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ALGEBRAIC INDEPENDENCE OF CERTAIN NUMBERS OVER A FIELD OF FINITE TRANSCENDENCE TYPE**

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Abstract

The transcendental degree of certain numbers concerned with the exponential function over a field of finite transcendence type is considered.

§ 1. Introduction

If $P\in \mathbf{C}[x_1, \cdots, x_s]$ is nonzero, say $P(x_1, \cdots, x_s)=\sum_{(i)} A(i)x_1^{i_1}\cdots x_s^{i_s}$, where $A(i)\in \mathbf{C}$, define $d(P)=\max_i \deg_{x_i} P$, $H(P)=\max_{(i)} |A(i)|$ and $t(P)=\max(1+d(P), \log H(P))$. Let $\mathbf{K}\subset \mathbf{C}$ be a field. $\mathbf{K}$ is said to have a transcendence type$\leqslant\tau$ over $\mathbf{Q}$ ([1]) if $\mathbf{K}$ has a finite transcendence degree q over $\mathbf{Q}$, and if $\{w_1, \cdots, w_q\}$ is a transcendence basis of $\mathbf{K}$ over $\mathbf{Q}$, then for all $w\in Z[w_1, \cdots, w_q]$ with $w\neq 0$ one has $\log|w|\geqslant -c(t(w))^\tau$, where the constant $c>0$ does not depend on w. It is well-known that $\tau\geqslant q+1$.

In 1949 Gelfond ([2]) proved that certain sets of numbers related to the exponential function cannot lie in a field of transcendence degree one over $\mathbf{Q}$. In 1966 Lang ([1]) and in 1972 and 1975 Brownawell ([3, 4]) proved that certain values of the exponential function cannot all be algebraic over a field of finite transcendence type. In the present paper we obtain more general results for the same subject.

Let $a_i, b_j\in \mathbf{C}$ $(1\leqslant i\leqslant m, 1\leqslant j\leqslant n)$, and $\mathbf{K}\subset\mathbf{C}$ be a field. We denote

$$S_1=\{a_i, b_j, \exp(a_ib_j)\,(1\leqslant i\leqslant m, 1\leqslant j\leqslant n)\},$$
$$S_2=\{a_i, \exp(a_ib_j)\,(1\leqslant i\leqslant m, 1\leqslant j\leqslant n)\},$$
$$S_3=\{\exp(a_ib_j)\,(1\leqslant i\leqslant m, 1\leqslant j\leqslant n)\},$$

and let $\mathbf{L}_i=\mathbf{K}(S_i)$ $(i=1, 2, 3)$. We put

$$d_1=\frac{mn+m+n}{m+n}, \quad d_2=\frac{mn+m}{m+n}, \quad d_3=\frac{mn}{m+n}.$$

We take the following common hypotheses:

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1. $\mathbf{K}\subset\mathbf{C}$ is a field of transcendence typhe $\leqslant\tau$ ove $\mathbf{Q}$;

2. $\{a_1, \cdots, a_m\}$ and $\{b_1, \cdots, b_n\}$ are two sets of complex numbers such that for any $\varepsilon>0$ these is $H_0=H_0(\varepsilon)>0$ having the following property: for any $(u_1, \cdots, u_m)\in\mathbf{Z}^m$ and $(v_1, \cdots, v_n)\in\mathbf{Z}^n$ with $\max_i |u_i|\geqslant H_0$ and $\max_j |v_j|\geqslant H_0$ one has

$$\left|\sum_{i=1}^{m} u_i a_i\right| > \exp(-(\max_i |u_i|^\varepsilon)),$$

$$\left|\sum_{j=1}^{n} v_j b_j\right| > \exp(-(\max_j |v_j|)^\varepsilon).$$

Our main results are the following:

Theorem. *We have*

$$d_i\leqslant 2^{t_i}\tau \quad (i=1, 2, 3),$$

where $t_i=\mathrm{tr}\cdot\deg_{\mathbf{K}}\mathbf{L}_i$.

Corollary. *If* $2^r\tau<d_i$, *where* $r\geqslant 0$ *is integral, then* $t_i\geqslant r+1$.

Remark. Our theorem improves the results of [5]. Furthermore, taking $r=0$ or 1, from Corollary we deduce the results of [1, 3, 4] under slightiy stronger technical assumption on a_i and b_j.

§ 2. A Criterion of Algebraic independndce over a Field of Finite Trascendence Type

Lemma 1.[6] *Let* $\mathbf{K}\subset\mathbf{C}$ *be a field of transcendence type* $\leqslant\tau$ *over* $\mathbf{Q}$, *and* $\{w_1, \cdots, w_q\}$ *be a transcendence basis of* $\mathbf{K}$ *over* $\mathbf{Q}$. *Let* $g>1$ *and* g_1, $g_2>0$ *be real numbers such that* $g_1>g_2>\left(1-\frac{\tau}{g}\right)g_1$, *and let* $\sigma(N)$ $(N\geqslant N_0)$ *be a monotonically increasing sequence of positive numbers such that* $\sigma(N)\to\infty$ $(N\to\infty)$ *and* $\sigma(N+1)\leqslant a\sigma(N)$, *where* $a>1$ *is a constant. Let* $\theta_1, \cdots, \theta_s$ *be some complex numbers. If for every integer* $N\geqslant N_0$, *there is a nonzero polynomial* $P_N\in\mathbf{Z}[x_1, \cdots, x_q, y_1, \cdots, y_s]$ *such that*

$$-c_1\sigma(N)^{g+g_1}<\log|P_N(w_1, \cdots, w_q, \theta_1, \cdots, \theta_s)|<-c_2\sigma(N)^{g+g_2}$$

and $t(P_N)\leqslant\sigma(N)$, *where* c_1, $c_2>0$ *are constants, then* $g<2^t\tau$, *where* $t=\mathrm{tr}\cdot\deg_{\mathbf{K}}\mathbf{K}(\theta_1, \cdots, \theta_s)$.

§ 3. Auxiliary Lemmas

Let $\mathbf{K}=\mathbf{Q}(w_1, \cdots, w_q, \zeta)\subset\mathbf{C}$ be a finitely generated extension field of $\mathbf{Q}$, where $w_1, \cdots, w_q, \zeta$ is a system of generators of $\mathbf{K}$ over $\mathbf{Q}$, i.e., $w_1, \cdots, w_q$ are algebraically independent over $\mathbf{Q}$, and ζ is integral over $\mathbf{Z}[w_1, \cdots, w_q]$. Then any element w of $\mathbf{K}$ can be written uniquely as

$$w=\sum_{i=1}^{d}\frac{Q_i}{R_i}\zeta^{i-1}.$$

where d is the degree of ζ over $\mathbf{Q}(w_1, \cdots, w_q)$, and Q_i, $R_i \in \mathbf{Z}[w_1, \cdots, w_q]$ have not common factors. Let P be the least common multiple of $R_1, \cdots, R_d$. Then

$$Pw = \sum_{i=1}^{d} P_i \zeta^{i-1},$$

where $P_i \in \mathbf{Z}[w_1, \cdots, w_q]$ $(1 \leqslant i \leqslant d)$. We put

$$t(w) = \max(t(P), t(P_1), \cdots, t(P_d)),$$

which is called the size of w.

Lemma 2. *Let $\mathbf{K} \subset \mathbf{C}$ be as above. Then there is a constant $c_0 > 0$ depending only on $w_1, \cdots, w_q, \zeta$ such that for all $(e_1, \cdots, e_s) \in \mathbf{K}^s$ one has*

$$t(e_1, \cdots, e_s) \leqslant c_0(t(e_1) + \cdots + t(e_s)).$$

Furthermore, if $e_i \in \mathbf{Z}[w_1, \cdots, w_q, \zeta]$ $(1 \leqslant i \leqslant s)$, then

$$t(e_1 + \cdots + e_s) \leqslant \max t(e_i) + \log s.$$

Proof See Lemma (4.2.5) of [7].

Lemma 3. *Let r and s be positive integers with $r < s$, and let $a_{ij} \in \mathbf{Z}$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s)$ have absolute values at most $A \geqslant 1$. Then the system of equations*

$$\sum_{j=1}^{s} a_{ij} z_j = 0 \quad (1 \leqslant i \leqslant r)$$

has a solution in integers z_j $(1 \leqslant j \leqslant s)$, not all zero, with

$$|z_j| \leqslant (sA)^{r/(s-r)}.$$

Proof See Lemma 5.1 of [4].

Lemma 4. *Let r, s, t, u, v be integers, and $\xi_\nu (0 \leqslant \nu \leqslant r)$ and η_σ $(0 \leqslant \sigma < t)$ be some complex numbers. Put*

$$E(z_1, \cdots, z_s) = \sum_{\substack{0 \leqslant \mu_i < v \\ 0 \leqslant \nu_i < r}} A(\mu_1, \cdots, \mu_s, \nu_1, \cdots, \nu_s)\, z_1^{\mu_1} \cdots z_s^{\mu_s} \exp(\xi_{\nu 1} z_1 + \cdots + \xi_{\nu s} z_s).$$

where $A(\mu_1, \cdots, \mu_s, \nu_1, \cdots \nu_s) \in \mathbf{C}$, and set

$$a = \max_{\nu}(|\xi_\nu|, 1), \quad a_0 = \min_{\mu \neq \nu}(|\xi_\theta - \xi_\nu|, 1),$$

$$b = \max_{\sigma}(|\eta_\sigma|, 1), \quad b_0 = \min_{\rho \neq \sigma}(|\eta_\rho - \eta_\sigma|, 1),$$

$$A = \max_{\mu_i, \nu_i} |A(\mu_1, \cdots, \mu_s, \nu_1, \cdots, \nu_s)|,$$

$$E = \max_{\substack{0 \leqslant \rho_i < u \\ 0 \leqslant \sigma_i < t}} \left| \frac{\partial^{\rho_1 + \cdots \rho_s}}{\partial z_1^{\rho_1} \cdots \partial z_s^{\rho_s}} E(\eta_{\sigma 1}, \cdots, \eta_{\sigma s}) \right|.$$

If $tu \geqslant 2rv + 13ab$, then

$$A \leqslant \left(t \cdot \left(\frac{6v}{a_0 \sqrt{r}} \max\left(6, \frac{rv}{b}\right) \right)^{rv} \cdot \left(\frac{72b}{b_0 \sqrt{t}} \right)^{tu} \right)^s \cdot E. \tag{1}$$

Proof Use the induction on s. If $s=1$, then Lemma 4 coincides with Theorem 2 of [8]. Assume that $s \geqslant 2$ and that for the exponential polynomial in $s-1$ variables Lemma 4 holds. We denote by $B^s \cdot E$ the expression in the right side of the inequality (1). For every pair (μ_s, ν_s), $0 \leqslant \mu_s < v$, $0 \leqslant \nu_s < r$, we define the function

$$E_{\mu_s,\nu_k}(z_1, \cdots, z_{s-1}) = \sum_{D_1} A(\mu_1, \cdots, \mu_s, \nu_1, \cdots, \nu_s)\, z_1^{\mu_1}\cdots z_{s-1}^{\mu_{s-1}} \exp(\xi_{\nu 1} z_1 + \cdots + \xi_{\nu_{s-1}} z_{s-1}),$$

where $D_1 = \{0 \leqslant \mu_i < v,\ 0 \leqslant \nu_i < r,\ 1 \leqslant i < s\}$. Then

$$E(z_1, \cdots, z_s) = \sum_{D_2} E_{\mu_s,\nu_s}(z_1, \cdots, z_{s-1}) z_s^{\mu_s} \exp(\xi_{\nu s} z_s),$$

where $D_2 = \{0 \leqslant \mu_s < v,\ 0 \leqslant \nu_s < r\}$. Applying Theorem 2 of [8] to the function

$$\frac{\partial^{\rho_1 + \cdots + \rho_{s-1}}}{\partial z_1^{\rho_1} \cdots \partial z_{s-1}^{\rho_{s-1}}} E(\eta_{\sigma_1}, \cdots, \eta_{\sigma_{s-1}}, z_s),\ 0 \leqslant \sigma_i < t,\ 0 \leqslant \rho_i < u,\ 1 \leqslant i < s,$$

we have

$$\max_{\substack{0 \leqslant \rho_i < u \\ 0 \leqslant \sigma_i < t \\ 1 \leqslant i < s}} \left| \frac{\partial^{\rho_1 + \cdots + \rho_{s-1}}}{\partial z_1^{\rho_1} \cdots \partial z_{s-1}^{\rho_{s-1}}} E_{\mu_s,\nu_s}(\eta_{\sigma_1}, \cdots, \eta_{\sigma_{s-1}}) \right| \leqslant BE. \tag{2}$$

Applying the inductive assmuption to the function $E_{\mu_s,\nu_s}(z_1, \cdots, z_{s-1})$, we see that the expression in the left side of the inequality (2) is greater than $AB^{-(s-1)}$. Therefore the inequality (1) follows.

Lemma 5. *Let $T_1, \cdots, T_s$ be subsets of $\mathbf{C}$, each of which contains at least t points. Let $D(0, r_1) = \{z = (z_1, \cdots, z_s) \in \mathbf{C}^s \mid \max_i |z_i| \leqslant r_1\}$ be a polycylinder containing the set $T = T_1 \times \cdots \times T_s$. If u is a positive integer and $f(z_1, \cdots, z_s)$ is a entire function such that*

$$\frac{\partial^{\rho_1 + \cdots + \rho_s}}{\partial z_1^{\rho_1} \cdots \cdots \partial z_s^{\rho_s}} f(\eta_1, \cdots, \eta_s) = 0$$

for $0 \leqslant \rho_i < u$ $(1 \leqslant i \leqslant s)$ and $(\eta_1, \cdots, \eta_s) \in T$, then for $R \geqslant r$ and $R > r_1$

$$\log |f|_r \leqslant \log |f|_R - tu \log \frac{R - r_1}{r + r_1},$$

where $|f|_r = \sup\{|f(z_1, \cdots, z_s)| \mid \max_{1 \leqslant i \leqslant s} |z_i| = r\}$.

Proof See Proposition 7.2.1 of [9].

§ 4. Proof of Theorem

Let $\mathbf{L}_i = \mathbf{K}(S_i) = \mathbf{Q}(w_1, \cdots, w_q, \alpha_1, \cdots, \alpha_{t_i}, \zeta)$ such that

(i) $\{w_1, \cdots, w_q\}$, is a transcendence basis of $\mathbf{K}$ over $\mathbf{Q}$;

(ii) $\{w_1, \cdots, w_q, \alpha_1, \cdots, \alpha_{t_i}\}$ is a transcendence basis of $\mathbf{L}_i$ over $\mathbf{Q}$;

(iii) ζ is integral over the ring $\mathbf{Z}[w_1, \cdots, w_q, \alpha_1, \cdots, \alpha_{t_i}]$.

For two points $\boldsymbol{u} = (u_1, \cdots, u_s)$ and $\boldsymbol{v} = (v_1, \cdots, v_s)$ of $\mathbf{C}^s$, we denote $\boldsymbol{uv} = u_1 v_1 + \cdots + u_s v_s$. Let $\boldsymbol{a} = (a_1, \cdots, a_m)$, $\boldsymbol{b} = (b_1, \cdots, b_n)$, $\boldsymbol{\lambda}_i = (\lambda_{i,1}, \cdots, \lambda_{i,m}) \in \mathbf{Z}^m_{\geqslant 0} (1 \leqslant i \leqslant s)$, $\boldsymbol{\delta}_i = (\delta_{i,1}, \cdots, \delta_{i,n}) \in \mathbf{Z}^n_{\geqslant 0} (1 \leqslant i \leqslant s)$.

Now we consider the case $i = 1$.

We will adopt reducton to absurdity. We assume that $d_1 > 2^{t_1}\tau$. Let $\mu = \left[\frac{2d_1}{\tau}\right]$ and s be a parameter such that

$$s > 2(q + t_1) \cdot \max\left(\frac{m+n}{mn},\ \frac{\mu}{d_1 - 2^{t_1}\tau}\right). \tag{3}$$

Let N be an integral parameter, which we will assume to be large enough. Put

$$P=\left[\frac{N^{1+\frac{m}{n}}}{\log N}\right],\quad J=[N^{\frac{m}{n}-\left(1+\frac{m}{n}\right)(q+t_1)/s_n}],\quad J_1=[3N^{\frac{m}{n}}].$$

Lemma 6. *There exists an exponential polynomial*

$$f_N(z_1,\cdots,z_s)\leqslant\sum_{\lambda_i,\mu_i}C(\boldsymbol{\lambda}_1,\cdots,\boldsymbol{\lambda}_s,\mu_1,\cdots,\mu_s)\prod_{i=1}^{s}z_i^{\mu_i}\exp((\lambda_i a)z_i),$$

where $0\leqslant\lambda_{i,j}<N(1\leqslant i\leqslant s,\ 1\leqslant j\leqslant m)$, $0\leqslant\mu_i<P(1\leqslant i\leqslant s)$, *such that*

(i) *the coefficients* $C(\lambda_1,\cdots,\lambda_s,\mu_1,\cdots,\mu_s)\in\mathbf{Z}$, *and*

$$0<\log\max_{\lambda_i,\mu_i}|C(\lambda_1,\cdots,\lambda_s,\mu_1,\cdots,\mu_s)|\ll N^{1+\frac{m}{n}};$$

(ii) *for* $0\leqslant\rho_i<P\ (1\leqslant i\leqslant s)$ *and* $0\leqslant\delta_{i,j}<J\ (1\leqslant i\leqslant s,\ 1\leqslant j\leqslant n)$

$$\frac{\partial^{\rho_1+\cdots+\rho_s}}{\partial z_1^{\rho_1}\cdots\partial z_s^{\rho_s}}f_N(\delta_1 b,\cdots,\delta_s b)=0.$$

Proof Let $A_i, B_j, C_{i,j}\in\mathbf{Z}[w_1,\cdots,w_q,\alpha_1,\cdots,\alpha_{t_1}]$ be denominators of a_i, b_j, $\exp(a_i b_j)$ $(1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n)$, respectively. Put $A=\prod_i A_i$, $B=\prod_j B_j$, $C=\prod_{i,j}C_{i,j}$. Consider the system of equations for $C(\boldsymbol{\lambda}_1,\cdots,\boldsymbol{\lambda}_s,\mu_1,\cdots,\mu_s)$

$$(AB)^{sP}C^{sNJ}\frac{\partial^{\rho_1+\cdots+\rho_s}}{\partial z_1^{\rho_1}\cdots\partial z_s^{\rho_s}}f_N(\boldsymbol{\delta}_1\boldsymbol{b},\cdots,\boldsymbol{\delta}_s\boldsymbol{b})=0,\ (0\leqslant\rho_i<P,\ 0\leqslant\delta_{i,j}<J).$$

The expressions in the left side can be rewritten as the polynomials in $w_1,\cdots,w_s,\alpha_1,\cdots,\alpha_{t_1},\zeta$, the coefficients of which are the linear forms in $C(\boldsymbol{\lambda}_1,\cdots,\boldsymbol{\lambda}_s,\mu_1,\cdots,\mu_s)$ with integral coefficients. Thus we obtain a new systerm of homogeneous linear equations with integral coefficients in $C(\boldsymbol{\lambda}_1,\cdots;\boldsymbol{\lambda}_s,\mu_1,\cdots\mu_s)$. The number of unknowns $=N^{sm}P^s$, and the number of equations $\ll(NJ+P)^{q+t_1}P^sJ^{sn}$. Further, by Lemma 2 the coefficients of equations have

$$\text{sizes}\ll s(NJ+P\log J+P\log N)\ll N^{1+\frac{m}{n}}.$$

Thus Lemma 6 follows by Lemma 3.

Lemma 7. *There are integers* $\rho_1^{(0)},\cdots,\rho_s^{(0)}$ *and integral points* $\boldsymbol{\delta}_1^{(0)},\cdots,\boldsymbol{\delta}_s^{(0)}$ *with* $0\leqslant\rho_i^{(0)}<P(1\leqslant i\leqslant s)$ *and* $0\leqslant\delta_{i,j}^{(0)}<J_1(1\leqslant i\leqslant s,\ 1\leqslant j\leqslant n)$ *such that*

$$\log\left|\frac{\partial^{\rho_1^{(0)}+\cdots+\rho_s^{(0)}}}{\partial z^{\rho_1^{(0)}}\cdots\partial z^{\rho_s^{(0)}}}f_N(\boldsymbol{\delta}_1^{(0)}\boldsymbol{b},\cdots,\boldsymbol{\delta}_s^{(0)}\boldsymbol{b})\right|\gg-N^{m+1+\frac{m}{n}+s}.$$

Proof Apply Lemma 4 to the exponential polynomial $f_N(z_1,\cdots,z_s)$, taking the parameters

$$\xi_i=\boldsymbol{\lambda}_i\boldsymbol{a},\ 0\leqslant\lambda_{i,j}<N(1\leqslant i\leqslant s,\ 1\leqslant j\leqslant m),$$
$$\eta_i=\boldsymbol{\delta}_i\boldsymbol{b},\ 0\leqslant\delta_{i,j}<J_1(1\leqslant i\leqslant s,\ 1\leqslant j\leqslant n).$$

It is easy to see that the values in Lemma 4

$$r=N^m,\ t=J_1^n,\ u=P,\ v=P,\ a\ll N,\ \exp(-J_1)\ll b\ll J_1,$$
$$\log a_0\gg-N^s,\ \log b_0\gg-J_1^s.$$

It is clear that the condition $tu\geqslant 2rv+13ab$ is satisfied here. Thus Lemma 7 follows.

Lemma 8. *We have*

$$\log\left|\frac{\partial^{\rho_1^{(0)}+\cdots+\rho_s^{(0)}}}{\partial z_1^{\rho_1^{(0)}}\cdots\partial z_s^{\rho_s^{(0)}}}f_N(\delta_1^{(0)}b,\cdots\delta_s^{(0)}b)\right|\ll -N^{1+m+\frac{m}{n}-2\left(1+\frac{m}{n}\right)(q+t_1)/s}.$$

Proof Taking $r=n\max\limits_i|b_i|\cdot J_1$, by the Cauchy inequality we have

$$\left|\frac{\partial^{\rho_1^{(0)}+\cdots+\rho_s^{(0)}}}{\partial z_1^{\rho_1^{(0)}}\cdots\partial z_s^{\rho_s^{(0)}}}f_N(\delta_1^{(0)}b,\ \cdots,\ \delta_s^{(0)}b)\right|\leqslant\rho_1^{(0)}!\cdots\rho_s^{(0)}!\,|f_N|_r.$$

In Lemma 5 taking $R=7r$ and $r_1=2r$, and noticing the values in Lemma 5, $t=J^n$, $u=P$, we have

$$\log|f_N|_r\ll -N^{1+m+\frac{m}{n}-2\left(1+\frac{m}{n}\right)(q+t_1)/s}.$$

Thereby Lemma 8 is proved.

Lemma 9. *There is a sequence of polynomials* $P_N\in[x,\ \cdots,\ x_q,\ y_1,\ \cdots,\ y_{t_1+1}]$ $(N\geqslant N_0)$ *such that* $t(P_N)\ll N^{1+\frac{m}{n}}$ *and*

$$-N^{m+1+\frac{m}{n}+2\left(1+\frac{m}{n}\right)(q+t_1)/s}\ll\log|P_N(w_1,\ \cdots,\ w_q,\ \alpha_1,\ \cdots,\ \alpha_{t_1},\ \zeta)|$$
$$\ll -N^{1+m+\frac{m}{n}-2\left(1+\frac{m}{n}\right)(q+t_1)/s}.$$

Proof Denote

$$P_N(w_1,\ \cdots,\ w_q,\ \alpha_1,\ \cdots,\ \alpha_{t_1},\ \zeta)=(AB)^{sP}C^s{}_N{}^{J_1}\frac{\partial^{\rho_1^{(0)}+\cdots+\rho_s^{(0)}}}{\partial z^{\rho_1^{(0)}}{}_1\cdots\partial z_s^{\rho_s^{(0)}}}f_N(\delta_1^{(0)}\boldsymbol{b},\ \cdots,\ \delta_s^{(0)}\boldsymbol{b}).$$

Lemma 9 follows from Lemma 6 and Lemma 7 with $\varepsilon=2\left(1+\dfrac{m}{n}\right)(q+t)/s$ and Lemma 8.

Proof of Theorem with $i=1$. In Lemma 1 taking

$$g=d_1-\mu g_0,\ g_1=(\mu+1)g_0,\ g_2=(\mu-1)g_0,\ \sigma(N)=c_3N^{1+\frac{m}{n}},$$

where $g_0=2(q+t_1)/s$, and $c_3>0$ is a constant, we deduce that

$$d_1-\mu g_0<2^{t_1}\tau.$$

It contradicts (3), therefore Theorem with $i=1$ is proved.

Since the proof for the cases $i=2$, 3 has much in common with the proof for the case $i=1$, we shall only give the form of the exponential polynomials and the main parameters.

For the case $i=2$, let

$$f_N(z_1,\ \cdots,\ z_s)=\sum_{\lambda_i}C(\boldsymbol{\lambda}_1,\ \cdots,\ \boldsymbol{\lambda}_s)\prod_{i=1}^{s}\exp((\boldsymbol{\lambda}_i\boldsymbol{a})z_i),\tag{4}$$

where $0\leqslant\lambda_{i,j}<N\ (1\leqslant i\leqslant s,\ 1\leqslant j\leqslant m)$, and put

$$P=\left[\frac{N^{\frac{m+n}{n+1}}}{\log N}\right],\quad J=[N^{\frac{n-1}{n+1}-\left(1+\frac{m}{n}\right)(q+t_1)/s(n+1)}],\quad J_1=[3N^{\frac{m-1}{n+1}}(\log N)^{\frac{1}{n}}].$$

For the case $i=3$, let $f_N(z_1,\ \cdots,\ z_s)$ be as in (4), and put

$$P=1,\quad J=[N^{\frac{m}{n}-\left(1+\frac{m}{n}\right)(q+t_1)/s_n}],\quad J_1=[3N^{\frac{m}{n}}].$$

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