$\left\|\langle x_1\rangle^sV\frac{\partial E_0}{\partial \lambda}(\lambda)\partial_xVR(\lambda\pm io)\langle x_1\rangle^{-s}\right\|_{\mathrm{tr}}\leqslant C_c\langle\lambda\rangle^{n/2+2+c-2\delta_1},\quad \lambda<0. \tag{4.10}$$

Here $\|\cdot\|_{\mathrm{tr}}$ denotes trace norm of operators in $L^2(\mathbf{R}^n)$.

Proof By Proposition 4.1, one has

$$\begin{aligned}&\left\|\langle x_1\rangle^s\partial_1V\frac{\partial E}{\partial \lambda}(\lambda)VR_0(\lambda\pm io)\langle x_1\rangle^{-s}\right\|_{\mathrm{tr}}\leqslant C\langle\lambda\rangle^{1/2+\varepsilon}\left\|\langle x_1\rangle^s\partial_1V\frac{\partial E}{\partial \lambda}(\lambda)V\langle x_1\rangle^s\right\|_{\mathrm{tr}}\\&\leqslant C'\langle\lambda\rangle^{1/2+\varepsilon}\left\|\langle D\rangle^r\langle x_1\rangle^{s-\delta_1}\frac{\partial E}{\partial \lambda}(\lambda)\langle x_1\rangle^{s-\delta_1}\right\|.\end{aligned}$$

Here $n<r<2\delta_1-1$ and $\langle D\rangle=\left(1<\sum_{j=1}^n|D_j|^2\right)^{1/2}$. We claim that for any $r\in[0,n+1]$, $s_1>r/2+1/4$,

$$\left\|\langle D\rangle^r\langle x_1\rangle^{-s_1}\frac{\partial E}{\partial \lambda}(\lambda)\langle x_1\rangle^{-s_2}\right\|\leqslant C\langle\lambda\rangle^{r/2+1+c-s_1-s_2},\ \lambda<0, \tag{4.11}$$

It suffices to prove (4.11) for $r\in\mathbf{N}$. Then an interpolation gives the desired result. We prove (4.11) by an induction. For $r=0$, (4.11) follows from proposition 4.1. Now suppose (4.11) to be true for any $0\leqslant r\leqslant k$. Let $r=k+1$. Remark that

$$(H-\lambda+i)^{-1}\frac{\partial E}{\partial \lambda}(\lambda)=i^{-1}\frac{\partial E}{\partial \lambda}(\lambda).$$

We can write

$$\begin{aligned}\langle x_1\rangle^{-s_1}\frac{\partial E}{\partial \lambda}(\lambda)&=i(H-\lambda+i)^{-1}\langle x_1\rangle^{-s_1}\frac{\partial E}{\partial \lambda}(\lambda)\\&+(H-\lambda+i)^{-1}(C_1D_1\langle x_1\rangle^{-s_1-1}+C_2\langle x_1\rangle^{-s_1-2})\frac{\partial E}{\partial \lambda}(\lambda),\end{aligned}$$

where C_1 and C_2 are some constants. Making use of the estimate

$$\|\langle D\rangle^r (H-\lambda+i)^{-1}\langle D\rangle^{-r+2}\langle x_1\rangle^{-1}\| \leqslant C\langle\lambda\rangle,\ \lambda\in R,\ r\leqslant n+1,$$

which can be easily proved under the condition (4.1) by commutator method, we obtain

$$\begin{aligned}&\left\|\langle D\rangle^{k+1}\langle x_1\rangle^{-s_1}\frac{\partial E}{\partial\lambda}(\lambda)\langle x_1\rangle^{-s_2}\right\|\\ &\leqslant C'\langle\lambda\rangle\langle D\rangle^{k-1}\left\{\left\|\langle x_1\rangle^{-s_1+1}\frac{\partial E}{\partial\lambda}(\lambda)\langle x_1\rangle^{-s_2}\right\|\right.\\ &\quad\left.+\left\|\langle D\rangle^{k}\langle x_1\rangle^{-s_1}\frac{\partial E}{\partial\lambda}(\lambda)\langle x_1\rangle^{-s_2}\right\|\right\}\\ &\leqslant C''\langle\lambda\rangle^{k+1+c-s_1-s_2},\quad \lambda<0.\end{aligned}$$

This proves (4.11) by induction. (4.9) follows immediately. (4.10) can be proved in the same way.

The main result of this section is the following

Theorem 4.3. *Let* (4.1) *be satified. Define* $F(\cdot)$ *by*

$$F(x_1)=\partial_{x_1}V(\partial_1),\ \text{if } n=1,\ (x=x_1)$$

and

$$F(x_1)=\int_{\mathbf{R}^{n-1}}\partial_{x_1}V(x_1,x')dx',\ \text{if } n\geqslant 2.$$

Assume that there exists $a\in\mathbf{R}$ *such that*

$$F(x_1)=a|x_1|^{-\delta_1-1}+O(|x_1|^{-\delta_1-2}),\ x_1\to-\infty. \tag{4.12}$$

Then the mean time-delay has the following asymptotic

$$\langle t_D\rangle(\lambda)=C_0|\lambda|^{n/2-1-\delta_1}+O(|\lambda|^{-\gamma_0}),\ \lambda\to-\infty \tag{4.13}$$

where

$$C_0=\frac{\sqrt{\pi}\ a}{(2\pi)^{n-1}}\ \frac{\Gamma(\delta_1+1-n/2)}{\Gamma(\delta_1+1)}\ b_n,$$

with $b_1=1$ *and* $b_n=(\sigma_{n-1}/2)\Gamma(n/2-1/2)$, *for* $n\geqslant 2$, σ_k *being the surface measure of the sphere* S^{k-1}, *and*

$$\gamma_0=\min(\delta_1+3/2-n/2,\ 2\delta_1-n-2-\varepsilon),\quad \textit{for any } \varepsilon>0.$$

Proof According to Lemma 4.2 $r(\lambda)=o(|\lambda|^{n/2+2+\varepsilon-2\delta_1})$, $\lambda\to-\infty$, for any $\varepsilon>0$. We need only to establish the asymptotic for $-2\pi\int_{\mathbf{R}^n}\partial_{x_1}V(x)\frac{\partial E_0}{\partial\lambda}(x,x;\lambda)dx$ (see (4.6)).

Consider first the case $n=1$. Then

$$\frac{\partial E_0}{\partial\lambda}(x,x;\lambda)=Ai(x-\lambda)^2,$$

where $Ai(\cdot)$ is the Airy function, which has well known asymptotics [11]

$$Ai(x)=\frac{e^{-2/3x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}}(1+0(x^{-3/2})),\quad x\to+\infty; \tag{4.14}$$

$$Ai(-x)=\frac{1}{\pi^{1/2}x^{1/4}}\left(\cos\left(\frac{2}{3}x^{3/2}-\frac{\pi}{4}\right)+o(x^{-3/2})\right),\quad x\to+\infty. \tag{4.15}$$

Let $\theta \in]0, 1[$, It can be easily estimated that

$$\int_{x \geq \lambda+|\lambda|^\theta} \partial_x V(x) Ai(x, \lambda)^2 dx = 0(e^{-4/3|\lambda|^{3/2\theta}}), \quad \lambda \to -\infty,$$

and for any $C>0$, one has

$$\int_{\lambda-C<x<\lambda+|\lambda|^\theta} \partial_x V(x) Ai(x-\lambda)^2 dx = 0(|\lambda|^{-\beta}), \quad \lambda \to -\infty.$$

Here $\beta=\delta_1+1$. To evaluate the integral for $x<\lambda-C$, we use (4.15) to obtain

$$Ai(x)^2 = \frac{1+\sin\frac{4}{3}|x|^{3/2}}{2\pi\sqrt{|x|}} + 0(|x|^{-2}), \; x<-C, \; C>0 \text{ large enough.}$$

Then

$$\int_{x<\lambda-C} \frac{a}{|x|^\beta} Ai(x-\lambda)^2 dx = \int_{-\infty}^{-C} \frac{a\left(1+\sin\frac{4}{3}|x|^{3/2}\right)}{2\pi|x+\lambda|^\beta\sqrt{|x|}} dx + O(|\lambda|^{-\beta})$$
$$= |\lambda|^{-\beta+1/2} \int_{-\infty}^{0} \frac{a}{2\pi|y-1|^\beta\sqrt{|y|}} dy + O(|\lambda|^{-\beta}), \quad \lambda \to -\infty.$$

Similarly we can show that if $G(x) = O(|x|^{-\beta-\varepsilon_0})$, $x \to -\infty$, then

$$\int_{x<\lambda-C} G(x) Ai(x-\lambda)^2 dx = O(|\lambda|^{-\beta-\varepsilon_0+1/2}), \quad \lambda \to -\infty.$$

Applying the above estimate with $G(x) = F(x) - (a/|x|^\beta)$, we obtain

$$\int_{\mathbf{R}} F(x) \frac{\partial E_0}{x\lambda}(x, x;\lambda) dx = \frac{a}{2\sqrt{\pi}} \cdot \frac{\Gamma(\beta-1/2)}{\Gamma(\beta)} |\lambda|^{-\beta+1/2} + O(|\lambda|^{-\beta}), \quad \lambda \to -\infty. \tag{4.16}$$

For $n \geq 2$, notice that $H_0 = -\Delta + x_1$ is unitarily equivalent to $(-\partial^2/\partial x_1) + x_1 + \xi'^2$, $\xi' \in \mathbf{R}^{n-1}$. It can be easily computed that

$$\frac{\partial E_0}{\partial \lambda}(x, x; \lambda) = (2\pi)^{1-n} \int \mathbf{R}^{n-1} Ai(x_1 + \xi'^2 - \lambda)^2 d\xi'.$$

By (4.14) and assumptions on V, one has

$$\int_{\mathbf{R}^n} \partial_{x_1} V(x) \frac{\partial E_0}{\partial \lambda}(x, x; \lambda) dx = \int_{\mathbf{R}^{n+1}} (2\pi)^{1-n} \left(\int_{\mathbf{R}^n} F(x_1) Ai(x_1 + \xi'^2 - \lambda)^2 dx_1 \right) d\xi'.$$

Utilizing (4.16) for $\mu = -\xi'^2 + \lambda \to -\infty$, we obtain

$$\int_{\mathbf{R}^n} \partial_{x_1} V(x) \frac{\partial e_0}{\partial \lambda}(x, x; \lambda) dx = K(\alpha, \beta) |\lambda|^{-\beta+n/2} + O(|\lambda|^{(-2\beta+n-1)/2})$$

for $\lambda \to -\infty$ with

$$K(\alpha, \beta) = \frac{1\sqrt{\pi}}{(2\pi)^n} \frac{\sigma_{n-2}}{2} \cdot \frac{\Gamma^{(n/2-1/2)} \Gamma(\beta - n/2)}{\Gamma(\beta)}.$$

Here σ_{n-2} is the area of S^{n-2}, with the convention $\sigma_0 = 2$. This proves (4.13).

Remark that the asymptotic formula (4.13) makes sense only when $\delta_1 > \frac{n}{2} + 3$. In the proof of Theorem 4.3, we take $\frac{\partial E_0}{\partial \lambda}$ as the first approximation for $\frac{\partial E}{\partial \lambda}$ (see

(4.5)), we can in fact improve the remainder estimates in (4.13) up to $\gamma_0=\delta_1+\frac{3-n}{2}$. The details are omitted.

References

[1] Agmon, S., Kannai, On the asymptotic behaviour of spectral functions and resolvent kernle of elliptic operators, *Israel J. of Math.*, **5**(1967), 1—30.

[2] Avron, J. E. & Herbst, I. W., Spectral and scattering theory of Schrodinger operators related to the Stark effect, *Commun. Math. Phys.*, **52**(1977), 239—254.

[3] Dauge, M. & Robert, D., Weyl formula for pseudodifferential operators of negative order, Lecture Note in Math. N 1256, Springer-Verlag, 1986.

[4] Gohberg, I. & Krein, M. G., Introduction à la théorie des opérateurs linéaires non auto-adjoints dans un espace hilbertien, Dunod, 1971.

[5] Jensen, A., Scattering theory for Hamiltonians with Stark effect, *Ann I. H. P, Peysique theorique*, **46**: 4(1987), 383—395.

[6] Jensen, A., Precise estimates for Stark effect Hamiltonians, Conference Holzhau, April, 1988.

[7] Jensen, A., Mourre E. & Perry P., Multiple commutator estimates and resolvent smoothness in quantum scattering theory, *Ann. I. H. P.*, **41A** (1984), 207—225.

[8] Kuroda, S. T., Scattering theory for differential operators, I. operator theory, *J. Math. Soc. Japan*, **25** (1973), 74—104.

[9] Lavine, R., Spectral density and sojourn times, in Atomic Scattering theory, J. Nuttall ed-Univ of Western Ontario-London-Ont-1978.

[10] Lax, P. D. & Phillips, R., The time-delay operator and a related trace formula, 197—215, in Topics in Functionol Analysis, eds. I. Gohberg and M. Kac, 1978.

[11] Olver, F., Asymtotics and Special Functions, Acad. Press, 1974.

[12] Robert, D. & Tamura, H., Asymptotique semiclassique pour la densité spectrale locale d'opérateurs de Schrödinger, Sémihaire Equations aux Dérivées Partielles, Ecole Polytechnique, 1985—1986,

[13] Robert, D. & Tamura, H., Semi-classical asymptotics for local spectral density and time-delay problem in scattering processes, *J. Funct. Analysis*, **80**: 1(1988), 124—147.

[14] Robert, D. & Wang, X. P., Time-delay and Spectral density for Stark Hamiltonians, I., Existence of Time-delay operator, *Commun. in P. D. E.*, **14**: 1(1989), 63—98.

[15] Rondeaux, C., Classes de Schatten d'operateurs pseudodifferentiels. These de 3e cocle, Reims 1980.

[16] Wang, X. P., Phase space description of time delay in scattering theory, *Comm. in P. D. E.*, **13**: 2 (1988), 223—259.

[17] Wang, X. P., Semiclassical estimates on resolvents of Schrodinger operators with homogeneous electric field, *J. of Diff. Equations*, **78**(1989), 354—373.

[18] Yajima, K., Spectral and scattering theory for Schrodinger operators with Stark effect, *J. Fac. Sc. Univ. Tokyo*, **28**(1981), 1—15.