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ON COUPLING OF JUMP PROCESSES

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Abstract

This is a sequel to the paper [3] in which the marginality, regularity and order-preservation for couplings of jump processes are studied. The main purpose of this note is to present a criterion and some practical sufficient conditions for the success of couplings and some criteria for the monotonicity of jump processes.

§ 1. Introduction

Let $(E, \mathscr{E})$ be an arbitrary measurable space, $(X_i(t))_{t\geq 0}$, $i=1, 2$ be two Markov processes valued in $(E, \mathscr{E})$. A Markovian coupling is simply to construct a Markov process $X(t)=(X_1(t), X_2(t))_{t\geq 0}$ on a common probability space with state space $(E^2, \mathscr{E}^2)\equiv(E\times E, \mathscr{E}\times\mathscr{E})$ having the marginality:

$$\begin{aligned}\mathbf{P}^{(x_1,x_2)}[X(t)\in A_1\times E]&=\mathbf{P}^{x_1}[X_1(t)\in A_1],\\ \mathbf{P}^{(x_1,x_2)}[X(t)\in E\times A_2]&=\mathbf{P}^{x_2}[X_2(t)\in A_2]\end{aligned}\tag{1}$$

for all $x_i\in E$ and $A_i\in\mathscr{E}$, $i=1, 2$. Define (formally)

$$T=\inf\{t\geq 0: X_1(t)=X_2(t)\}.$$

Then the coupling is called successful if

$$\mathbf{P}^{(x_1,x_2)}[T<\infty]=1\tag{2}$$

and

$$\mathbf{P}^{(x_1,x_2)}[X_1(t)=X_2(t),\ \forall t\geq T]=1\tag{3}$$

for all, $x_1, x_2\in E$.

In some applications of coupling methods, one of the key points is the success. To see this, let $(X_i(t))_{t\geq 0}$, $i=1, 2$ be two copies of a process with different starting points. Suppose that the process has a stationary distribution μ and a successful coupling does exist. For any starting probability measure λ, let $\mathbf{P}^{(\lambda,\mu)}$ denote the distribution of the above coupling process starting at (λ, μ), We have

$$\|\lambda P(t)-\mu\|_{\text{Var}}\leq 2\mathbf{P}^{(\lambda,\mu)}[T>t]\to 0$$

as $t\to\infty$, where $P(t)$ is the transition probability function of the original process.

The coupling methods for Markov processes have been studied by many authors. For recent references, see Liggett [11], Lindvall [13], Lindvall and Rogers [14], and

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Thorisson [17]. However, it seems to us that the success problem for couplings has not been well understood even for birth-death processes [12]. One aim of this note is to solve this problem in the context of jump processes.

First of all, let us recall some notation. Let $(E, \mathscr{E})$ be a separable measurable space with the property that all the singletons $\{x\}$ $(x\in E)$ belong to $\mathscr{E}$. Let $P(t, x, dy)$ be a sub-Markovian transition function on $(E, \mathscr{E})$. It is called a jump process if

$$\lim_{t\downarrow 0} P(t, x, A)=P(0, x, A)=I_A(x) \tag{4}$$

for all $x\in E$ and $A\in\mathscr{E}$. In the discrete case we call it a Markov chain and we use the matrix $(P_{ij}(t))$ instead of $P(t, x, dy)$. Associated with a jump process $P(t, x, dy)$ we have a q-pair $q(x)-q(x, dy)$ (Q-matrix $Q=(q_{ij})$): $q(x, \{x\})=0$, $0\leqslant q(x, \cdot)\leqslant q(x)\leqslant +\infty$, $x\in E$. Unless otherwise stated, we restrict ourselves to totally stable and conservative q-pairs:

$$q(x, E)=q(x)<\infty,\ x\in E.$$

Then, it was proved long ago by Kendall[10] that

$$\lim_{t\downarrow 0}\frac{P(t, x, A)-\delta(x, A)}{t}=q(x, A)-q(x)I_A(x) \tag{5}$$

for all $x\in E$ and $A\in\mathscr{E}$.

A q-pair is called regular if it determines uniquely a jump process satisfying (5). For a given q-pair, the uniqueness problem is now well understood (see Hou[8], Reuter [15], Chen and Zheng[5]). Recently, some more practical sufficient conditions for uniqueness were given in [3].

Let $(X, \mathscr{B})$ be an arbitrary measurable space and $b\mathscr{B}$ be the set of all bounded $\mathscr{B}$-measurable functions. A q-pair $q(x)-q(x, dy)$ on $(X, \mathscr{B})$ corresponds uniquely to an operator Ω on $b\mathscr{B}$:

$$\Omega f(x)=\int q(x, dy)(f(y)-f(x)),\ x\in X.$$

Now, suppose that we are given two jump processes $P_i(t, x_i, dy_i)$ with q-pair $q_i(x_i)-q_i(x_i, dy_i)$ $(i=1, 2)$ respectively on the same state space $(E, \mathscr{E})$. It is reasonable to assume that the coupling of them is again a jump process $P(t; x_1, x_2; dy_1, dy_2)$ with q-pair $q(x_1, x_2)-q(x_1, x_2; dy_1, dy_2)$. We use Ω_i $(i=1, 2)$ and Ω to denote the operators corresponding to the above three q-pairs respectively. Then, by using (1) and regarding $f\in b\mathscr{E}$ as a bivariable function, it is easy to check that

$$\begin{aligned} \Omega f(x_1, x_2)&=\Omega_1 f(x_1) \quad \text{independent of } x_2,\\ \Omega f(x_1, x_2)&=\Omega_2 f(x_2) \quad \text{independent of } x_1. \end{aligned} \tag{6}$$

Such an operator Ω (q-pair) defined on $b\mathscr{E}'$ is called a coupling operator (q-pair). Certainly, there are many choices of Ω for the given marginal operators Ω_1 and Ω_2. As an example, we mention the basic coupling [8]:

$$\Omega f(x_1, x_2) = \int (q_1(x_1, \cdot) - q_2(x_2, \cdot))^+(dy)(f(y, x_2) - f(x_1, x_2))$$
$$+ \int (q_2(x_2, \cdot) - q_1(x_1, \cdot))^+(dy)(f(x_1, y) - f(x_1, x_2))$$
$$+ \int (q_1(x_1, \cdot) \wedge q_2(x_2, \cdot))\,(dy)(f(y, y) - f(x_1, x_2)),$$
$$x_1, x_2 \in E, \quad f \in b\mathscr{E}^2, \tag{7}$$

where $(\mu_1 - \mu_2)^\pm$ is the Jordan–Hahn decomposition of $\mu_1 - \mu_2$ and $\mu_1 \wedge \mu_2 = \mu_1 - (\mu_1 - \mu_2)^+$. One basic fact for couplings of jump processes is that Ω_1 and Ω_2 are regular if and only if any coupling Ω is regular [8]. From now on, we will assume that Ω_1 and Ω_2 are regular and fixed.

Let us begin with the case that $\Omega_1 = \Omega_2$ for the study of successful coupling (from Theorem (1) to Corollary (4)). Put

$$\Delta = \{(x, x) : x \in E\}.$$

Clearly, the q-pair corresponding to the basic coupling satisfies

$$q(x, x; \Delta^c) = 0, \ x \in E \tag{8}$$

which is equivalent to the condition (3) from the uniqueness conclusion. In other words, for any successful coupling q-pair $q(x_1, x_2) - q(x_1, x_2; dy_1, dy_2)$ which satisfies (8), denoting by $p(t; x_1, x_2; dy_1, dy_2)$ the corresponding jump process, we have

$$P(t; x, x; \Delta) = 1, \quad t \geqslant 0, \quad x \in E. \tag{9}$$

Hence

$$\mathbf{P}^{(x_1, x_2)}\,[T \leqslant t] = \mathbf{P}^{(x_1, x_2)}[(X_1(t), X_2(t)) \in \Delta] = P(t; x_1, x_2; \Delta)$$

and condition (2) is now deduced to

$$P(t; x_1, x_2; \Delta) \to 1 \quad \text{as} \quad t \to \infty \tag{10}$$

for all $x_1 \neq x_2$. Thus, we may say that a coupling q-pair is successful if the corresponding process satisfies (9) and (10).

(1) **Theorem.** *A coupling q-pair $q(x_1, x_2) - q(x_1, x_2; dy_1, dy_2)$ is successful if and only if the following conditions hold:*

(i) $q(x, x; \Delta^c) = 0, \ x \in E$;

(ii) $q(x_1, x_2) > 0, \ (x_1, x_2) \in E^2 - \Delta$;

(iii) *the equation*

$$\begin{cases} Y(x_1, x_2) = \displaystyle\int_{E^2 - \Delta} \frac{q(x_1, x_2; dy_1, dy_2)}{q(x_1, x_2)} Y(y_1, y_2), \\ 0 \leqslant Y(x_1, x_2) \leqslant 1, \ (x_1, x_2) \in E^2 - \Delta \end{cases} \tag{11}$$

has solution zero only.

Clearly, the first two conditions are necessary. Hence the condition (iii) is essential. The general procedure to obtain the maximal solution $\{\overline{Y}(x_1, x_2): (x_1, x_2) \in E^2 - \Delta\}$ to equation (11) is as follows: Set

$$Y^{(0)}(x_1, x_2) = 1, \ (x_1, x_2) \in E^2 - \Delta,$$

$$Y^{(n+1)}(x_1, x_2) = \int_{E^2-\Delta} \frac{q(x_1, x_2; dy_1, dy_2)}{q(x_1, x_2)} Y^{(n)}(y_1, y_2),$$
$$(x_1, x_2) \in E^2 - \Delta, \ n \geq 0.$$

Then

$$Y^{(n)}(x_1, x_2) \downarrow \overline{Y}(x_1, x_2) \text{ as } n \uparrow \infty$$

for each $(x_1, x_2) \in E^2 - \Delta$. However, this procedure is sometimes not very practical, so we would like to propose some more effective sufficient conditions.

Take $\theta \notin E^2$ and set $E_\theta = (E^2 - \Delta) \cup \{\theta\}$, $\mathscr{E}_\theta = \sigma\{\mathscr{E}^2|_{E^2-\Delta}, \{\theta\}\}$. Define a transition probability on $(E_\theta, \mathscr{E}_\theta)$ as follows:

$$P(\theta, \theta) = 1,$$
$$P(x_1, x_2; A) = [q(x_1, x_2; A \setminus \{\theta\}) + q(x_1, x_2; \Delta) I_A(\theta)] / q(x_1, x_2).$$
$$x_1 \neq x_2, \quad A \in \mathscr{E}_\theta. \tag{12}$$

(2) **Corollary.** *Assume that the conditions* (i) *and* (ii) *of Theorem* (1) *hold. Let h be a $\mathscr{E}_\theta$-measurable non-negative function with $h(\theta) = 0$. Suppose that there exist constants $C \geq 0$ and $0 \leq c < 1$ such that*

$$\int P(x_1, x_2: dy_1, dy_2)\, h(y_1, y_2) \leq C + ch(x_1, x_2), \quad (x_1, x_2) \in E_\theta, \tag{13}$$

and there exist constants $k \in [0, 1)$ and $K > C/[(1-c)(1-k)]$ such that

$$P(x_1, x_2; \Delta) \geq 1 - k \tag{14}$$

for all $(x_1, x_2) \in E_\theta$: $h(x_1, x_2) \leq K$. Then the coupling q-pair is successful. In particular, if (14) *is satisfied for all $x_1 \neq x_2$, then the same conclusion holds without using condition* (13).

(3) **Corollary.** *Let E be a separable complete metric space with metric ρ. Assume that the conditions* (i) *and* (ii) *of Theorem* (1) *hold. Suppose that $q(x)$ is bounded on bounded sets and for every $r > 0$ there exists a bounded function φ: $[0, r] \to [0, \infty)$ such that*

$$\int q(x_1, x_2; dy_1, dy_2)[\varphi \circ \rho(y_1, y_2) - \varphi \circ \rho(x_1, x_2) + 1] \leq 0, \ 0 < \rho(x_1, x_2) \leq r. \tag{15}$$

Given a function f: $[0, \infty) \to \mathbf{R}$,

$$f(r) \uparrow f(\infty) = \infty \text{ as } r \uparrow \infty, \ f(0) = 0. \tag{16}$$

If

$$\int q(x_1, x_2; dy_1, dy_2)[f \circ \rho(y_1, y_2) - f \circ \rho(x_1, x_2)] \leq 0, \quad (x_1, x_2) \in E_\theta, \tag{17}$$

then the coupling q-pair is successful.

In the case that the state space is countable, we have a more precise criterion.

(4) **Corollary.** *Let E be countable. Assume that the condition* (i) *of Theorem* (1) *and the following condition hold*

(ii)$'$. *for every $(i_1, i_2) \in E^2 - \Delta$, there exists a sequence $(i_1^{(m)}, i_2^{(m)})$, $m = 0, 1, \cdots, M+1$ such that $(i_1^{(0)}, i_2^{(0)}) = (i_1, i_2)$, $(i_1^{(M+1)}, i_2^{(M+1)}) \in \Delta$, $(i_1^{(k)}, i_2^{(k)}) \notin \Delta$, $k = 0, 1, \cdots M$ and $q(i_1^{(k)}, i_2^{(k)}; i_1^{(k+1)}, i_2^{(k+1)}) > 0$, $k = 0, 1, \cdots, M$.*

If there exists a sequence $\{A_n\}_1^\infty\}$ of subsets of $E^2\backslash\Delta$, $|A_n|<\infty$, $A_n\downarrow E^2-\Delta$ and a positive function ϕ, such that

$$\lim_{n\to\infty}\inf_{(i_1,i_2)\notin A_n}\phi(i_1, i_2)=+\infty, \tag{18}$$

$$\phi(i_1, i_2)\geqslant\sum_{\substack{(j_1,j_2)\in E^2-\Delta\\(j_1,j_2)\neq(i_1,i_2)}}\frac{q(i_1, i_2; j_1, j_2)}{q(i_1, i_2)}\phi(j_1, j_2),\quad (i_1, i_2)\in E^2-\Delta, \tag{19}$$

then the coupling Q-matrix is successful.

If we consider only whether $P^{(x_1,x_2)}[T<\infty]=1$ hold for all $x_1\neq x_2$ or not, then we can allow $\Omega_1\neq\Omega_2$.

Now we turn to our second problem. Assume that E is endowed with a measurable semi-order "$\leqslant$" and that

$$F=\{(x_1, x_2): x_1\leqslant x_2\}\in\mathscr{E}^2.$$

We call a real function f on E monotone if

$$x_1\leqslant x_2\Rightarrow f(x_1)\leqslant f(x_2).$$

Following [11], for two probability measures μ_1 and μ_2, we define $\mu_1\leqslant\mu_2$ provided for every non-negative monotone function f,

$$\int f(x)\mu_1(dx)\leqslant\int f(x)\mu_2(dx).$$

Similarly, for the semigroups $P_i(t)$ on $b\mathscr{E}_i$ induced by the process $P_i(t, x_i, dy_i)$, $i=1, 2$, we say that $P_1(t)\leqslant P_2(t)$ if for every monotone function f,

$$x_1\leqslant x_2\Rightarrow P_1(f(x_1)\leqslant P_2(t)f(x_2),\quad t\geqslant 0. \tag{20}$$

If $P_1(t)=P_2(t)=P(t)$ and (20) holds, we say that the process $P(t)$ is monotone.

If $\mu_1\leqslant\mu_2$, under a general assumption on E (cf. [16]), there exists a coupling measure $\mathbf{P}$ on $(E^2, \mathscr{E}^2))$ such that

$$\mathbf{P}(F)=1.$$

Because of this property, the monotonicity is also called stochastic monotonicity. This topic has received a lot of attention. The readers who are interested in it may find some recent references from Aldous [1], Doorn [7] and Stoyan andDaley [16].

By using the coupling technique, in [3], we proved the following result:

(5) **Theorem.** *Set*

$$F_1(x)=\{y: y\leqslant x\}\text{ and }F_2(x)=\{y: x\leqslant y\}.$$

If

$$(q_1(x_1, \cdot)-q_2(x_2, \cdot))^+(F_1(x_2)^c)=0$$

$$(q_2(x_2, \cdot)-q_1(x_1, \cdot))^+(F_2(x_1)^c)=0\quad\text{for all } x_1\leqslant x_2, \tag{21}$$

then $P_1(t)\leqslant P_2(t)$.

We call a set $A\in\mathscr{E}$ monotone if so is the indicator function I_A. That is,

$$x_1\leqslant x_2,\ x_1\in A\Rightarrow x_2\in A.$$

For the monotonicity, an easier case to handle is that

$$\sup_x q_i(x)<\infty,\ i=1, 2. \tag{22}$$

(6) **Theorem.** *Given two bounded q-pairs (i. e., (22) holds), possibly non-*

conservative. (i) *If the q-pairs are the same, then* $P_1(t)\leqslant P_2(t)$ *if and only if, for every monotone set A,*

$$\begin{aligned} &x_1\leqslant x_2,\ x_1,\ x_2\notin A\Rightarrow q_1(x_1, A)\leqslant q_2(x_2, A),\\ &x_1\leqslant x_2,\ x_1,\ x_2\in A\Rightarrow q_1(x_1, A)-q_1(x_1)\leqslant q_2(x_2, A^c)-q_2(x_2). \end{aligned} \tag{23}$$

(ii) *If for every* $A\in\mathscr{M}$, *there is* $g_A\in\mathscr{M}$ *such that* $\Omega_1 I_A\leqslant g_A\leqslant \Omega_2 I_2$, *then* (23) *implies that* $P_1(t)\leqslant P_2(t)$

(iii) *If one of* $P_1(t)$ *and* $P_2(t)$ *being monotone, then* (23) *again implies that* $P_1(t)\leqslant P_2(t)$

For the general case, we need a technical assumption.

(H) **Hypothesis.** *There exist* $E_n\uparrow E$ $(n\geqslant 1)$ *such that*

(i) $\sup\limits_{x\in E_n} q_i(x)<\infty,\ n\geqslant 1,\ i=1,\ 2,$

(ii) $H_n\equiv\{y\in E\backslash E_n: \exists x\in E_n$ *such that* $x\leqslant y\}$ *is a monotone set in E. If* $H_n\neq\phi$, *there is a* $b_n\in H_n$ *so that*

$$x\leqslant b_n, \text{ for all } x\in E_n.$$

In the case that $E=\mathbf{R}^d$, $\mathbf{Z}^d$, $\mathbf{R}^d_+$ or $\mathbf{Z}^d_+$ with the ordinary semi-order and $q_i(x)$ $(i=1, 2)$ are locally bounded, (H) is trivial. We simply take

$$\begin{aligned} E_n&=\{x\in E:\ -n\leqslant x\leqslant n\} \text{ if } E=\mathbf{R}^d \text{ or } \mathbf{Z}^d\\ &=\{x\in E:\ 0\leqslant x\leqslant n\} \text{ if } E=\mathbf{R}^d_+ \text{ or } \mathbf{Z}^d_+ \end{aligned}$$

and

$$b_n=(n+1,\ \cdots,\ n+1)\in E,\ n\geqslant 1.$$

(7) **Theorem.** *If (H) holds, then the conclusion* (i)—(iii) *of Theorem* (6) *hold.*

The proofs of the above results are given in section 3. In the next section, we illustrate some applications of our results.

§ 2. Examples

(8) *Example.* Take $E=\{0, 1, 2, \cdots\}$ and let $Q=(q_{ij})$ be a regular Q-matrix on E. The basic coupling is

$$\begin{aligned} \Omega f(i_1, i_2)=\sum_{k=0}^{\infty}\{&(q_{i_1k}-q_{i_2k})^+(f(k, i_2)-f(i_1, i_2))\\ &+(q_{i_2k}-q_{i_1k})^+(f(i_1, k)-f(i_1, i_2))+q_{i_1k}\wedge q_{i_2k}(f(k,k)-f(i_1, i_2))\}. \end{aligned} \tag{24}$$

Here we have used the convention $q_{kk}=0$. If $q_i=0$ for same $i\in E$, then we must assume that for every $j\in E\backslash\{i\}$, there exist $j_1, \cdots, j_m\in E$ such that

$$q_{jj_1}q_{j_1j_2}\cdots q_{j_mi}>0, \tag{25}$$

which implies the condition (ii)′ of Corollary (4). Now, we choose

$$A_n=\{(i_1, i_2): i_1\neq i_2,\ i_1+i_2\leqslant n\},$$

$$\varphi(i_1, i_2)=i_1+i_2+C,$$

where C is a constant. Then condition (18) is automatic and condition (19) becomes

$$\sum_{k=1}^{\infty}|q_{i_1k}-q_{i_2k}|k\leqslant q_{i_1}i_1+q_{i_2}i_2+C\left(\sum_{k=0}^{\infty}q_{i_1k}\wedge q_{i_2k}\right)$$

for all $i_1\neq i_2$. For a birth-death process, if

$$q_{0,1}+q_{i,i+1}\leqslant q_{i,i-1},\ i\geqslant 1,$$

then the above conditions hold and hence the coupling is successful.

Now, we turn to Corollary (3). Consider again the basic coupling. Let $i_1=i\geqslant 0$, $i_2=i+k$, $k\geqslant 1$. Then, we have

$$\begin{aligned}&\sum_{(j_1,j_2)\neq(i_1,i_2)}q(i_1,\ i_2;j_1,\ j_2)\psi(\rho(j_1,\ j_2))\\ &=\sum_{j=-\infty}^{\infty}\{(q_{i,i+j}-q_{i+k,i+j})^+\psi(|k-j|)+(q_{i+k,i+j}-q_{i,i+j})^+\psi(|j|)\\ &\quad+q_{i,i+j}\wedge q_{i+k,i+j}\psi(0)\}\\ &=(q_{i,\ i+k}+q_{i+k,i}+\sum_{j=-i}^{\infty}q_{i,i+j}\wedge q_{i+k,i+j})\psi(0)\\ &\quad+\sum_{j=1}^{\infty}[(q_{i,i+k\pm j}-q_{i+k,i+k\pm j})+(q_{i+k,i\pm j}-q_{i,i\pm j})^+]\psi(j),\end{aligned}$$

where $q_{ij}=0$ if $j=i$ or $j\leqslant 0$. In particular, for birth-death process, we have

$$\begin{aligned}&\sum_{(j_1,j_2)\neq(i_1,i_2)}q(i_1,\ i_2;\ j_1,j_2)\psi(\rho(j_1,\ j_2))\\ &=(q_{i,i+1}+q_{i+k,i+k-1})\psi(k-1)+(q_{i,i-1}+q_{i+k,i+k+1})\psi(k+1)\quad \text{if } k=1 \text{ or } k\geqslant 3\\ &=(q_{i,i+1}\wedge q_{i+2,i+1})\psi(0)\\ &\quad+((q_{i,i+1}-q_{i+2,i+1})^++(q_{i+2,i+1}-q_{i,i+1})^+)\psi(1)\\ &\quad+(q_{i,i-1}+q_{i+2,i+3})\psi(3),\quad \text{if } k=2.\end{aligned}$$

From this, it is easy to check that conditions (15) and (17) hold with $\varphi(r)=\alpha\log(1+r)$ or $\alpha r/(1+r)$ $(\alpha>0)$ and $f(r)=r$ provided

$$q_{i+1,i}-q_{i+1,i+2}\geqslant q_{i,i-1}-q_{i,i+1}.$$

(9) *Example*. Take E and $Q(q_{ij})$ as above. By Theorem (5), a sufficient condition for the monotonicity of the Markov chain $P(t)=(P_{ij}(t))$ is the following:

$$\begin{aligned}&q_{i_1k}\leqslant q_{i_2k}\quad \text{for all}\quad i_1\leqslant i_2<k\\ \text{and}\qquad &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (26)\\ &q_{i_1k}\geqslant q_{i_2k}\quad \text{for all}\quad k<i_1\leqslant i_2.\end{aligned}$$

In other words, for each fixed k,

$$q_{ik}\downarrow \text{as } i\downarrow,\ i<k,$$

$$q_{ik}\uparrow \text{as } i\uparrow,\ i>k.$$

This picture is very easy to remember. From this, we see immediately that the birth-death processes are monotone.

However, Theor m (7) gives us a necessary and sufficient condition for the monotonicity of $P(t)$:

$$\begin{aligned}&\sum_{j\geqslant k}q_{i_1j}\leqslant\sum_{j\geqslant k}q_{i_2j}\quad \text{for all}\quad i_1\leqslant i_2<k,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (27)\\ &\sum_{j<k}q_{i_1j}\geqslant\sum_{j<k}q_{i_2j}\quad \text{for all}\quad k<i_1\leqslant i_2.\end{aligned}$$

Obviously, (26) is stronger than (27). The last result is the only known complete answer to the monotonicity for jump processes (cf. [16, Chapter 4]). Thus, our Theorem (7) is an improvement on [3] and an extension to [16].

(10) *Example.* Lotka-Volterra model.

$$E=\mathbf{Z}_+^2,$$

$$q(i_1, i_2; j_1, j_2)=\begin{cases}\lambda_1 i_1, & \text{if } j_1=i_1+1,\ j_2=i_2,\\ \lambda_2 i_2, & \text{if } j_1=i_1,\quad j_2=i_2-1,\\ \lambda_3 i_1 i_2, & \text{if } j_1=i_1-1,\ j_2=i_2+1.\end{cases}$$

Consider the monotone set

$$A=\{(i_1, i_2): i_1\geqslant 1\}.$$

Then

$$(1, 1),\ (1, 2)\in A$$

and

$$q(1, 1; A^c)=\lambda_3,\ q(1, 2; A^c)=2\lambda_2,$$

and so

$$q(1, 1; A^c)<q(1, 2; A^c).$$

By Theorem (7), the process is not monotone.

(11) *Example.* Finite dimensional generalized Potlatch process.

$$E=\mathbf{R}_+^d,$$

$$\Omega f(x)=\sum_{i=1}^d\int_0^\infty\Big[f\Big(x-e_ix_i+\xi\sum_{j=1}^d P_{ij}x_ie_j\Big)-f(x)\Big]dF(\xi),$$

where F is a distribution on $[0, \infty)$ with mean 1, e_i is the i-th unit vector in $\mathbf{R}^d$ and (P_{ij}) is a Markovian trasition matrix on $\{1, 2, \cdots, d\}$.

We now prove that the process generated by Ω is monotone. Clearly,

$$q(x, B)=\sum_{i=1}^d\int_0^\infty I_{B\setminus(x)}\Big(x-e_ix_i+\xi\sum_{j=1}^d P_{ij}x_ie_j\Big)F(d\xi),$$

$$q(x)=q(x, E)\leqslant d,\ x\neq 0.$$

Let A be a monotone set in E. Fix $x^{(1)}, x^{(2)}\in E$, $x^{(1)}\leqslant x^{(2)}$. Since for each i and $\xi\geqslant 0$,

$$y^{(1)}(i, \xi)\equiv(x^{(1)}-x_i^{(1)}e_i)+\xi x_i^{(1)}\sum_{j=1}^d P_{ij}e_j$$

$$\leqslant(x^{(2)}-x_i^{(2)}e_i)+\xi x_i^{(2)}\sum_{j=1}^d P_{ij}e_j\equiv y^{(2)}(i, \xi),$$

we have

$$y^{(1)}(i, \xi)\in A\Rightarrow y^{(2)}(i, \xi)\in A,$$

$$y^{(2)}(i, \xi)\notin A\Rightarrow y^{(1)}(i, \xi)\notin A.$$

Thus, if $x^{(1)}, x^{(2)}\notin A$, then

$$q(x^{(1)}, A)=\sum_{i=1}^d\int_0^\infty I_A(y^{(1)}(i, \xi))F(d\xi)$$

$$\leqslant\sum_{i=1}^d\int_0^\infty I_A(y^{(2)}(i\xi))F(d\xi)=q(x^{(2)}, A).$$

If $x^{(1)}, x^{(2)}\in A$, then

$$q(x^{(1)}, A^c)=\sum_{i=1}^d\int_0^\infty I_{A^c}(y^{(1)}(i, \xi))F(d\xi)\geqslant q(x^{(2)}, A^c).$$

Our conclusion follows from Theorem (6).

An alternate way to prove the monotonicity of the process is using the following coupling

$$\Omega f(x, y)=\sum_{i=1}^{d}\int_0^\infty \left[f\left(x-x_ie_i+\xi x_i\sum_{j=1}^{d}P_{ij}e_j,\ y-y_ie_i+\xi y_i\sum_{j=1}^{d}P_{ij}e_j\right)\right.$$
$$\left.-f(x, y)\right]F(d\xi),\quad x,\ y\in E, \tag{29}$$

which gives us

$$\mathbf{P}^{x,y}[X_t\leqslant Y_t,\ t\geqslant 0]=1.$$

(12) *Example*. Let us consider a special case of the above example: $d=1$, $P_{11}=1$ and $F(d\xi)=e^{-\xi}d\xi$. Now, the coupling in (29) is reduced as follows:

$$\Omega f(x_1, x_2)=\int_0^\infty [f(\xi x_1, \xi x_2)-f(x_1, x_2)]e^{-\xi}d\xi,$$

$$q(x_1, x_2;B)=\int_0^\infty I_{B\setminus\{(x_1,x_2)\}}(\xi x_1, \xi x_2)e^{-\xi}d\xi=\int_0^\infty I_B(\xi x_1, \xi x_2)e^{-\xi}d\xi,$$

$$q(x_1, x_2)=q(x_1, x_2;\mathbf{R}_+^2)=1 \text{ if } x_1\neq x_2,$$

$$q(x, x;\Delta^c)=0,\ x\in\mathbf{R}_+.$$

However, this coupling is not successful. Indeed,

$$Y^{(0)}(x_1, x_2)\equiv 1,$$

$$Y^{(n+1)}(x_1, x_2)=\int_{E^2\setminus\Delta}\frac{q(x_1, x_2;\ dy_1,dy_2)}{q(x_1, x_2))}Y^{(n)}(y_1, y_2)$$
$$=\int_0^\infty Y^{(n)}(\xi x_1, \xi x_2)e^{-\xi}d\xi,\ n\geqslant 0,\ x_1\neq x_2.$$

Byinduction, we have $Y^{(n)}(x_1, x_2)=1$ and hence $Y(x_1;x_2)=1$ for all $x_1\neq x_2$. By Theorem (1), this gives us the conclusion.

§ 3. Proofs of Results

(13) *Proof of Theorem* (1). As we mentioned in the first section, the first twoconditions of Theorem (1) are necessary. Hence, we need only to show that the condition (iii) is equivalent to (10) under the assumptions (i) and (ii). Now, we assume that (i) and (ii) hold. Define

$$P(\lambda;x_1,x_2;A)=\int_0^\infty e^{-\lambda t}P(t;x_1,x_2;A)dt,\quad \lambda>0.$$

Then (9) becomes

$$\lambda P(\lambda;x,x;\Delta)=1,\quad x\in E,\ \lambda>0 \tag{30}$$

and we can rewrite (10) as

$$\lambda P(\lambda;x_1,x_2;\Delta)\to 1 \text{ as } \lambda\downarrow 0,\quad x_1\neq x_2. \tag{31}$$

Thus we only need to prove that (iii) and (31) are equivalent.

Fix $\lambda>0$. It is well known [4] that $\{\lambda P(\lambda;x_1,x_2;\Delta):\ x_1,\ x_2\in E\}$ is the minimal (non-negative) solution to the equation:

$$X(\lambda;x_1, x_2)=\int_{E^2}\frac{q(x_1,x_2;dy_1,dy_2)}{\lambda+q(x_1,x_2)}X(\lambda;y_1,y_2)+\frac{\delta(x_1,x_2;\Delta)}{\lambda+q(x_1, x_2)},\quad x_1,\ x_2\in E,$$

where δ is the point mass measureon E^2. From (30) and by a localization procedue we see that $\{\lambda P(\lambda;x_1,x_2;\Delta):x_1\neq x_2\}$ is the minimal solution to

$$X(\lambda;x_1,x_2)=\int_{E^2-\Delta}\frac{q(x_1,x_2;dy_1,dy_2)}{\lambda+q(x_1,x_2)}X(\lambda;y_1,y_2)+\frac{q(x_1,x_2;\Delta)}{\lambda+q(x_1,x_2)},\quad x_1\neq x_2.$$

Noting that

$$\frac{1}{\lambda+q(x_1,\ x_2)}\uparrow\frac{1}{q(x_1,\ x_2)}\text{ as }\lambda\downarrow 0,\quad x_1\neq x_2$$

and using the condition (ii), we obtain

$$\lambda P(\lambda;x_1,x_2;\Delta)\uparrow\text{some }\overline{X}(x_1,\ x_2)\text{ as }\lambda\downarrow 0,\quad x_1\neq x_2,$$

and $\{\overline{X}(x_1,\ x_2):(x_1,\ x_2)\in E^2-\Delta\}$ is the minimal solution to

$$X(x_1,\ x_2)=\int_{E^2-\Delta}\frac{q(x_1,x_2;dy_1,dy)}{q(x_1,x_2)}X(y_1,y_2)+\frac{q(x_1,x_2;\Delta)}{q(x_1,x_2)},\quad (x_1,x_2)\in E^2-\Delta. \tag{32}$$

Thus, $\{\overline{Y}(x_1,x_2)\equiv 1-\overline{X}(x_1,x_2):(x_1,x_2)\in E^2-\Delta\}$ is the maximal solution to equation (11). This completes our proof.

(14) *Proof of Corollary* (2). Consider the process $(X_1(n),\ X_2(n))_{n\geq 0}$ on a probability space $(\Omega,\ \mathscr{F},\ \mathbf{P})$ valued in $(E_\theta,\ \mathscr{E}_\theta)$ with transition probability $P(x_1,x_2;dy_1,dy_2)$. What we need is to show that

$$\mathbf{E}I_{[X_1(n)\neq X_2(n)]}\to 0,\text{ as }n\uparrow\infty.$$

Put

$$I_n=I_{[X_1(n)\neq X_2(n)]},\quad n\geq 0,$$
$$J_n=I_nh(X_1(n),X_2(n)).$$

Since θ is an absorbing state and $I_{n-1}=0\Rightarrow I_n=0$, by (13), we get

$$\begin{aligned}\mathbf{E}J_n&=\mathbf{E}[I_{n-1}\mathbf{E}(J_n|X_1(n-1),\ X_2(n-1))]\\&\leq\mathbf{E}[I_{n-1}\int P(X_1(n-1),\ X_2(n-1);dy_1,dy_2)h(y_1,y_2)]\\&\leq\mathbf{E}[I_{n-1}(C+ch(X_1(n-1),X_2(n-1)))]\\&=C\mathbf{E}I_{n-1}+c\mathbf{E}J_{n-1}.\end{aligned}\tag{33}$$

On the other hand, by (14), we have

$$\begin{aligned}\mathbf{E}I_n&=\mathbf{E}[I_{n-1}P(X_1(n-1),\ X_2(n-1);E^2-\Delta)]\\&\leq\mathbf{E}[I_{n-1}P(X_1(n-1),\ X_2(n-1);\ E^2-\Delta);\ h(X_1(n-1),\ X_2(n-1))<K]\\&\quad+\mathbf{E}[I_{n-1}P(X_1(n-1),\ X_2(n-1);E^2-\Delta);\ h(X_1(n-1),\ X_2(n-1))\geq K]\\&\leq k\mathbf{E}I_{n-1}+K^{-1}\mathbf{E}J_{n-1}.\end{aligned}\tag{34}$$

Combining (33) and (34), we obtain

$$\begin{bmatrix}\mathbf{E}I_n\\\mathbf{E}J_n\end{bmatrix}\leq\begin{bmatrix}k & K^{-1}\\C & c\end{bmatrix}\begin{bmatrix}\mathbf{E}I_{n-1}\\\mathbf{E}J_{n-1}\end{bmatrix},\quad n\geq 0.$$

Since the eigenvalues of the matrix on the right hand side are smaller than 1, we see that the left hand side goes to zero as $n\to\infty$.

The same proof, even simpler, will give us the last assertion of Corollary (2).

The above idea is due to Dobrushin and Pecherski [8]. Here we emphasize on the coupling behavior rather than the marginals.

(15) *Proof of Corollary* (3) Use the same notations as above and set

$$T=\min\{n\geqslant 1:\ \rho(X_1(n),\ X_2(n))=0\},$$
$$S_r=\min\{n\geqslant 1:\ \rho(X_1(n),\ X_2(n))>r\},$$
$$T_{0,r}=T\wedge S_r.$$

Without any confusion, we use the same notation $\mathbf{P}^{(x_1,x_2)}$(resp. $\mathbf{E}^{(x_1,x_2)}$) to denote the distribution (resp. expectation) of the embeded Markov process $P(x_1,x_2;dy_1,dy_2)$ starting from (x_1,x_2).

First we prove that

$$\mathbf{P}^{(x_1,x_2)}\ [T_{0,r}<\infty]=1 \text{ for all } 0<\rho(x_1,\ x_2)\leqslant r. \tag{35}$$

Choose and fix a reference point $0\in E$, set

$$\tau_N=\min\ \{n\geqslant 1:\rho(X_1(n),\ 0)^2+\rho(X_2(n),\ 0)^2>N^2\},\quad N\geqslant 1.$$

For each bounded $\mathscr{E}$-measurable function g, define $P^0g=g$, $P^1g=Pg$:

$$Pg(x_1,x_2)=\int P(x_1,x_2;dy_1,dy_2)g(y_1,y_2)$$

and $P^ng=P(P^{n-1}g)$. Since

$$P^ng-g=\sum_{k=0}^{n-1}(P^{k+1}g-P^kg),$$

by the conditions (i) and (ii) of Theorem (1) and condition (15), if we take $g=\Phi_r=\varphi\circ\rho$ and set $Z(n)=(X_1(n),\ X_2(n))$, then

$$\mathbf{E}^{(x_1,x_2)}\ \Phi_r(Z(n\wedge\tau_N\wedge T_{0,r}))-\Phi_r(x_1,x_2)=\mathbf{E}^{(x_1,x_2)}\sum_{k=0}^{n\wedge\tau_N\wedge T_{0,r}-1}[P\Phi_r(Z(k))-\Phi_r(Z(k))]$$
$$\leqslant-\mathbf{E}^{(x_1,x_2)}(n\wedge\tau_N\wedge T_{0,r}).$$

That is

$$\mathbf{E}^{(x_1,x_2)}(n\wedge\tau_N\wedge T_{0,r})\leqslant\Phi_r(x_1,x_2).$$

Let $n\uparrow\infty$ and then $N\uparrow\infty$ to get

$$\mathbf{E}^{(x_1,x_2)}(T_{0,r})\leqslant\Phi_r(x_1,x_2)<\infty,\ 0<\rho(x_1,\ x_2)\leqslant r. \tag{36}$$

This certainly implies (35).

Next, fix $r_2>0$ and set $F=f\circ\rho$. From (i) and (ii) of Theorem (1) and (17), it follows that

$$P^nF\leqslant F,\ n\geqslant 1.$$

Hence, for $\rho(x_1,\ x_2)\equiv r\in(0,\ r_2)$,

$$\mathbf{E}^{(x_1,x_2)}F(Z(n\wedge T_{0,r}))\leqslant F(x_1,x_2).$$

Letting $n\uparrow\infty$, by (35), we see that

$$F(x_1,\ x_2)=f(r)\geqslant\mathbf{E}^{(x_1,x_2)}[F(Z(T)):T<S_{r_2}]+\mathbf{E}^{(x_1,x_2)}[F(Z(S_{r_2}):T>S_{r_2}]$$
$$\geqslant f(r_2)\mathbf{P}^{(x_1,x_2)}[T>S_{r_2}].$$

Thus

$$\mathbf{P}^{(x_1,x_2)}[T>S_{r_2}]\leqslant\frac{f(r)}{f(r_2)}.$$

Finally, letting $r_2\uparrow\infty$ and using (16), we obtain

$$\mathbf{P}^{(x_1,x_2)}[T=\infty]=0.$$

This shows that our coupling is successful.

The above technique is the modification of a proof for the recurrence of one-dimensional diffusion processes, and will be also used to study the couplings of diffusion processes jointly with S. F. Li.

(16) *Proof of Corollary* (4). The condition (ii)′ is clearly necessary. Now, assume that (i) and (ii) hold. We need to show that (iii) holds under our assumptions. Equivalently, equation (29) has solution

$$\overline{X}(i_1, i_2)=1, \quad (i_1,i_2)\in E^2-\Delta. \tag{37}$$

Consider the subset Δ as a single absorbing state θ. Setting $\phi(\theta)=0$ and using the notation defined by (12), we have

$$\sum_{(j_1,j_2)\in E} P(i_1,i_2;j_1,j_2)\phi(j_1,j_2)\leqslant\phi(i_1,i_2),\ (i_1,i_2)\in E_\theta \tag{38}$$

and

$$\lim_{n\to\infty}\inf_{(i_1,i_2)\notin A_n\cup\{\theta\}}\phi(i_1,i_2)=+\infty.$$

These are enough to conclude that (37) holds. Indeed, by (38), we obtain

$$\begin{aligned}1&=\sum_{j_1,j_2}p^m(i_1,i_2;j_1,j_2)\\ &\leqslant\sum_{(j_1,j_2)\in A_n\cup\{\theta\}}p^m(i_1,i_2;j_1,j_2)\\ &\quad+\phi(i_1,i_2)/\inf_{(k_1,k_2)\notin A_n\cup\{\theta\}}\phi(k_1,k_2).\end{aligned}$$

Letting $m\to\infty$, then $n\to\infty$, we get.

$$1\leqslant\overline{X}(i_1,i_2), i_1\neq i_2.$$

The idea goes back to Kendall [9].

We use the same notation $\mathscr{M}$ to denote both the set of monotone functions and the set of monotone sets.

The following simple result enable us to simplify our consideration.

(17) **Lemma.** *The following statements are equivalent.*

(i) $x_1\leqslant x_2\Rightarrow P_1(t,x_1,A)\leqslant P_2(t,x_2,A)$ *for all* $A\in\mathscr{M}$,

(ii) $x_1\leqslant x_2\Rightarrow P_1(t)f(x_1)\leqslant P_2(t)f(x_2)$ *for all bounded* $f\in\mathscr{M}_+\equiv\{g\in\mathscr{M}:g\geqslant 0\}$,

(iii) $x_1\leqslant x_2\Rightarrow P_1(t)f(x_1)\leqslant P_2(t)f(x_2)$ *for all* $f\in\mathscr{M}_+$,

(iv) $x_1\leqslant x_2\Rightarrow P_1(t)f(x_1)\leqslant P_2(t)f(x_2)$ *for all* $f\in\mathscr{M}$ *for which the integrals exist.*

Proof Clearly, (iv)⇒(iii)⇒(ii)⇒(i). To prove (i)⇒(iii), simply use

$$\begin{aligned}f_n&=\sum_{k=1}^{\infty}\frac{1}{2^n}I_{[f>\frac{k}{2^n}]}=\sum_{k=1}^{\infty}\frac{1}{2^n}\sum_{j=k}^{\infty}I_{[\frac{j}{2^n}<f<\frac{j+1}{2^n}]}\\ &=\sum_{j=0}^{\infty}\frac{j}{2^n}I_{[\frac{j}{2^n}<f<\frac{j+1}{2^n}]}\uparrow f, \text{ as } n\to\infty.\end{aligned}$$

(iii)⇒(iv): Let $f\in\mathscr{M}$ such that $P_i(t)f(x_i)$ exists, $i=1, 2$. Fix $x_1\leqslant x_2$ and set $\mu_i=P_i(t,x_i,\cdot)$, $i=1, 2$. Without loss of generality, we assume that $\int fd\mu_1>-\infty$ and $\int fd\mu_2<\infty$. Since $f^+=f\vee 0$, $f^-=f\wedge 0\in\mathscr{M}$ and

$$\int f^+ d\mu_1 \leqslant \int f^+ d\mu_2 < \infty \quad \text{(by (iii))},$$

it suffices to consider $f \in \mathscr{M}$, $f \leqslant 0$ and $\int f d\mu_1 > -\infty$. Take

$$f_n = -nI_{[f<-n]} + fI_{[f\geqslant -n]}.$$

Then $f_n + n \in \mathscr{M}_+$. Applying (iii) to $f_n + n$ we get

$$\int f_n d\mu_1 \leqslant \int f_n d\mu_2 \leqslant \int_{[f\geqslant -n]} f d\mu_2.$$

Thus

$$\int (-f) d\mu_2 = \lim_{n\to\infty} \int_{[f\geqslant -n]} (-f) d\mu_2$$
$$\leqslant \lim_{n\to\infty} \int (-f_n) d\mu_1 = -\int f d\mu_1 < \infty.$$

This completes our proof.

More generally, even though $P_1(t)$ and $P_2(t)$ are only sub-Markovian, the first three statements of Lemma (17) are still equivalent.

For the subsequent use, we now prove an extension to Theorem (6).

(18) *Proof of Theorem* For every bounded q-pair $q(x) - q(x, dy)$, the corresponding process $P(t, x, dy)$ is unique and

$$\left.\frac{dP(t, x, A)}{dt}\right|_{t=0} = q(x, A) - q(x) I_A(x), \tag{39}$$

for all $x \in E$ and all $A \in \mathscr{E}$ (see [4] or [5]). By Lemma (17), our condition (23) is necessary. We now prove the sufficiency.

(a). Observe

$$\begin{aligned} P(t) &= \exp[t\Omega] \\ &= \exp[-\lambda t] \exp[t(\lambda I + \Omega/\lambda)] \\ &= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{t^n}{n!} (P^{(\lambda)})^n, \end{aligned} \tag{40}$$

where

$$P_k^{(\lambda)} = \lambda I + \Omega_k, \; k = 1, 2. \tag{41}$$

Since

$$P_1^{(\lambda)} \leqslant P_2^{(\lambda)}$$

for all $\lambda \geqslant \sup (q_1(x) + q_2(x))$, it is easy to check that the comparability of $P_1(t)$ and $P_2(t)$ for $f \in \mathscr{M}_+$ can be reduced to the one of the corresponding $P_1^{(\lambda)}$ and $P_2^{(\lambda)}$ defined by (41).

(b). The proof for the assertion (ii) is similar.

(c). The last assertion follows by using the integration by parts formula

$$\begin{aligned} P_1(t)f - P_2(t)f &= \int_0^t P_1(s)(\Omega_1 - \Omega_2) P_2(t-s) f ds \\ &= \int_0^t P_1(s)(P_1^{(\lambda)} - P_2^{(\lambda)}) P_2(t-s) f ds, \; f \in b\mathscr{E}. \end{aligned}$$

For unbounded q-pairs, we need an approximation procedure.

(19) **Lemma.** *Let $q(x)-q(x,dy)$ be a regular q-pair and $q_n(x)-q_n(x,dy)$ $(n\geq 1)$ be a sequence of bounded q-pairs. If*

$$q_n(x)\to q(x),\ q_n(x,A)\to q(x,A),\ \text{as } n\to\infty$$

for all $x\in E$ $A\in\mathscr{E}$, then

$$\lim_{n\to\infty} P_n(t,x,A)=P(t,x,A)\quad x\in E,\ A\in\mathscr{E}, \tag{41}$$

where $P_n(t)$ and $P(t)$ are the processes determined by $q_n(x)-q_n(x,dy)$ and $q(x)-q(x,dy)$ respectively.

Proof Omitted (cf. [4; Lemma 5.2.3]).

(20) *Proof of Theorem* (7) Define

$$q_i^{(n)}(x,B)=I_{E_n}(x)\left[q_i(x,B\backslash\{b_n\})+I_B(b_n)q_i(x,H_n)\right]$$
$$x\in E_n+\{b_n\},\ B\in(E_n+\{b_n\})\cap\mathscr{E},\ n\geq 1,\ i=1,2,$$
$$q_i^{(n)}(x)=\begin{cases} q_i(x), & \text{if } x\in E_n,\\ 0, & \text{if } x\notin E_n,\ n\geq 1.\end{cases}$$

Then

$$\sup_{x\in E_n+\{b_n\}} q_i^{(n)}(x)<\infty,\ i=1,2,\ n\geq 1,$$
$$q_i^{(n)}(x)\to q_i(x),\ \text{as } n\to\infty.\ x\in E,\ i=1,2,$$
$$q_i^{(n)}(x,A\cap E_n)\to q_i(x,A),\ \text{as } n\to\infty,\ x\in E,\ A\in\mathscr{E},\ i=1,2.$$

Thus, by Lemma (19), we have

$$\lim_{n\to\infty} P_i^{(n)}(t,x,A\cap E_n)=P_i(t,x,A),\ t\geq 0,\ x\in E,\ A\in\mathscr{E},\ i=1,2. \tag{42}$$

Clearly the semi-order on E induces a semi-order on $E_n+\{b_n\}$. Let $\mathscr{M}_n$ denote the set of all monotone sets (functions) in $E_n+\{b_n\}$. Of course, $A\in\mathscr{M}\Rightarrow A\cap(E_n+\{b_n\})\in\mathscr{M}_n$. Now, by (42), $P_1(t)\leq P_2(t)$ follows immediately from

$$P_1^{(n)}(t)f\leq P_2^{(n)}(t)f,\ f\in\mathscr{M}_+,\ n\geq 1. \tag{43}$$

To prove this, we should to check that $q_i^{(n)}(x)-q_i^{(n)}(x,dy)$ $(i=1,2)$ satisfy (39).

Let $x_1,x_2\in E_n+\{b_n\}$, $x_1<x_2$ and $B\in\mathscr{M}_n$. Then $b_n\in B$.

(a) If $x_2\neq b_n$ and $x_1,x_2\notin B$, since $B\cup H_n\in\mathscr{M}$, by the first condition of (23), we have

$$q_1^{(n)}(x_1,B)=q_1(x_1,B\backslash\{b_n\})+q_1(x_1,H_n)=q_1(x_1,B\cup H_n)$$
$$\leq q_2(x_2,B\cup H_n)=q_2^{(n)}(x_2,B).$$

(b) If $x_2\neq b_n$ and $x_1,x_2\in B$, then $x_1,x_2\in B\cup H_n$. Again, since $B\cup H_n\in\mathscr{M}$, by the second condition of (23), we have

$$q_1^{(n)}(x_1,B)-q_1^{(n)}(x_1)=q_1(x_1,B\cup H_n)-q_1(x_1)$$
$$\leq q_2(x_2,B\cup H_n)-q_2(x_2)=q_2^{(n)}(x_2,B)-q_2^{(n)}(x_2).$$

(c) If $x_2=b_n\in B$, we need only to consider the case that $x_1\in B$. Then

$$q_2^{(n)}(b_n,B)\leq q_2^{(n)}(b_n)=0,$$
$$q_1^{(n)}(x_1,B)-q_1^{(n)}(x_1)\leq 0=q_2^{(n)}(b_n,B).$$

Combining (a), (b) with (c), we see that $q_i^{(n)}(x) - q_i^{(n)}(x, dy)$, $i=1, 2$, satisfy (23). Therefore, we have proved the sufficiency for the case (i) and case (iii). As for case (ii), we need to construct for each $B \in \mathscr{M}_n$ a function $g_n = g_B \in \mathscr{M}_n$ such that

$$\Omega_1^{(n)} I_B \leqslant g_n \leqslant \Omega_2^{(n)} I_B \text{ on } E_n \cup \{b_n\}.$$

But by assumption, since $B \cup H_n \in \mathscr{M}$, there is $g = g_{B\cup H_n} \in \mathscr{M}$ such that

$$\Omega_1 I_{B\cup H_n} \leqslant g \leqslant \Omega_2 I_{B\cup H_n},$$

Hencef $g_n := g I_{E_n}$ is a required function.

Because of the regularity of the q-pair $q(x) - q(x, dy)$, the necessity can be proved by the same approach as in the proof of Theorem (18).

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