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DAVENPORT'S THEOREM IN SHORT INTERVALS

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Abstract

Let μ be te Möbius function. It is proved that for any $A>0$

$$\sum_{x<n\leqslant x+y}\mu(n)e(n\alpha)\ll_{\varepsilon,A}y(\log y)^{-A}$$

holds uniformly for real α and $y\geqslant x^{2/3+\varepsilon}$ $(\varepsilon>0)$, which generalizes H. Davenport's theorem for Möbius function to short intervals.

§1. Statement of the Results

In 1937 H. Davenport[1] proved the following theorem for Möbius μ-function

Theorem[1]. *Let R denote the set of real numbers. For any given $A>0$*

$$\sum_{n\leqslant x}\mu(n)e(n\alpha)\ll(\log x)^{-A} \tag{1.1}$$

holds uniformly for $\alpha\in R$, where $e(\mu)=\exp(2\pi iu)$ and the $\ll$ constant depends only on A.

Using Vaughan's identity the author in [2] showed that the analogous result of (1.1) holds for arithmetic progressions. However, no results of type (1.1) in short intervals are known, and such a kind of result will be shown in the present paper. The main theorem is as follows

Theorem 1. *For any given $A>0$, $\varepsilon>0$,*

$$\sum_{x<u\leqslant x+y}\mu(n)e(n\alpha)\ll y(\log y)^{-A} \tag{1.2}$$

holds uniformly in $\alpha\in R$ and $y\geqslant x^{2/3+\varepsilon}$, where the $\ll$ constant depends only on A and ε.

We shall use the following notations. $L=\log y$, $\varepsilon'>0$ is a constant which may be taken arbitrarily small and may be different at each occurence. $\rho(q)$ is the function defined by

$$\rho(q)=\prod_{p\mid q}\left(1+\frac{1}{\sqrt{p}}\right),$$

p always denotes a prime. We also use the notaion $\sum_{\chi_l}^{*}$ which represents the summation over all primitive characters modulus l; in particular, if $l=1$, $\chi_{l(n)}^{*}\equiv1$ for all n.

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The proof of Theorem 1 is based on Heath-Brown's identity, Ramachandra's method in [3] and the mean square estimate of Dirichlet L-functions in short intervals which is due to the autnor [4]. Precisely speaking, we start from the well-known Dirichlet's lemma, which states that given $\tau>0$, then for any $\alpha\in R$ there exist a pair of integers a and q, $(a, q)=1$, $1\leqslant q\leqslant\tau$ such that

$$\left|\alpha-\frac{a}{q}\right|\leqslant\frac{1}{q\tau}. \tag{1.3}$$

(We shall choose $\tau=x^{1/3}$ in the proof.)

Lemma 7 below (§ 2) shows that our task in proving Theorem 1 is reduced to the estimation of the sum

$$\sum_{\chi_l}{}^{*}\Big|\sum_{\substack{x/d<m\leqslant(x+y)/d\\(m,q)=1}}\mu(m)\chi(m)e(m\lambda d)\Big| \tag{1.4}$$

with $l\geqslant1$, $ld|q$ and $\lambda=\alpha-\frac{a}{q}$ $\left(|\lambda|\leqslant\frac{1}{q\tau}\right)$.

For α satisfying (1.3) with "large" $q(q\geqslant L^{c})$ and $l>1$, Heath-Brown's identity and the mean square estimate of L-functions in short intervals enable us to give an estimate for sum (1.4), namely,

Theorem 2. *Let $\tau=x^{1/3}$, $1\leqslant q\leqslant\tau$, $|\lambda|\leqslant\frac{1}{q\tau}$ and $y=x^{2/3+\varepsilon}$ ($\varepsilon>0$ sufficiently small). Then for any $A>0$, $ld|q$ and $l>1$ we have*

$$\sum_{\chi_l}{}^{*}\Big|\sum_{\substack{x/d<m\leqslant(x+y)/d\\(m,q)=1}}\mu(m)\chi(m)e(m\lambda d)\Big|\ll d^{-1/2}l^{(1-\varepsilon)/2}yL^{c_1}$$

where $c_1>0$ is an absolute constant and indepent of A and ε, the $\ll$ constant depends only on A and ε.

While α satisfying (1.3) with "small" $q(q\leqslant L^{c})$, or with "large" q and $l=1$, we apply Ramachandra's method in [3] and zero density theorem for L-functions (Lemma 4 and Lemma 5 in § 2) to the sum (1.4) and obtain

Theorem 3. *Let $\tau=x^{1/3}$, $1\leqslant q\leqslant\tau$, $|\lambda|\leqslant\frac{1}{q\tau}$ and $y=x^{2/3+\varepsilon}$ ($\varepsilon>0$ sufficiently small). If*

i) $q\leqslant L^{c_2}$

($c_2>0$ is any given constant) or

ii) $q\geqslant L^{c_2}$ *and* $l=1$,

then for $ld|q$ and $A>0$ we have

$$\sum_{\chi_l}{}^{*}\Big|\sum_{\substack{x/d<m\leqslant(x+y)/d\\(m,q)=1}}\mu(m)\chi(m)e(m\lambda d)\Big|\ll d^{-1/2}\rho(q)yL^{-A},$$

where the $\ll$ constant depends only on A and ε.

Now it is easy to see that (1.2) follows from Lemma 7 (below), Theorem 2 and Theorem 3 if $y=x^{2/3+\varepsilon}$ and ε is sufficiently small. As for $y>x^{2/3+\varepsilon}$ we could find an integer K such that $y=Kx^{2/3+\varepsilon_1}$ ($\varepsilon/2\leqslant\varepsilon_1\leqslant\varepsilon$). Then we have

$$\sum_{x<n\leqslant x+y}\mu(n)e(n\alpha)=\sum_{0\leqslant k\leqslant K-1}\ \sum_{x+kx^{5/8+\varepsilon_1}<y\leqslant x+(k+1)x^{5/8+\varepsilon_1}}\mu(n)e(n\alpha)\ll_{\varepsilon,A}yL^{-A}.$$

Hence Theorem 1 is proved.

§ 2. Some Lemmas

Several preliminary results will be needed in the proofs. We state them here as lemmas.

Lemma 1.[5] *Let $F(u)$ and $G(u)$ be real functions in (a, b), $|G(u)|\leqslant M$ and $G(u)/F'(u)$ be monotonic.*

a) *If $|F'(u)|\geqslant m>0$, we have*

$$\int_a^b G(u)\exp(iF(u))du\ll\frac{M}{m}.$$

b) *If $|F''(u)|\geqslant r>0$, we have*

$$\int_a^b G(u)\exp(iF(u))du\ll\frac{M}{\sqrt{r}}.$$

Lemma 2.[6] *Let*

$$H(s,\chi)=\sum_{n=M+1}^{M+N}a_n\chi(n)n^{-s},\ \chi=\chi \bmod l,\ s=\sigma+it.$$

Then for all $T\geqslant 1$ and real U we have

$$\int_U^{U+T}\sum_{\chi_l}|H(s,\chi)|^2dt\ll\sum_{n=M+1}^{M+N}(lT+n)|a_n|^2n^{-2\sigma}.$$

Lemma 3.[4] *For $H\geqslant T^{1/3}$ and $l\geqslant 1$,*

$$\sum_{x_l}{}^{*}\int_T^{T+H}|L(1/2+it,x)|^2dt\ll\varphi(l)H\log(lH).$$

Lemma 4.[7] *Let $N(\alpha, T, l)$ be the number of zeros of $\prod_{x_l}L(s,\chi_l)$ in*

$$\mathrm{Re}s\geqslant\alpha \text{ and } |\mathrm{Im}s|\leqslant T.$$

For $l\leqslant(\log T)^c$ there exists $D=D(c)>0$ such that

$$N(\alpha,T,l)\ll T^{1600(1-\alpha)^{3/2}}(\log T)^D,\ 1/2\leqslant\alpha\leqslant 1.$$

Lemma 5.[4] *Use the notation in Lemma 4. Let*

$$N(\alpha,T,H,l)=N(\alpha,T+H,l)-N(\alpha,T,l).$$

Then for $H\leqslant T^{1/3}$ and $l\geqslant 1$ we have

$$N(\alpha,T,H,l)\ll(lH)^{8(1-\alpha)/3}(\log lH)^{216},\ 1/2\leqslant\alpha\leqslant 1.$$

Lemma 6.[8] *For $l\geqslant 1$, $\prod_{\chi_l}L(s,\chi)$ $(s=\sigma+it)$ has no zeros in*

$$\lambda\geqslant 1-\frac{C_0}{\log q+(\log(T+2))^{4/5}} \text{ and } |t|\leqslant T\ (C_0>0)$$

except the only excepoional zero $\tilde{\beta}$. And for $l\leqslant(\log T)^c$ no such exceptional zero exists.

Lemma 7. *Let*

$$S(\alpha,x,y)=\sum_{x<u\leqslant x+y}\mu(n)e(n\alpha),\ \alpha=\frac{a}{q}+\lambda,\ (a,q)=1.$$

Then

$$S(\alpha,x,y)\ll L\sum_{ld|q}\left(\frac{q}{d}\right)^{-1/2}\mu^2(d)\sum_{\chi_l}{}^{*}\Big|\sum_{\substack{x/d<m<(x+y)/d\\(m,q)=1}}\mu(m)\chi(m)e(m\lambda d)\Big|.$$

Proof

$$\begin{aligned}S(\alpha,x,y)&=\sum_{d|q}\sum_{\substack{x<u\le x+y\\(u,q)=d}}\mu(n)e\left(\frac{a}{q}n\right)e(n\lambda)\\&=\sum_{d|q}\sum_{\substack{x/d<m<(x+y)/d\\(m,q/d)=1}}\mu(dm)e\left(\frac{a}{q/d}m\right)e(md\lambda)\\&=\sum_{d|q}\sum_{l=1}^{q/d}{}'e\left(\frac{a}{q/d}l\right)\sum_{\substack{x/d<m<(x+y)/d\\m\equiv l(\mathrm{mod}\,q/d)\\(m,q/d)=1}}\mu(dm)e(md\lambda)\\&=\sum_{d|q}\frac{1}{\varphi(q/d)}\sum_{l=1}^{q/d}{}'e\left(\frac{a}{q/d}l\right)\sum_{x_{q/d}}\bar\chi(l)\sum_{x/d<m<(x+y)/d}\mu(dm)\chi(m\lambda d)\\&=\sum_{d|q}\frac{1}{\varphi(q/d)}\sum_{x_{q/d}}\sum_{l=1}^{q/d}\bar\chi(l)e\left(\frac{a}{q/d}l\right)\cdot\sum_{x/d<m<(x+y)/d}\chi(m)\mu(dm)\lambda(m/d)\\&\ll L\sum_{d|q}\left(\frac{q}{d}\right)^{-1/2}\sum_{\chi_{q/d}}\Big|\sum_{x/d<m\le(x-y)/d}\mu(dm)\chi(m)e(m\lambda d)\Big|\\&=L\sum_{d|q}\left(\frac{q}{d}\right)^{-1/2}\sum_{l|q/d}\sum_{\chi_l}{}^{*}\Big|\sum_{\substack{x/d<m\le(x+y)/d\\(m,q/d)=1}}\mu(dm)\chi(m)e(m\lambda d)\Big|\\&=L\sum_{ld|q}\left(\frac{q}{d}\right)^{-1/2}\mu^2(d)\sum_{\chi_l}{}^{*}\Big|\sum_{\substack{x/d<m\le(x+y)/d\\(m,q)=1}}\mu(m)\chi(m)e(m\lambda d)\Big|\end{aligned}$$

The lemma is proved.

§ 3. The Proof of Theorem 2

In this section we assume all the conditions of Theorem 2.

Let $S_1(\chi)=\sum_{\substack{x_1<m\le x_1+y_1\\(m,q)=1}}\mu(m)\chi(m)e(m\lambda d)$ (χ is primitive character modulus l)

and

$$S_1(l,d)=\sum_{\chi_l}{}^{*}|S_1(\chi)|,$$

where

$$x_1=\frac{x}{d},\ y_1=\frac{y}{d}.$$

In Heath-Brown's identity for Möbius function

$$\frac{1}{\zeta}=\sum_{j=1}^{k}(-1)^{j-1}\binom{k}{j}\zeta^{j-1}M^j+\frac{1}{\zeta}(1-\zeta M)^k$$

we choose $k=3$, $M=M_X(s)=\sum_{u\le X}\mu(n)n^{-s}$ and $X=\sqrt[3]{2x_1}$. It is easy to see that for $x_1\le u\le 2x_1$, the sum

$$\sum_{\substack{x_1<m\le u\\(m,q)=1}}\mu(m)\chi(m)\tag{3.1}$$

is a liner combination of $O(1)$ sums of the form

$$S_2=\sum_{\substack{x_1<n_1n_2\cdots n_5<u\\ u_j\leqslant X,3\leqslant j\leqslant 5}} a_1(n_1)\chi(n_1)a_2(n_2)\chi(n_2)\cdots a_5(n_5)\chi(n_5)$$

with

$$a_j(n)=\begin{cases}1, & 1\leqslant j\leqslant 2,\\ \mu(n), & 3\leqslant j\leqslant 5,\end{cases} \quad \text{if } (n, q)=1,$$

and $a_j(n)=0$ otherwise (in fact, some n_k in S_2 may only take the value 1, however, this does not affect the following proof).

By Perron's summation formula we have

$$S_2=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_{\substack{x_1<n_1\cdots n_5<2x_1\\ u_j\leqslant X,3\leqslant j\leqslant 5}}\frac{a_1(n_1)\chi(n_1)\cdots a_5(n_5)\chi(n_5)}{(n_1n_2\cdots n_5)^s}\frac{u^s-x_1^s}{s}\,ds+O\left(\frac{x_1^{1+\varepsilon'}}{T}\right). \quad (3.2)$$

On splitting up each range of the summation in (3.2) into intervals of the form $N<n\leqslant 2N$, we find that S_2 is the sum of $O(L^5)$ sums of the following form

$$S_3=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\sum_{n_j\in I_j}\frac{a_1(n_1)\chi(n_1)\cdots a_5(n_5)\chi(n_5)}{(n_1\cdots n_5)^s}\cdot\frac{u^s-x_1^s}{s}\,ds+O\left(\frac{x_1^{1+\varepsilon'}}{T}\right), \quad (3.3)$$

where

$$I_j=(N_j, 2N_j),\ x_1\ll\prod_{j=1}^{5}N_j\ll x_1 \text{ and } N_j\leqslant X,\ 3\leqslant j\leqslant 5.$$

Letting

$$f_j(s, \chi)=\sum_{n\in I_j}a_j(n)n^{-s}\chi(n) \quad (1\leqslant j\leqslant 5),$$

$$F(s, \chi)=\prod_{j=1}^{5}f_j(s, \chi)$$

and shifting the integral line on the right-hand side of (3.3) to Re $s=1/2$, we get

$$S_3=\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}F(s, \chi)\,\frac{u^s-x_1^s}{s}\,ds+O\left(\frac{x_1^{1+\varepsilon'}}{T}\right)+O\left(\max_{1/2\leqslant\sigma\leqslant c}\frac{x_1^{\sigma}\cdot x_1^{1-\sigma}}{T}\right)$$
$$=\frac{1}{2\pi}\int_{-T}^{T}F(1/2+it, \chi)\frac{u^{1/2+it}-x_1^{1/2+it}}{1/2+it}\,dt+O\left(\frac{x_1^{1+\varepsilon'}}{T}\right).$$

The above discussion shows that the sum (3.1) is a liner combination of $O(L^5)$ sums of form S_3. Hence

$$S_1(\chi)=\int_{x_1}^{x_1+y_1}e(\lambda du)d\sum_{\substack{x_1<m\leqslant u\\ (m,q)=1}}\mu(m)\chi(m)$$
$$\ll L^5\max_{(I_j)}\left|\int_{x_1}^{x_1+y_1}e(\lambda du)du\int_{-T}^{T}F(1/2+it, \chi)u^{-1/2+it}dt\right|+\frac{1+|\lambda|y}{T}L^5x_1^{1+\varepsilon'}.$$

Taking

$$T=(dl)^{1/2+\varepsilon}y^{-1}x_1^{1+\varepsilon}+(dl)^{1/2+\varepsilon}|\lambda|x_1^{1+\varepsilon}\ll x^{2/3+\varepsilon}$$

we have

$$S_1(\chi)\ll L^5\max_{(I_j)}\left|\int_{-T}^{T}F(1/2+it, \chi)dt\int_{x_1}^{x_1+y_1}u^{-1/2}e\left(\frac{t}{2\pi}\log u+\lambda\,du\right)du\right|$$
$$+d^{-1/2}l^{-1/2-\varepsilon}y.$$

Now we apply Lemma 1 to the inner integral on the right-hand side of (3.4) and obtain

$$\int_{x_1}^{x_1+y_1} u^{-1/2} e\left(\frac{t}{2\pi}\log u+\lambda du\right) du \ll x_1^{-1/2}\min\left(y_1, \frac{x_1}{\sqrt{|t|}}, \frac{x_1}{\min\limits_{x_1<u\leqslant x_1+y_1}|t+2\pi\lambda du|}\right).$$

Therefore

$$S_1(\chi)\ll L^5 x_1^{-1/2}\max_{(I_j)}\int_{-T}^{T}\min\left(y_1, \frac{x_1}{\sqrt{|t|}}, \frac{x_1}{\min\limits_{x_1<u\leqslant x_1+y_1}|t+2\pi\lambda du|}\right)|F(1/2+it\chi)|dt$$

$$+yd^{-1/2}l^{-1/2-\varepsilon}.$$

Let $H=xy^{-1}+10|\lambda|y$. Since for $|t+2\pi\lambda dx_1|\leqslant H$

$$\min\left(y_1, \frac{x_1}{\sqrt{|t|}}\right)\ll\min\left(y_1, \sqrt{\frac{x_1}{|\lambda|d}}\right)$$

holds and for

$$|t+2\pi\lambda dx_1|\geqslant kH \quad (k\geqslant 1)$$

$$|t+2\pi\lambda du|\geqslant kH-2\pi|\lambda|y\gg kH$$

holds, we have

$$\int_{-T}^{T}\min\left(y_1, \frac{x_1}{\sqrt{|t|}}, \frac{x_1}{\min\limits_{x_1\leqslant u\leqslant x_1+y_1}|t+2\pi\lambda du|}\right)|F(1/2+it, \chi)|dt$$

$$\ll\int_{|t+2\pi\lambda dx_1|\leqslant H}\min\left(y_1, \frac{x_1}{\sqrt{|t|}}\right)|F(1/2it, \chi)|dt$$

$$+\sum_{\substack{k\\ kH\leqslant 2T}}\int_{kH<|t2+\pi\lambda dx_1|\leqslant(k+1)H}\frac{x_1}{\min|t+2\pi\lambda du|}|F(1/2+it, \chi)|dt$$

$$\ll L\max_{|T_1|\leqslant 2T}\int_{T_1}^{T_1+H}\left(\min\left(y_1, \sqrt{\frac{x_1}{|\lambda|d}}\right)+\frac{x_1}{H}\right)|F(1/2+it, \chi)|dt_{x}.$$

Since it is a simple matter to show that

$$\min\left(y_1, \sqrt{\frac{x_1}{|\lambda|d}}\right)+\frac{x_1}{H}\ll\sqrt{\frac{x_1y_1}{H}}=\frac{1}{d}\sqrt{\frac{xy}{H}},$$

we have

$$S_1(\chi)\ll L^6\max_{(I_j)}\max_{|\pi|\leqslant 2T} d^{-1/2}(yH^{-1})^{1/2}\int_{T_1}^{T_1+H}|F(1/2+it, x)|dt+yd^{-1/2}l^{-1/2-\varepsilon}.$$

To prove Theorem 2, now it is sufficient to show that for all possible (I_j) and $|T_1|\leqslant 2T$

$$H^{-1/2}\sum_{x_l}{}^{*}\int_{T_1}^{T_1+H}|F(1/2+it, \chi)|dt\ll y^{1/2(\varepsilon)}l^{(1-2}L^{c_1-6}. \qquad (3.5)$$

We consider three cases separately.

Case I. The indices $j=1, 2, \cdots, 5$ may be divided into two blocks J_1 and J_2 such that

$$M=\max\left(\prod_{j\in J_1}N_j, \prod_{j\in J_2}N_j\right)\ll yl^{-\varepsilon}.$$

Let

$$M_i=\prod_{j\in J_i}N_j, \ M_1M_2\ll x_1 \quad (i=1, 2),$$

$$F_i(s, \chi)=\prod_{j\in J_i}f_j(s, \chi)=\sum_{n\ll M_i}b_i(n)\chi(n)n^{-s},$$

where

$$|b_i(n)|\ll d_5(n)\ll d^5(n)\quad (i=1,\ 2).$$

By Cauchy's inequality, Lemma 2 and the well-known estimates

$$\sum_{n\leqslant N} d^{10}(n)\ll N(\log N)^{1023}$$

and

$$\sum_{u\leqslant N}\frac{d^{10}(n)}{n}\ll(\log N)^{1024},$$

we obtain

$$\begin{aligned}
&H^{-1/2}\sum_{\chi_l}{}^{*}\int_{T_1}^{T_1+H}|F(1/2+it,\ \chi)|dt\\
&\leqslant H^{-1/2}\Big(\sum_{\chi_l}{}^{*}\int_{T_1}^{T_1+H}|F_1(1/2+it,\ \chi)|^2dt\Big)^{1/2}\Big(\sum_{\chi_l}{}^{*}\int_{T_1}^{T_1+H}|F_2(1/2+it,\ \chi)|^2dt\Big)^{1/2}\\
&\ll H^{-1/2}((lH)^{1/2}+M_1^{1/2})((lH)^{1/2}+M_2^{1/2})L^{1024}\\
&\ll(lH^{1/2}+l^{1/2}M^{1/2}+H^{-1/2}x_1^{1/2})L^{1024}\\
&\ll y^{1/2}(l^{(1-s)/2}\tau^{(1+s)/2}x^{1/2}y^{-1}+l^{1/2}\tau^{-1/2}+l^{(1-s)/2})L^{1024}\\
&\ll y^{1/2}l^{(1-s)/2}L^{1402}.
\end{aligned}$$

So in this case (3.5) holds with $c_1=1030$.

Case II. There exists $N_{j_0}\geqslant 2x^{1/3}$.

Since for $3\leqslant j\leqslant 5$, $N_t\leqslant X<2x^{1/3}$, we must have $j_0=1,\ 2$. Without loss of generality we suppose $j_0=1$.

Now we first show

$$\sum_{\chi_l}{}^{*}\int_{T_1}^{T_1+H}|f_1(1/2+it,\ \chi)|^2dt\ll L^3lH\rho^2(q). \tag{3.6}$$

By Perron's summation formula it follows that

$$f_1(1/2+it,\ \chi)=\frac{1}{2\pi i}\int_{c'-iT}^{c'+iT}L_1(1/2+it+\omega,\ \chi)\,\frac{(2N_1)^{\omega}-N_1^{\omega}}{\omega}\,d\omega+O\Big(\frac{N_1^{1/2}}{T_0}L\Big), \tag{3.7}$$

where

$$L_1(s,\ \chi)=\sum_{\substack{n=1\\(n,q)=1}}^{\infty}\chi(n)n^{-s}(\mathrm{Re}s>1)\ \text{ and } c'=\frac{1}{2}+\frac{1}{L}.$$

For $\mathrm{Re}s>1$

$$L_1(s,\ \chi)=\prod_{p|q}\Big(1-\frac{\chi(p)}{p^s}\Big)^{-1}=L(s,\chi)\prod_{p|p}\Big(1-\frac{\chi(p)}{p^s}\Big). \tag{3.8}$$

This gives an analytic continuation for $L_1(s,\ \chi)$ in the whole s-plane, since $L(s,\ \chi)$ is an entire function for primitive character $\chi_{\bmod l}(l>1)$. It follows that (3.8) holds for all s and

$$|L_1(\sigma+it,\ \chi)|\ll\rho(q)\,|L(\sigma+it,\ \chi)|,\ \sigma\geqslant 1/2. \tag{3.9}$$

It is known that for $0\leqslant\sigma\leqslant 1$, $|t|\geqslant 2$

$$L(\sigma+it,\ \chi_l^*)\ll(l|t|)^{(1-\sigma)/2}\log l|t|. \tag{3.10}$$

Now moving the integral line in (3.7) to $\mathrm{Re}\omega=0$, by (3.9), (3.10) and the fact $\rho(q)\ll q^{\varepsilon'}$ we get

$$f_1(1/2+it,\chi)=\frac{1}{2\pi}\int_{-T}^{T}L_1(1/2+it+iv,\chi)\frac{(2N_1)^{iv}-N_1^{iv}}{iv}dv$$
$$+O\Big(\max_{0<\alpha\leqslant c'}\frac{(l(T+T_0))^{1-\sigma/2}}{T_0}N_1^{\sigma}\log l(T+T_0)\Big)+O\Big(\frac{N_1^{1/2}}{T_0}L\Big).$$

On taking

$$T_0=x^{2/3+\varepsilon}l^{1/3}+T\ll x^{7/9+\varepsilon};$$

it follows

$$|f_1(1/2+it,\chi)|^2\ll\Big(\int_{-T_0}^{T_0}\rho(g)\,|L(1/2+it+iv,\chi)|\frac{dv}{1+|v|}\Big)^2+1$$
$$\leqslant\rho^2(q)L\int_{-T_0}^{T_0}|L(1/2+it+iv,\chi)|^2\frac{dv}{1+|v|}+1.$$

Hence

$$\sum_{\chi_l}^{*}\int_{T_1}^{T_1+H}|f_1(1/2+it,\chi)|^2dt$$
$$\ll L\rho^2(q)\sum_{\chi_l}^{*}\int_{T_1}^{T_1+H}dt\int_{-T_0}^{T_0}|L(1/2+it+i\sigma,\chi)|^2\frac{dv}{1+|v|}+lH$$
$$=L\rho^2(q)\int_{-T_0}^{T_0}\frac{dv}{1+|v|}\sum_{\chi_l}^{*}\int_{T_1}^{T_1+H}|L(1/2+it+iv,\chi)|^2dt+lH$$
$$=L\rho^2(q)\int_{-T_0}^{T_0}\frac{dv}{1+|v|}\int_{T_1+v}^{T_1+v+H}\sum_{\chi_l}^{*}|L(1/2+iu,\chi)|^2du+lH.$$

By Lemma 3 we obtain

$$\sum_{\chi_l}^{*}\int_{T_1}^{T_1+H}|f_1(1/2+it,\chi)|^2dt\ll\rho^2(q)lHL^3,$$

which is (3.6).

From Lemma 2 and (3.6) it follows that

$$H^{-1/2}\int_{T_1}^{T_1+H}\sum_{\chi_l}^{*}|F(1/2+it,\chi)|dt$$
$$\leqslant H^{-1/2}\Big(\sum_{\chi_l}^{*}\int_{T_1}^{T_1+H}|f_1(1/2+it,\chi)|^2dt\Big)^{1/2}\Big(\sum_{\chi_l}^{*}\int_{T_1}^{T_1+H}\Big|\prod_{j=2}^{5}f_j(1/2+it,\chi)\Big|^2dt\Big)^{1/2}$$
$$\ll H^{-1/2}(lH)^{1/2}\Big((lH)^{1/2}+\Big(\frac{x_1}{N_1}\Big)^{1/2}\Big)L^{514}$$
$$\ll y^{1/2}(lH^{1/2}y^{-1/2}+l^{1/2}y^{-1/2}x^{1/3}d^{-1/2})L^{514}$$
$$\ll y^{1/2}l^{(1-\varepsilon)/2}L^{514}.$$

Hence (3.5) is true in case II with $c_1\geqslant520$.

Case III. all $N_j\leqslant2x^{1/3}$ $(1\leqslant j\leqslant5)$.

Suppose that $N_{j_1}\leqslant N_{j_2}\leqslant\cdots\leqslant N_{j_5}$. Since $x_1\ll\prod_{i=1}^{5}N_j\ll x_1$, there exists r, $1\leqslant r\leqslant4$ and

$$N_{j_1}N_{j_2}\cdots N_{j_r}\leqslant2x^{1/3},\ N_{j_1}\cdots N_{j_r}N_{j_{r+1}}>2x^{1/3}.$$

Let

$$M_1=N_{j_1}\cdots N_{j_{r+1}},\ M_2=N_{j_{r+2}}\cdots N_{j_5},$$

We have

$$M_1\ll x^{2/3}=yx^{-\varepsilon}\ll yl^{-\varepsilon}$$

and

$$M_2 \ll x^{2/3} \ll yl^{-8}.$$

So $N_j (1 \leqslant j \leqslant 5)$ satisfy the condition of case I, and (3.5) follows.

Now we have finished the proof of Theorem 2.

§ 4. The Proof of Theorem 3

In this section we assume all the conditions of Theorem 3 and use the notations $S_1(\chi)$ and $S_1(l, d)$ in § 3.

We start from Perron's summation formula. For $x_1 < u \leqslant 2x_1$

$$\sum_{\substack{x_1 < m \leqslant x_1 + y_1 \\ (m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} L_2(s, \chi)\frac{u^s - x_1^s}{s}\,ds + O\left(\frac{x^{1+\varepsilon'}}{T}\right), \qquad (4.1)$$

where $c = 1 + \frac{1}{L}$ and

$$L_2(s, \chi) = \sum_{\substack{n=1 \\ (m,q)=1}}^{n} \mu(m)\chi(m)m^{-s}, \ \text{Re}\, s > 1.$$

For Re $s > 1$,

$$L_2(s, \chi) = \prod_{p,d}\left(1 - \frac{\chi(p)}{p^s}\right) = L^{-1}(s, \chi)\prod_{p|q}\left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

In fact, this gives a continuation for $L_2(s, \chi)$ in the s-plane. Moreover,

$$L_2(\sigma + it, \chi) \ll |L^{-1}(\sigma + it, \chi)| \prod_{p|q}\left(1 - \frac{1}{\sqrt{p}}\right)^{-1} \ll L\rho(q)|L^{-1}(\sigma + it, \chi)|$$

for $\text{Re}\, s = \sigma > 1/2$.

Let M be the so-called Huxley-Hooley contour as described by K. Ramachandra[3]. Briefly speaking, we take the rectangle

$$\frac{1}{2} \leqslant \sigma \leqslant 1, \ |t| \leqslant T + 2000(\log T)^2,$$

and divide it into equal rectangles of height $400(\log T)^2$ (the real line cuts one of these rectangles into two equal parts, we denote this rectangle by R_0). Let R^n ($n = -n_1, \cdots, 0, \cdots n_1$) be all these rectangles. In R^n we fix a new right side and obtain a new rectangle as follows. Consider R^{n-1}, R^n and R^{n+1} whenever all of the three are defined. Pick out a zero of $\prod_{x_1}^{*} L(s, \chi)$ with the greatest real part β_n and Re $s = \beta_n$ is the new right side of R^n. Now we join all the right edges of the new rectangles by horizontal lines. These form the contour M'.

The Huxley-Hooley contour, or briefly, the H-H contour is obtained by making the following changes on M'.

Let a, b and θ be positive constants to be chosen later, satisfying $0 < \theta < 1$, a shall be small and b shall be close to 1. If $\beta_n < \theta$, then in place of β_n we take $\beta_n' = \beta_n + 3a(1 - \beta_n)$. If $\beta_n \geqslant \theta$, then β_n is replaced by $\beta_n' = \beta_n + b(1 - \beta_n)$. These form the

H-H contour.

Now we join the points $c\pm iT$ to M by horizontal lines H_1 and H_2. T will be chosen as a suitable power of x. Since

$$|L^{-1}(s,\chi)|\ll T^{\varepsilon'}$$

for s on H_1 and H_2, as shown in [3], shifting the integral line in (4.1) to M we get

$$\sum_{\substack{x_1<m<u\\(m,q)=1}}\mu(m)\chi(m)=\frac{1}{2\pi i}\int_M L_2(s,\chi)\frac{u^s-x_1^s}{s}\,ds+O\Big(\frac{x^{1+\varepsilon'}}{T}\Big).$$

Therefore

$$\begin{aligned}s_1(\chi)&=\int_{x_1}^{x_1+y_1}e(\lambda du)d\sum_{\substack{x_1<m<u\\(m,p)=1}}\mu(m)\chi(m)\\&=\frac{1}{2\pi i}\int_{x_1}^{x_1+y_1}e(\lambda du)du\int_M L_2(s,\chi)u^{s-1}ds+O\Big(\frac{1+|\lambda|y}{T}x^{1+\varepsilon'}\Big)\\&=\frac{1}{2\pi i}\int_M L_2(s,\chi)ds\int_{x_1}^{x+y_1}u^{s-1}e(\lambda du)du+O\Big(\frac{1+|\lambda|y}{T}x^{1+\varepsilon'}\Big).\end{aligned}$$

Take

$$T=d^{1/2}x^{1+\varepsilon}y^{-1}+d^{1/2}x^{1+\varepsilon}|\lambda|\quad(x^{1/3}\ll T\ll x^{2/3+\varepsilon}),$$
$$H=xy^{-1}+10|\lambda|y.$$

Let $M(H)$ denote the part of M satisfying

$$T_1\leqslant \operatorname{Re}s\leqslant T_1+H,\ |T_1|\leqslant 2T.$$

In the same way as in § 3 we could obtain

$$s_1(\chi)\ll Ld^{-1/2}\rho(q)(xy^{-1}H)^{1/2}\max_{|T_1|\leqslant 2T}\int_{M(H)}x^{\sigma-1}|L^{-1}(s,\chi)||ds|+yd^{-1/2}x^{-\varepsilon/2}.$$

To prove Theorem 3, now it is sufficient to show that for $|T_1|\leqslant 2T$,

$$\sum_{x_l}{}^{*}\int_{M(H)}x^{\sigma-1}|L^{-1}(s,\chi)||ds|\ll(Hyx^{-1})^{1/2}L^{-A-1},$$

or, since $H\geqslant xy^{-1}$ we only have to prove

$$\sum_{x_l}{}^{*}\int_{M(H)}x^{\sigma-1}|L^{-1}(s,\chi)||ds|\ll L^{-A-1}.\tag{4.3}$$

To p rove (4.3), we just follow the method of Ramachandra[3]. It is shown in [3] that

$$|L^{-1}(s,\chi)|\ll T^{\varepsilon'},\ \text{if } s\in M(H) \text{ and } \operatorname{Re}s\leqslant\theta+b(1-\theta),$$
$$|L^{-1}(s,\chi)|\ll\exp(\log T)^{2(1-b)})\ \text{if } s\in M(H) \text{ and } \operatorname{Re}s>\theta+b(1-\theta).$$

We divide the smallest vertical strip containing $M(H)$ into vertical strips of width $1/\log T$. Consider the bit of $M(H)$, say $M(H,\sigma')$, in the vertical strip about the abscissa σ'. Then we have

$$\int_{M(H,\sigma')}|ds|\ll N(\sigma,T_1,H,l)(\log T)^{10},$$

where σ' is $\sigma+3a(1-\sigma)$ or $\sigma+b(1-\sigma)$ according as $\sigma'\leqslant\theta$ or $\sigma'>\theta$. By the above discussion and Lemmas 4—6 we obtain ($l\ll L^{c_1}$)

$$\sum_{\chi_l}{}^{*}\int_{M(H)} x^{\sigma-1}|L^{-1}(s,\ \chi)|\,|ds|$$
$$=\sum_{\chi_l}{}^{*}\int_{\substack{M(H)\\ \sigma'<\theta}} x^{\sigma'-1}|L^{-1}(s,\ x)|\,|ds|+\sum_{\chi_l}{}^{*}\int_{\substack{M(H)\\ \theta\leqslant\sigma'\leqslant\theta+b(1-\theta)}} x^{\sigma'-1}|L^{-1}(s,\ \chi)|\,|ds|$$
$$+\sum_{\chi_l}{}^{*}\int_{\substack{M(H)\\ \sigma'>\theta+b(1-\theta)}} x^{\sigma'-1}|L^{-1}(s,\ \chi)|\,|ds| \quad (s=\sigma'+it)$$
$$\ll T^{\varepsilon'}\left(\frac{H^{8/3(1-3a)}}{x}\right)^{1-\theta}+T^{\varepsilon'}\left(\frac{T^{1600(1-b)^{-3/2}(1-\theta)^{1/2}}}{x}\right)^{(1-b)(1-\theta)}$$
$$+\exp\left((\log T)^{3(1-b)}\right)\cdot\left(\frac{T^{1600(1-\theta)-(1-b)^{-1/2}}}{x}\right)^{c_0(1-b)(\log T)^{-4/5}}$$

proivded a, b and θ satisfy

$$H^{8/3(1-3a)}\leqslant x^{1-\varepsilon}, \tag{4.4}$$
$$T^{2000(1-\theta)^{1/2}(1-b)^{-3/2}}\leqslant x. \tag{4.5}$$

In fact, we may first choose a such that

$$\frac{8}{3}\left(\frac{1}{3}+\varepsilon\right)(1-3a)^{-1}<1-\varepsilon \quad (H\leqslant x^{1/3+\varepsilon}),$$

b such that $3(1-b)=\dfrac{1}{100}$ and then θ such that (4.5) holds. Hence

$$\sum_{\chi_l}{}^{*}\int_{M(H)} x^{\sigma-1}|L^{-1}(s,\ \chi)|\,|ds|\ll\exp(-c_0'L^{1/6}) \quad (c_0'>0)$$

and (4.3) follows.

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