Chin. Ann. of Math.
12B:4(1989), 407—420.

DIRICHLET PROBLEMS FOR MONGE–AMPÈRE EQUATION DEGENERATE ON BOUNDARY**

Hong Jiaxing (洪家兴)*

Abstract

The existence of C^∞ -solutions to Dirichlet problem for Monge-Ampère equation degenerate on boundary is proved, and some applications to the equation prescribed Gaussian curvature are also given.

§ 1. Introduction.

The purpose of the present paper is concerned with the Dirichlet problem for the equations of Monge–Ampère type

$$\det(u_{ij}) = K(x) \quad \text{in}\, \Omega \tag{1.1}$$

with

$$u=0, \text{ on } \partial\Omega, \tag{1.2}$$

where Ω is a smooth convex bounded domain in R^2, i. e. there is a strictly convex function $\psi(x)$ in $C^\infty(R^2)$ such that $\Omega=\{x\in R^2 | \psi(x)<0\}$. Throughout the present paper we always assume that

$$K \text{ in } C^\infty(\overline{\Omega}) \text{ and } K>0 \text{ in } \Omega \tag{1.3}$$

and

$$K=0 \text{ and } dK\neq 0 \text{ on } \partial\Omega. \tag{1.4}$$

As is well known, there have been many works [2, 3, 5, 6] devoted to the elliptic case and particularly, the latter three to the degenerate elliptic case. For the degenerate case, however, so far only solutions in $C^{0+1}(\overline{\Omega}) \cap C^2(\Omega)$ have been obtained if there is no additional assumption on K. Naturally, one can ask whether the solutions have much better regularity. This is just the motivation of the present paper. Our main result is as follows.

Theorem 1. *Let* (1.3) (1.4) *be fulfilled. Then* (1.1) (1.2) *admits a unique convex solution smooth up to the boundary.*

From the later argument, reader can find that if (1.2) is replaced by a prescribed smooth function φ on $\partial\Omega$, Theorem 1 is still true, provided that there exists a $C^2(\overline{\Omega})$ convex supersolution $\bar{u}$ in which $\partial\Omega$ is noncharacteristic for $\det(\bar{u}_{ij})$.

Manuscript received March 3, 1989.
* Institute of Mathematics, Fudan University, Shanghai, China.
** The project supported by the National Natural Science Foundation.

The crucial difficulties we ran into are the C^2-estimate and the estimate of lower bound for the maximal eigenvalue of the matrix (u_{ij}). We overcome those obstacles by locally constructing an upper barrier like function to Problem (1.1) (1.2) and using it to evaluate the upper and lower bound of Δu in section 2. The $C^{2+\alpha}$-estimate is presented in section 3 with the aid of Aubin's techniques in [1]. The existence of C^∞-solutions is based on a regularity theorem for the boundary problems of nonlinear degenerate elliptic equations recently given in [7].

With a similar argument we also provide in section 4 the existence of C^∞-solutions for the following problem

$$\det(u_{ij}) = K(x)f(x, u, \partial u) \quad \text{in } \Omega \tag{1.5}$$

and

$$u=0 \text{ on } \partial\Omega, \tag{1.6}$$

which is studied by many authors, for example, [4] for non homogeneous boundary value problems. Here f is a smooth positive function subject to the following conditions

$$f \text{ is convex in } p \text{ and } f_u \geqslant 0 \text{ in } \bar{\Omega}\times R^1\times R^2, \tag{1.7}$$

$$f(X, N, p) \leqslant h^{-1}(p) \text{ for all } x \text{ in } \bar{\Omega} \text{ and } p \text{ in } R^2, \tag{1.8}$$

$$f(X, 0, p) \leqslant C(1+p^2)^2, \tag{1.9}$$

where N, C are nonnegative constants and h is a positive function in $L^1_{loc}(R^2)$ satisfying structure conditions

$$\int_\Omega K dx < \int_{R^2} h dp. \tag{1.10}$$

Theorem 2. *Let* (1.3) (1.4) (1.7) (1.8) (1.9) (1.10) *be fulfilled. Then* (1.5) (1.6) *admits a unique convex solution smooth up to the boundary.*

Remark 3. If $f=(1+p^2)^2$, (1.5) is just the equation prescribed Gaussian curvature in two dimensional space, Theorem 2 tells us that one can find a smooth graph with any given Gaussian curvature K and meeting any given plane curve over $\partial\Omega$.

The discussion fo: nonhomogeneous boundary problems will be published in another paper.

§ 2. C^2-Estimates of Solutions for the Regularized Problem

The idea we shall attack Theorem 1 is to approximate Problem (1.1) (1.2) by the following regularized problem

$$\det(u_{ij}) = K+s, \text{ in } \Omega \text{ with } u=0 \text{ on } \partial\Omega,\ 0<s\leqslant 1. \tag{1.1'}$$

This section is devoted to the estimates of all the derivatives up to order 2 of the solutions to (1.1′), and the lower bound of Δu.

From the result in [2], it follows that (1.1′) always admits a unique smooth convex solution for all $0<s\leqslant 1$. Denote by $\underline{u}$ the solution for the case $s=1$ which is the subsolution of them for all the cases $0<s<1$.

C^1-estimates. It is easy to see, from the convexity of $\underline{u}$ and u and the maximum principle,

$$\min_{\bar{\Omega}} \underline{u} \leqslant \underline{u} \leqslant u \leqslant 0. \tag{2.1}$$

The condition (1.3) and the convexity of u guarantee that $|\partial u|$ attains its maximum on $\partial\Omega$. On the other hand (2.1) implies

$$\underline{u}_n(p) \leqslant u_n(p) \quad \text{and} \quad \underline{u}_\nu(p) \leqslant u_\nu(p) \tag{2.2}$$

for all p on $\partial\Omega$, where n is the interior normal to $\partial\Omega$ and ν is any direction at the angle $<\pi/2$ with n. To estimate from above we assume that p is the origin, n the Y axis and the Y axis meets $\partial\Omega$ at points p and $\bar{p}$. In view of the convexity of u and (2.2) it follows that

$$u_n(p) = u_y(p) \leqslant u_y(\bar{p}) \leqslant \underline{u}_y(\bar{p}). \tag{2.3}$$

From (2.2) (2.3) we can infer

$$|u_n(p)| \leqslant \max |\partial \underline{u}|.$$

Now we have completed the estmates of u and its first derivatives.

C^2-estimates. We begin with the estimates on $\partial\Omega$. Let p be any point on $\partial\Omega$. Suppose that under the coordinate system mentioned previously the defining function $\psi(x)$ is subject to

$$\psi_x(0)=0,\ \psi_y(0)=-1 \text{ and } \psi_{xx}(0)>0. \tag{2.4}$$

With $X=\psi_y\partial_x-\psi_x\partial_y$ we have, at the origin,

$$0 = X^2 u = u_{xx} + \psi_{xx}(0) u_y. \tag{2.5}$$

So $u_{XX}(0)$ is controlled. To evaluate $u_{xy}(0)$ we consider, after differentiating both sides of (1.1), the equation

$$a^{ij}(Xu)_{ij} = K(\text{bounded term}) + a^{ij}(\text{bounded terms}), \tag{2.6}$$

where $a^{ij}=\delta^{ij}\Delta u - u_{ij}$ and the bounded terms are referred to all the terms controlled by some constants independent of s; sometimes, we call them the constants under control. Thus we have

$$a^{ij}(Xu)_{ij} = f, \tag{2.7}$$

where the absolute value of f is controlled by

$$|f| \leqslant C(K + \operatorname{trac}(a^{ij})) \leqslant C_1(\operatorname{trac}(a^{ij})), \tag{2.8}$$

since $XK = X\det(u_{ij}) = 0$ on $\partial\Omega$ and $\operatorname{trac}(a^{ij}) \geqslant 2\sqrt{K}$. So we have

$$a^{ij}((Xu) \mp \gamma(ay-(x^2+y^2)))_{ij} \gtrless 0 \text{ in } \Omega$$

if γ is chosen big enough. By means of the convexity of Ω one can choose a suitable constant a such that

$$Xu \pm \gamma(ay-(x^2+y^2)) \lesseqgtr 0 \text{ on } \partial\Omega.$$

An application of the maximum principle yields

$$|u_{xy}(0)|\leqslant C.$$

There is somewhat difficulty in estimating $u_{xy}(0)$. We first claim that $\partial\Omega$ is noncharacteristic for the linearized operators of (1.1′) in u. It is easy to see that the solution u to (1.1) is subharmonic function since $\Delta u\geqslant 2(K+\varepsilon)^{1/2}\geqslant 2K^{1/2}$ in Ω. Let us solve a boundary value problem

$$\Delta\bar{u}=CK \text{ in } \Omega \text{ with } C=(\max_{\bar{\Omega}} K^{1/2})^{-1} \text{ and } \bar{u}=0 \text{ on } \partial\Omega. \tag{2.9}$$

Obviously, its solution is also a subharmonic function smooth up to the boundary. The strong maximum principle shows $\partial\bar{u}/\partial n\leqslant -C<0$ on $\partial\Omega$. Hereafter, all the letters C, C_1, $\cdots$ denote various constants under control. A similar argument provides $\partial(u-\bar{u})/\partial n\leqslant 0$ on $\partial\Omega$ since $\Delta(u-\bar{u})\geqslant K^{1/2}$. Moreover under the coordinate system we have used previously in (2.4) one can obtain

$$\bar{u}_{xx}(0)=X^2\bar{u}-\psi_{xx}(0)\bar{u}_y\geqslant C>0 \tag{2.10}$$

for another constant C and hence

$$u_{xx}(0)=\bar{u}_{xx}(0)+X^2(u-\bar{u})-\psi_{xx}(0)(u-\bar{u})_y\geqslant C. \tag{2.10′}$$

This means that $u_{xx}(0)$ has a lower bound away from zero which is independent of ε and $\partial\Omega$ is noncharacteristic for the linearized operator of (1.1′) in u. Using (1.1′) we can complete the estimates on $\partial\Omega$ of $\partial^\alpha u$ with $|\alpha|=2$.

Remark 2.1. Let $X=b^i\partial_i$ be a tangential vector with respect to $\partial\Omega$. Then $b^ib^ju_{ij}=b^ib^j\bar{u}_{ij}-b^ib^j_{,i}(u-\bar{u})_j+X^2(u-\bar{u})\geqslant b^ib^j\bar{u}_{ij}\geqslant C$ on $\partial\Omega$ since $b^ib^j_{,i}\partial_{X_j}$ is a vector at the angle less than $\pi/2$ with the interior normal. Here $b_{,i}=\partial b/\partial X_i$.

The next part of this section is devoted to the interior estimates of all derivatives of second order. It suffices to estimate the upper bound of Δu since (u_{ij}) is positive in Ω. With $W=\Delta u+x_1^2+x_2^2$ we have

$$LW=a^{ij}W_{ij}=2\sum_s(u_{12s}^2-u_{11s}u_{22s})+2\Delta u+\Delta K. \tag{2.11}$$

Lemma 2.2. *Let (1.3) be fulfilled and let u be a soluition to (1.1′). Then there exists a constant C independent of ε such that $\Delta u\leqslant C$.*

Proof From the C^2-boundary estimates carried out previously, it follows that there is a constant C which is independent of ε and satisfies $W\leqslant C$ on $\partial\Omega$. If W attains its maximum on $\partial\Omega$, this lemma is proved. Let us consider the case that W attains its maximum at a point $p\in\Omega$. Obviously we have, at p

$$u_{11s}=-(u_{22s}+2x_s),\ s=1.2 \tag{2.12}$$

and

$$LW\leqslant 0. \tag{2.13}$$

Substituting (2.12) into (2.11) and observing (2.13) we can arrive, at p,

$$\begin{aligned}0&\geqslant 2\sum_s(u_{12s}^2-u_{11s}u_{22s})+2\Delta u+\Delta K\\&=2[\sum_s(u_{12s}^2+u_{22s}^2)+2x_su_{22s}]+2\Delta u+\Delta K\\&\geqslant 2\Delta u-8|x|^2+\Delta K.\end{aligned}$$

Hence

$$W(p)=\max_{\Omega} W\leqslant \max(5|x|^2+|\Delta K|/2).$$

This completes the proof of the present lemma.

Lemma 2.3. *Let* (1.3)(1.4) *be fulfilled. Then Δu has a louwer bound which is away from zero and independent of s. Furthermore, for each C^∞ vector field $X=b^i\partial_i$ tangential to $\partial\Omega$, so does $H=b^ib^ju_{ij}$ in a neighbourhood of $\partial\Omega$;*

Proof By means of the interior estimates for elliptic Monge-Ampere equion in [9] it suffices to prove the assertion in Lemma 2.3 to be true in $\bar\Omega\backslash\Omega_\delta=\{x\in\bar\Omega|-\psi(x)<\delta\}$ for some positive constant δ independent of s. For any point $p\in\partial\Omega$, we take the coordinate system as before, namely, p as the origin and the interior normal to $\partial\Omega$ at p as the axis x_2. Evidently, under the present circumstance, we can assume the defining fuction of the form

$$-\psi=x_2-g(x_1) \text{ with } g(0)=\dot g(0)=0 \text{ and } \ddot g(0)>0. \tag{2.14}$$

Differentiation of (1.1) yields at once

$$L(u_{11})=a^{ij}(u_{11})_{ij}=2(u_{121}^2-u_{111}u_{122})+K_{11}. \tag{2.15}$$

Introduce an auxiliary function $W=u_{11}+(x_2^2-h^2)$ where the constant h is yet to be determined and hence

$$LW=2(u_{112}^2-u_{111}u_{122})+K_{11}+2u_{11}. \tag{2.16}$$

We shall restrict our attention on $D_h=\Omega\cap\{0<x_2<h\}$. It is easily seen that $W\geqslant 0$ on $\Omega\cap\partial D_h$ since $u_{11}\geqslant 0$ there. To prove $W\geqslant 0$ too on $\partial\Omega\cap\partial D_h$ we take $X=\partial_{x_1}+\dot g\partial_{x_2}$ in Remark 2.1 and obtain

$$u_{11}+2\dot g u_{12}+\dot g^2u_{22}\geqslant \bar u_{11}+2\dot g\bar u_{12}+\dot g^2\bar u_{22}\geqslant C>0$$

for some positive constant C under control. As a consequence, the inequality

$$u_{11}\geqslant C-\sqrt{h}\,C_1 \quad \text{on} \quad \partial\Omega\cap\partial D_h$$

follows immediately from (2.14) and the C^2-boundary estimates. Therefor $u_{11}>0$ on $\partial\Omega\cap\partial D_h$ if h is chosen small enough.

Suppose that W attains its minimum at $p\in D_h$. Then we have

$$LW\geqslant 0,\ u_{111}=0 \text{ and } u_{112}=-2x_2 \text{ at } p. \tag{2.17}$$

Substituting them into (2.16) we can arrive

$$8x^2+K_{11}(p)+2u_{11}(p)\geqslant 0. \tag{2.18}$$

On the other hand, observing $K=0$ on $\partial\Omega$ we havo

$$K_{11}(p)\leqslant X^2K-\ddot gK_2(p)+C\sqrt{h}. \tag{2.19}$$

From (1.4)(2.14) and the continuity of K_2, (2.18)(2.19) enable us to derive

$$W(p)=(x_2^2-h^2)|_p+u_{11}(p)\geqslant \ddot g(0)K_2(0)/2>0$$

if h is chosen small enough. So far we have proved $W\geqslant 0$ on D_h, namely

$$u_{11}\geqslant (h^2-x_2^2)\geqslant \frac{3}{4}h^2 \text{ on } D_{h/2}. \tag{2.19'}$$

The first assertion in the present lemma is proved if we observe the compactness of

$\partial\Omega$ and the fact that $\Delta u \geqslant u_{11}$ everywhere.

To prove the second assertion in Lemma 2.3 we fix h in (2.19'). Under the present circumstance $X=\partial_{x_1}+\dot{g}\partial_{x_2}=b^i\partial_i$ from Remark 2.1 it follows that

$$H=b^ib^ju_{ij}=u_{11}+2\dot{g}u_{12}+\dot{g}^2u_{22}\geqslant\frac{3}{4}h^2-\sqrt{\rho}C_1 \text{ on } D_\rho$$

if $\rho<h/2$ for some positive constant C_1 independent of h and s, and hence $H\geqslant h^2/2$ on D_{ρ_0} if ρ_0 is small enough.

We now proceed to verify the required assertion for any C^∞-tangential vectror field $\tilde{X}=\tilde{l}^i\partial_i$. Write $N=-\dot{g}\partial_{x_1}+\partial_{x_2}$ which is perpendicular to X. It is easy to express $\tilde{X}$ in terms of X and N

$$\tilde{X}=(\mu_1+(x_2-g)\mu_2)x+(x_2-g)\mu_3N$$

where $\mu_i(i=1, 2, 3)$ are smooth functions defined in D_h and μ_1 is strictly positive. Then

$$\tilde{H}=\tilde{b}^i\tilde{b}^ju_{ij}\geqslant\mu_1^2b^ib^ju_{ij}-|x_2-g|C_1$$
$$\geqslant\mu_1^2h^2/2-C\rho_0 \text{ on } D_{\rho_0}.$$

Choosing a small ρ_0 one can at once complete the proof of the present lemma.

§ 3. Existence of C^∞-Solutions

This section is concerned with the existence of smooth solutions to Problem (1.1)(1.2). To do so we shall give the estimates of modulus of continuity for the second derivatives of the solutions u to (1.1'). Let $X=b^i\partial_i$ be a smooth vector field defined in a neighbourhood of $\partial\Omega$. Differentiation of the both sides of (1.1') yields

$$a^{ij}((Xu)_{ij}-G_{ij}(u))=XK+2K_s\,\mathrm{div}X, \tag{3.1}$$

where $G_{ij}(u)=b^k_{,ij}\partial_k u$ and $K_s=(K+s)$. Furthermore,

$$\begin{aligned}a^{ij}(X^2u)_{ij}=&K_su^{is}u^{tj}((Xu)_{st}-G_{st}(u))((Xu)_{ij}-G_{ij}(u))\\&+a^{ij}(XG_{ij}(u)-G_{ij}(Xu)+b^k_{,j}G_{ki}(u)+b^k_{,i}G_{kj}(u))\\&+K_sX(XK/K)+2K_s\mathrm{div}\,X.\end{aligned} \tag{3.2}$$

From (3.1) we have

$$\begin{aligned}&K_s^{-1}(XK+2K_s\mathrm{div}X)^2\\&=K_su^{ij}u^{st}((Xu)_{ij}-G_{ij}(u))((Xu)_{st}-G_{st}(u))\\&=K_su^{is}u^{jt}((Xu)_{ij}-G_{ij}(u))((Xu)_{st}-G_{st}(u))\\&\quad-2[((Xu)_{12}-G_{12}(u))^2-((Xu)_{11}-G_{11}(u))((Xu)_{22}-G_{22}(u))].\end{aligned} \tag{3.3}$$

Solving $(Xu)_{ij}-G_{ij}(u)$ from the following system

$$b^k((Xu)_{ki}-G_{ki}(u))=(X^2u)_i+h^i=D^i,\ i=1, 2 \tag{3.4}$$

and

$$a^{ij}((Xu)_{ij}-G_{ij}(u))=XK+2K_s\mathrm{div}X=D^0, \tag{3.5}$$

where $h^i=-b^kG_{ki}(u)-b^k_{,i}b^s_{,k}u_s-b^k_{,i}b^su_{sk}(i=1, 2)$ are bounded terms, one can get, with $H=b^sb^ku_{sk}$,

$$(Xu)_{11}-G_{11}(u)=H^{-1}[D^0(b^2)^2+a^{22}(D^1b^1-D^2b^2)-2a^{12}D^1b^2],$$
$$(Xu)_{12}-G_{12}(u)=H^{-1}[a^{11}D^1b^2-D^0b^1b^2+a^{22}b^1D^2],$$
$$(Xu)_{22}-G_{22}(u)=H^{-1}[a^{11}(b^2D^2-b^1D^1)-2b^1a^{12}D^2+(b^1)^2D^0]. \tag{3.6}$$

Suppose that X is a C^∞ vector field which is tangential to $\partial\Omega$ and defined on $\bar{\Omega}\backslash\Omega_\delta$ for some positive constant δ. Then Lemma 2.3 shows $H\geqslant C>0$ on $\bar{\Omega}\backslash\Omega_\delta$. Inserting (3.3)(3.6) into (3.2) we have

$$L(X^2u)=2H^{-1}[a^{ij}(X^2u)_i(X^2u)_j+2a^{ij}(X^2u)_ih^j-D^0b^i(X^2u)_i]+\text{bounded term}. \tag{3.7}$$

Similarly, solving u_{ijk} from the following system

$$\begin{cases} u_{11s}+u_{22s}=(\Delta u)_s, \\ a^{ij}u_{ijs}=K_s, \end{cases} \quad s=1,2, \tag{3.8}$$

and substituting them into

$$L(\Delta u)=a^{ij}(\Delta u)_{ij}=2\sum_s(u_{12s}^2-u_{11s}u_{22s})+\Delta K, \tag{3.9}$$

we can derive, near the boundary,

$$L(\Delta u)=2\Delta u((\Delta u)^2-4K)^{-1}[a^{ij}(\Delta u)_i(\Delta u)_j+B_i(\Delta u)_i+B]+\text{bounded term} \tag{3.10}$$

for some continuous functions B_i, B whose maximal norms are under control since Lemma 2.3 enables us to choose δ so small that $(\Delta u)^2-4K\geqslant C>0$ on $\bar{\Omega}\backslash\Omega_\delta$.

Lemma 3.1. Let Ω be a C^{2+1} convex bounded domain. Suppose that $v\in C^2(\bar{\Omega})$ satisfies

$$Lv=a^{ij}v_{ij}+a^iv_i+av=f \text{ in } \Omega \text{ with } v=0 \text{ on } \partial\Omega, \tag{3.11}$$

where L is elliptic in Ω and a^{ij}, a^i, a, f are continuous and $a\leqslant 0$, and $a^{ij}\psi_i\psi_j$ is strictly positive near the boundary for a defining function ψ of $\partial\Omega$. Then

$$|v(p)|\leqslant C\ \text{dis}(p,\partial\Omega) \tag{3.12}$$

for some constant C depending only on the maximal norms of a^{ij}, a^i, a, f, v, $(\Sigma a^{ii})^{-1}$ and $(a^{ij}\psi_i\psi_j)^{-1}$.

Lemma 3.2. *suppose that* $f=f_1+ha^{ij}v_iv_j$ *where* h, f_1 *are continuous in* $\bar{\Omega}$ *and* $h\geqslant 0$. *et the assumption in Lemma* 2.3 *be fulfilled. Then* (3.12) *is continuously valid if max* $|f|$ *is replaced by* $\max|f_1|+\max|h|$.

Proof Let us first claim that under the assumption of the present lemma for any defining function $\tilde{\psi}$, i. e., $\tilde{\psi}=\sigma\psi$ where σ is a C^{2+1} positive function, there is a constant $\delta>0$ such that $a^{ij}\tilde{\psi}_i\tilde{\psi}_j$ is positive on $\bar{\Omega}\backslash\tilde{\psi}_\delta$. In fact $a^{ij}\tilde{\psi}_i\tilde{\psi}_j=a^{ij}\psi_i\psi_j\sigma^2+2a^{ij}\sigma_i\psi_j\sigma\psi+a^{ij}\sigma_i\sigma_j\psi^2\geqslant(a^{ij}\psi_i\psi_j\sigma^2)/2-C|\psi|^2\geqslant C_1>0$ on $\bar{\Omega}\backslash\Omega_\delta$ if δ is chosen small enough. This proves this claim. Now we proceed to prove Lemma 3.1. Let us consider any point $p\in\partial\Omega$ which is taken to be the origin and let the interior normal at point p be chosen as the axis Y. Suppose that near the point p, the boundary $\partial\Omega$ may be expressed in the form

$$y-bx^2/2+O(x^3)=\psi=0 \text{ on } \partial\Omega \text{ with } b>0.$$

Evidently, we have

$$C\leqslant a^{ij}\psi_i\psi_j=a^{22}+O(|x|)$$

in the intersection of the disk $B_\delta=[(x, y)\in R^2|x^2+y^2<\delta]$ and $\bar{\Omega}$. Without loss of generality, we may assume, if nesessary, taking smaller δ,

$$a^{22}\geqslant C/2 \text{ in } B_\delta. \tag{3.13}$$

Let us study the following auxiliary function

$$W=bx^2/2+My^2-2y,$$

where the constant M is to be determined. A direct computation gives at once

$$LW=a^{11}b+2a^{22}M-2a^2+a^2(2My)+a^1bx+aW.$$

If M is chosen so big that $a^{22}M\geqslant 2|a^2|+|a^1bx|+1$ and δ so small that $\delta>C(4\max|a^2|)^{-1}$and $W<0$ in $B_\delta\cap\Omega$ we have $L(W)\geqslant 1$. Hence $L(v\pm\gamma W)\gtrless 0$ if γ is big enough. On the boundary $B_\delta\cap\partial\Omega$, the inequalities

$$v\pm\gamma W=\gamma(\mp bx^2/2+O(x^3))\lesseqgtr 0$$

and, on the boundary $\partial B_\delta\cap\Omega$,

$$v\pm\gamma W=\pm\gamma[(bx^2/2+My^2)-2y]+v\lesseqgtr 0$$

hold if δ is suitably small and γ is chosen big enough. From the maximum principle it is not difficult to get

$$|v(0,y)|\leqslant\gamma|W(0, y)|\leqslant C_2|y|,$$

which completes the proof of Lemma 3.1.

Proof of Lemma 3.2 By means of the same argument as in proving Lemma 3.1 and the fact that $Lv\geqslant f_1$, without difficulty, one can get

$$v(p)\leqslant C \operatorname{dis}(p, \partial\Omega) \tag{3.13}$$

for some constant C. It remains only to present another side estimate. Without loss of generatlty, we may assume $|v|\leqslant 1/2$ in the domain under consideration. Consider a function $V=(1+V)^{-q}-1$ which satisfies

$$\begin{aligned}LV&=a^{ij}V_{ij}+a^iV_i+aV\\&=q(1+v)^{-q-2}\{((q+1)-(1+v)h)a^{ij}v_iv_j-f_2\},\end{aligned} \tag{3.14}$$

where f_2 is only involved in q, v, and f_1. After taking q as an integer$>|\max(1+v)h|+1$, (3.14) is reduced to the case proved previously. So we have

$$\begin{aligned}(1+v)^q-1&\geqslant-(1+v)^qC\operatorname{dis}(p, \partial\Omega)\\&\geqslant-C_1\operatorname{dis}(p, \partial\Omega).\end{aligned}$$

This implies

$$\begin{aligned}v&\geqslant-C_2[(1+v)^{q-1}+\cdots+1]^{-1}\operatorname{dis}(p, \partial\Omega)\\&\geqslant-C_3\operatorname{dis}(p, \partial\Omega).\end{aligned} \tag{3.15}$$

Lemma 3.3. *Let the assumption in Lemma 3.2 be fulfilled except for the hypothesis:* $v=0$ *on* $\partial\Omega$. *Then we have*

$$-C^{-1}(\operatorname{dis}(x, \partial\Omega))^{1/2}\leqslant v(x)-v(Px)\leqslant C(\operatorname{dis}(x, \partial\Omega))^{1/2}, \tag{3.16}$$

where C *depends only on maximal norms of* a^{ij}, a^i, f, v, $(\Sigma a^{ii})^{-1}(a^{ij}\psi_i\psi_j)^{-1}$ *and* *maximum of* $|\partial v|$ *over* $\partial\Omega$ *and* $PX\in\Omega$, $\mathrm{dis}(X, \partial\Omega)=|x-Px|$.

Proof For any point $p\in\Omega$, take the coordinates as done in proving Lemma 3.1. Furthermore, without loss of generality, we assume $v(0)=0$. Let us write $\xi=2y-bx^2/2-My^2$. Then from the argument in proving Lemma 3.1, we have $\xi>0$ and $L\xi<-1$ in $B_\delta\cap\Omega$ for some positive constant δ. Now instead of the barrier function used before, we take $W=-\xi^{1/2}$. A calculation provides

$$LW=\frac{1}{4}\xi^{-3/2}a^{ij}\xi_i\xi_j-\frac{1}{2}\xi^{-1/2}L\xi-\frac{a}{2}\xi^{1/2}\geqslant\frac{1}{2}\xi^{-1/2}. \tag{3.17}$$

Thus we have $L(v\pm\gamma W)\gtrless 0$ in $B_\delta\cap\Omega$ and $v\pm\gamma W\lessgtr 0$ on $\partial B_\delta\cap\Omega$ if γ is big enough. Now let us check the situation on the rest bouhdary. Indeed, on $B_\delta\cap\partial\Omega$,

$$v\pm\gamma W\lessgtr\pm C_1|x|\mp\gamma(bx^2/2+O(|x|^3)^{1/2}\lessgtr 0$$

if γ is big enough. So the maximum principle tells us

$$|v(0, y)|\leqslant\gamma|W(0, y)|\leqslant Cy^{1/2}. \tag{3.18}$$

The remainder of the proof is the same and need not be repeated. This completes the proof of the present lemma.

The equicontinuity of $\partial^\alpha u$, $|\alpha|=2$. We first discuss the both sided bounds on $\partial\Omega$ of all third derivatives of u. From the fact that $u=0$ on $\partial\Omega$ it turns out $|X^3u|=0$ there for any tangential vector field $X=b^i\partial_i$. Now we take $X=\psi_2\partial_{x_1}-\psi_1\partial_{x_2}$ and hence Lemma 2.3 tells us $a^{ij}\psi_i\psi_j=b^ib^ju_{ij}$ strictly positive near the boundary. An application of Lemma 3.2 to (3.7) provides at once

$$|X^2u|\leqslant C_1\mathrm{dis}(x, \partial\Omega),\ x \text{ near } \partial\Omega. \tag{3.19}$$

Consequenely,

$$|\partial(X^2u)/\partial n|\leqslant C_2 \text{ on } \partial\Omega. \tag{3.20}$$

The fact that $\partial\Omega$ is noncharacteristic guarantees ∂^2u/∂^2n can be expressed in terms of other second derivatives. So the bounds on $\partial\Omega$ of all the rest third derivatives can be controlled.

Now we are in a position to prove the equicontinuity of all the second derivatives. As done before, by means of Lemma 3.3 to (3.10) one can get

$$|\Delta u(x)-\Delta u(Px)|\leqslant C(\mathrm{dis}(x, \partial\Omega))^{1/2}. \tag{3.21}$$

Under the coordinate system we worked with in (2.14), with $X=\partial_{x_1}+\dot g\partial_{x_2}$ we have, if $x=(0, x_2)$ and $Px=\{0\}$,

$$\begin{aligned}|u_{11}(x)-u_{11}(Px)|&=|X^2u-2\dot g u_{12}-\dot g^2u_{22}-\ddot g u_2-(X^2u-\ddot g u_2)|_{x=0}|\\&=|X^2u-(X^2u)|_{X=0}|+|\ddot g(0)||u_2-u_2(0)|\\&\leqslant C_3|x_2|=C_3\mathrm{dis}(x, \partial\Omega)\end{aligned} \tag{3.22}$$

since $\dot g(0)=0$. Therefore, the inequality

$$|u_{22}(0, x_2)-u_{22}(0, 0)|\leqslant C|x_2|^{1/2} \tag{3.23}$$

follows immediately if we combine (3.21) with (3.22). Now we proceed to estimate $|u_{12}(0, x_2)-u_{12}(0, 0)|$. Distinguish two cases.

Case(1), $u_{12}(0, x_2)$ and $u_{12}(0, 0)$ have the same sign. From (1.1) it is easy to see

$$|u_{12}-u_{12}(0, 0)|\leqslant \frac{|(u_{11}u_{22}-K)-(u_{11}u_{22}-K)|_{y=0}|^{1/2}}{\sqrt{u_{11}u_{22}-K}+\sqrt{(u_{11}u_{22}-K)|_{x=0}}}$$
$$\cdot|(u_{11}u_{22}-K)-(u_{11}u_{22}-K)|_{x=0}|^{1/2}.$$

The first factor of the right-hand side is less than or equal to 1. Hence

$$|u_{12}(0, x_2)-u_{12}(0, 0)|\leqslant C|x_2|^{1/4}. \tag{3.24}$$

Case(2), $u_{12}(0, x_2)$ and $u_{12}(0, 0)$ have different signs. Then there exists a $X^*=(0, X_2^*)$ with $u_{12}(0, x_2^*)=0$. Hence

$$\begin{aligned}|u_{12}(0, 0)|^2&=|(u_{11}u_{22}-K)|_{x=0}-(u_{11}u_{22}-K)|_{x=x^*}|\\&\leqslant C|x^*|^{1/2}\\&\leqslant C|x_2|^{1/2},\end{aligned}$$

and analogously

$$\begin{aligned}|u_{12}(0, x_2)|^2&=|(u_{11}u_{22}-K)-(u_{11}u_{22}-K)|_{x=x^*}|\\&\leqslant C|x_2|^{1/2}.\end{aligned}$$

So far we have proved

$$|u_{12}(x)-u_{12}(Px)|\leqslant C|\mathrm{dis}(x, \partial\Omega)|^{1/4}. \tag{3.25}$$

Summarizing up all the conclusions obtained we have

Lemma 3.4. *Let the assumption in Lemma 2.3 be fulfilled. Then for the solution u to Problem (1.1′) the inequalities*

$$|\partial^\beta u|\leqslant C \text{ on } \partial\Omega,\ |\beta|=3 \tag{3.26}$$

and

$$|\partial^\alpha u(x)-\partial^\alpha u(Px)|\leqslant C|x-Px|^{1/4},\ X\in\Omega,\ |\alpha|=2 \tag{3.27}$$

hold for some constant C independent of s.

Before completing the verification of Theorem 1, we also need a result on regularity of solutions to Dirichlet problem for degenerate elliptic Monge-Ampere equation. For the proof, see [7].

Lemma 3.5. *Any solution u in $C^2(\bar\Omega)$, to the Dirichlet problem:*

$$\det(u_{ij}+A_{ij}(x, u, \partial u))=K(x)f(x, u, \partial u) \text{ in } \Omega\subset R^2 \tag{3.28}$$

with

$$u=\varphi \text{ on } \partial\Omega \tag{3.29}$$

where f is strictly positive and K satisfies (1.3)(1.4), is smooth up to the boundaay if f, A, φ and ∂Ω are smooth and ∂Ω noncharacteristcs and the matrix $(u_{ij}+A_{ij}(x, u, \partial u))$ nonnegative.

The end of the proof of Theorem 1. The uniqueness is the immediate consequence of the maximum principle. Therefore it suffices to prove the existence of C^∞-solutions. In view of The Arzela theorem and Lemma 3.5 we need only verify the equicontinuity of second derivatives of solutions with respect to s. By means of the interior estimates given by [9] for elliptic Monge-Ampere equation, we know

$$|\partial^\beta u(X)| \leqslant (\rho(d(X, \partial\Omega))^{-1} X \in \Omega \text{ and all } |\beta| = 3, \tag{3.30}$$

where $\rho(t)$ is a bounded monotone-increasing, positive function defined in $[0, d^*]$, $2d^*$ = diameter of Ω, and moreover, $\rho(t) \to 0$ if $t \to 0$. For any point $p \in \partial\Omega$, we assume the defining function satisfies (2.14). A change of variables from $(x_1, x_2) \to (x, y) = (x_1, x_2 - g(x_1))$ makes the boundary flatten and the properties (3.26) (3.27)(3.30) are invariant for another constant C. Let $W(t)$ be the inverse functson of $t = W\rho(W^4)$. It is easy to see

$$W(t) \to 0 \text{ if } t \to 0. \tag{3.31}$$

Distinguish three cases.

Case(a). Let $p_i(x_i, y_i)$ satisfy $y_1 \leqslant y_2$ and $|p_1 - p_2| \leqslant \rho(y_1) W(|p_1 - p_2|)$. From (3.30)

$$\begin{aligned} |\partial^\alpha u(p_1) - \partial^\alpha u(p_2)| &\leqslant (\rho(y_1))^{-1} |p_1 - p_2| \\ &\leqslant W(|p_1 - p_2|) \text{ for all } |\alpha| = 2 \end{aligned} \tag{3.32}$$

follows at once.

Case(b). Let $p_i(x_i, y_i)$ satisfy $x_1 = x_2$, $y_1 \leqslant y_2$ and $|p_1 - p_2| \geqslant \rho(y_1) \cdot W(|p_1 - p_2|)$. Then by (3.27) we have

$$|\partial^\alpha u(p_1) - \partial^\alpha u(p_2)| \leqslant C |y_1^{1/4} + y_2^{1/4}| \text{ for all } |\alpha| = 2,$$

which is controlled by $3C|p_1 - p_2|^{1/4}$ if $y_2 \geqslant 2y_1$ or by

$$3C(\rho^{-1}(t/W(t)|_{t=|p_1-p_2|})^{1/4} = 3CW(|p_1 - p_2|) \text{if } y_2 \leqslant 2y_1.$$

Crse(c). Let $p_i(x_i, y_i)$ satisfy $|p_1 - p_2| \geqslant \rho(y_1) W(|p_1 - p_2|)$ and $y_1 \leqslant y_2$. Then with $p^*(x_2, y_1)$, for all $|\alpha| = 2$, it is easy to see

$$\begin{aligned} &|\partial^\alpha u(p_1) - \partial^\alpha u(p_2)| \leqslant |\partial^\alpha u(p_2) - \partial^\alpha u(p^*)| + |\partial^\alpha u(p^*) - \partial^\alpha u(p_1)| \\ &\leqslant C\max(W(|p_2 - p_1|), |p_2 - p^*|^{1/4}) + 2Cy_1^{1/4} + |\partial^\alpha u(x_2, 0) - \partial^\alpha u(x_1, 0)| \\ &\leqslant C\max(W(|p_1 - p_2|), |p_1 - p_2|^{1/4}). \end{aligned}$$

In getting the last ineqality, we have used (3.26). This completes the proof of Theorem 1.

§ 4. The Proof of Theorem 2

In this section we shall use a similar argument to prove Theorem 2.

The first step is to truncate the functi on f in such a way that f^m, $C^\infty(\bar{\Omega} \times R^1 \times R^2)$, $m = 1, 2, \cdots$, are subject to the following conditions

$$\begin{gathered} 0 < f^m \leqslant C_m, \; f_u^m \geqslant 0, \text{ and } f^m(x, 0, p) \leqslant C(1 + p^2)^2 \\ \text{with } C_m \to \infty \text{ if } m \to \infty, \end{gathered} \tag{4.1}$$

$$f^m(x, N, p) \leqslant h(p)^{-1} \text{ if } |p| \leqslant m \tag{4.2}$$

and

$f^m \cdot \partial^\alpha f^m$ uniformly approximate to f, $\partial^\alpha f$ in any compact set of $\overline{\Omega}\times R^1\times R^2$ for each $\alpha\in Z_+^5$. (4.3)

Here the constants N, C and the function $h(p)$ are mentioned in Theorem 2 and C_m are some constants depending only on m. To construct such f^m we first truncate f by multiplying a cutoff function $\varphi(p)$

$$f^m=f\varphi\left(\frac{p}{m}\right)+m\left(1-\varphi\left(\frac{p}{m}\right)\right),$$

where $0\leqslant\varphi\leqslant1$ and $\varphi=1$ if $|p|\leqslant1$, $\varphi=0$ if $|p|\geqslant2$. Therefore, f^m satisfies (4.1)(4.2) if $C(1+m^2)^{3/2}\geqslant1$. Now we truncate f^m with respect to u. Write

$$[f^m]_m=\begin{cases} f^m, & |u|\leqslant m,\\ f^m(x,\ \pm m,\ p) & \text{if } u>m \text{ or } u<-m.\end{cases}$$

It is apparent that $[f^m]_m$ still satisfies (4.1)(4.2) almost everywhere in u. Let $j(t)\in C_0^\infty(R^1)$ be a kernel of a mollifier operator, namely, $\int j(t)dt=1$, $j(t)\geqslant0$. Define

$$f^m=m\int j(m(u-\bar{u}))\,[f^m]_m(x,\ \bar{u},\ p)d\bar{u}. \tag{4.4}$$

It is easy to check that f^m defined in (4.4) are the desired functions.

The second step is to study the following regularized Dirichlet problem

$$\det(u_{ij})=(K(x)+1/C_m)f^m \text{ in } \Omega \text{ with } u=0 \text{ on } \partial\Omega. \tag{1.5$_m$}$$

From [8, Theorem 2.12] it follows that Problem $(1.5)_m$ admits a unique convex solution u^m in $C^{2+\alpha}(\overline{\Omega})$ for some $0<\alpha<1$. The regularity theorem for solutions to nonlinear elliptic boundary value problem tells us $u^m\in C^\infty(\overline{\Omega})$.

The third step, which is the crucial one of our proof, is to estimate the C^1 and C^2-bounds of u^m, and the modulus of continuity of its all second derivatives. These estimates can be accomplished with only minor modification to the method presented in [4]. We insert the outline of these estimatsons here for completeness, most of which are quoted from [8]. The convexity of u guarantees $u^m\leqslant0$ in Ω. For the lower bound of u^m, we choose R so large that

$$\mu(\Omega)C_m^{-1}+\int_\Omega K dx<\int_{B_R(0)} h\,dp \text{ for sufficiently large } m. \tag{1.10$'$}$$

This is possible from (1.10). The gradient mapping Du^m: $\Omega\to R^2$ is one to one with $\det(u_{ij})>0$. We obtain

$$\int_{Du^m(\Omega_N)} hdp=\int_{\Omega_N}(K+1/C_m)dx<\int_{B_R(0)} hdp,$$

where $\Omega_N=\{x\in\Omega\,|\,u(x)<N\}$. So there is a $p\in B_R(0)\setminus Du^m(\Omega_N)$. We then have, by the convexity of u^m,

$$\begin{aligned}\inf_\Omega u^m=\inf_{\Omega_N} u^m&\geqslant\inf_{\partial\Omega_N} u^m-|p|\operatorname{diam}\Omega\\ &\geqslant\min(0,\ N)-R\operatorname{diam}\Omega\end{aligned}$$

for sufficiently large m. This is just the lower bound we require.

The next step is the estimation of the gradient of u^m. From the convexity of u^m, and the boundary condition $u^m=0$ on $\partial\Omega$, it suffices to evaluate the lower bound of $\partial u^m/\partial n$ for inward normal direction n. Next we shall construct a lower barrier for Problem $(1.5)_m$: Let $\psi(x)$ be a defining function of $\partial\Omega$ and let ψ be strictly convex with $\det(\psi_{ij})\geqslant 1$. Define

$$W=\psi-\int_0^d (a+bt^2)^{-1/2}dt,$$

where $d=\mathrm{dis}(x, \partial\Omega)$. Choose a suitable b and so small a and ξ that

$$\int_0^\varepsilon (a+bt^2)^{-1/2}dt>2\max|u^m|$$

$$(a+b\varepsilon^2)^{-1/2}\geqslant \sup(1+|\partial\psi|^2)$$

and

$$\begin{aligned}\det(W_{ij})&\geqslant 1+\varkappa bd(a+bd^2)^{-2}\\ &\geqslant (K+1/C_m)f^m(x, W^m, \partial W^m),\end{aligned}$$

where $\varkappa$ is the curvature of $\partial\Omega$. An argument of the maximum principle for u^m-W yields at once the lower bound of $\partial u^m/\partial n$ and hence, the C^1-estimation since u^m is convex.

We shall next present the rest estimates. Before doing so we should emphasize that for sufficiently large m,

$$f^m(x, u^m(x), \partial u^m(x))=m\int j(m(u^m(x)-\bar{u}))f(x, \bar{u}, \partial u^m(x))d\bar{u}.$$

Consequently,

$$(f^m_{p_ip_j}(x, u^m(x), \partial u^m(x)))\geqslant 0.$$

Let us focus our attention on (3.1). Replacing $K(x)$ by $(K+1/C_m)f^m=K_mf^m$ we have

$$\begin{aligned}L_1(Xu)&=a^{ij}(Xu)_{ij}-K_mf^m_{,p_i}(Xu)_i-K_mf^m_{,u}(Xu)\\ &=K_m(\text{bounded term})+a^{ij}(\text{bounded term}).\end{aligned}\tag{4.5}$$

We shall next claim that in $B_\delta\cap\Omega$, the function $v=bx^2/2+y^2-2y$ with $b>0$ can be regarded as a lower barrier function for Xu at the origin p. In fact, a direct calculation gives

$$\begin{aligned}L_1v&=(a^{11}+2a^{22})-K_mf^m_{,p_1}bx-K_mf^m_{,p_2}2(y-1)-K_mf^m_{,u}v\\ &\geqslant \min\{b, 1\}\Delta u^m-C_1K_m\geqslant C_2\Delta u^m, \text{ in } B_\delta\cap\Omega,\end{aligned}$$

if δ is small enough. This proves the required claim, and hence the inequality

$$|\partial(Xu^m)/\partial y(0)|\leqslant C$$

holds from a quite similar discussion in section 2. As for the estimate of $u_{yy}(0)$, analogously, we must first evaluate the lower bound of $u_{xx}(0)$. From $(1.5)_m$ it follows that

$$\Delta u^m\geqslant 2\sqrt{\det(u^m_{ij})}\geqslant 2\sqrt{K(x)C_1},$$

where C_1 = minimum of $f^m(x, u, p)$ over $x\in\overline{\Omega}$ $|u|+|p|\leqslant C^*=\sup\{|u^m|+|\partial u^m|\}$. The solution $\bar{u}$ to Problem (2.9), if necessary, changing the constant C, can be regarded as an upper barrier function of u^m and the fact that $\partial\bar{u}/\partial n\leqslant -C<0$ on implies $\bar{u}_{xx}(0)\geqslant CR^{-1}$ where R is the curvature radius at the point p of $\partial\Omega$. So $u^m_{xx}(0)\geqslant\bar{u}_{xx}(0)-R^{-1}.\partial(u^m-\bar{u})/\partial n\geqslant CR^{-1}$, and hence $u^m_{yy}(0)$ is controlled too.

We must explain the C^2-estimate in Ω. In fact, a calculation yields, with $W=\Delta u^m+x^2+y^2$,

$$\begin{aligned} L_1(W) &= a^{ij}W_{ij}-K_m f^m_{,p_i}W_i-K_m f^m_{,u}W \\ &= 2\sum_s(u^2_{12s}-u_{11s}u_{22s})+K_m f^m_{,p_ip_j}u_{is}u_{js}+2\Delta u^m+\text{bounded term} \\ &\geqslant 2\sum_s(u^2_{12s}-u_{11s}u_{22s})+2\Delta u^m+\text{bounded term} \end{aligned}$$

since $f^m(x.\ u.\ p)$ is convex in p. Repeating the argument in proving Lemma 2.2 provides at once the upper bound of Δu^m. If we repeat the same argument as done in (2.17)—(2.19) there is no difficulty in getting the lower bounds of Δu^m and $H=b^ib^ju^m_{ij}$ for any given vector field $X=b^i\partial_i$ tangential to $\partial\Omega$. The remainder of the proof is the same as before and is left to the readers.

References

[1] Aubin, T., Equations de Monge-Ampère reellee, *J. Funct. Anal.*, **41**: 3(1881), 354—377.

[2] Caffarelli, L., Nirenberg, L., Spruck, J., The Dirichlet problem for nonlinear second order elliptic equations I–Monge-Ampère equation, *Comm. Pure Appl. Math.*, **37**(1984)., 369—402.

[3] Trucdinger, N. S., Fully nonlinear uniformly elliptic equations under natural structure conditions, *Trans. Amer. Math Soc.*, **278**: 2(1983), 751—769.

[4] Trudinger N. S., & Urbas, J. I. E., The Dirichlet problem for the equation of prescribed Gauss curvature, *Bull. Aust. Math. Soc.*, **28**(1983), 217—231.

[5] Lions, P. L., Sur les equatios de Monge-Ampère I. Manuscripta Math., **41**(1983), 1—43.

[6] Krylov, N. V., On degenerate nonlinear elliptic equations, *Math. USSR. Sbornik*, **48**: 2(1984), 307-632.

[7] Hong J. X. & Zuily, C., L_p and Hölder estimates for a class of degenerate elliptic boundary value prob ems(to appear).

[8] Urbas, J. I- E., Elliptic equations of Monge-Ampère type., Thesis in The Australian National University, 1984.

[9] Evans, L, Classical solutions of fully nonlinear convex second-order elliptic equtions, *Comm. Pure Appl. Math.*, **25**(1982), 333—363.