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PARAMETER ESTIMATION OF SPATIAL AR MODEL

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Abstract

Consider a stable AR model of two parameter spatial series $\{X_t, t\in N^2\}$, i. e. $\{X_t, t\in N^2\}$ is homogeneous and satisfies the following difference equation

$$X_t - \sum_{s\in<0,p]} a_s X_{t-s} = W_t \quad (t\in N^2),$$

where $\{W_t, t\in N^2\}$ is a two parameter white noise and the notation $<0, p]$ expresses the set of two dimentional lattice points $\{(k_1, k_2): 0\leqslant k_1\leqslant p_1, 0\leqslant k_2\leqslant p_2$ but $(k_1, k_2)\neq(0, 0)\}$, and furthermore the two-variable polynomial

$$1-\sum_{(s_1,s_2)\in<0,p]} a_{(s_1,s_2)} Z_1^{s_1} Z_2^{s_2} \neq 0 \,(|Z_1|\leqslant 1, \ |Z_2|\leqslant 1).$$

In this paper, under frirly general conditions (it is required that $\{W_t\}$ satisfies the conditions of two-parameter martingale difference, which is much weaker than supposing $\{W_t\}$ to be i. i. d.), the author obtains strong consistency and asymptotic normality of the Y-W (LS) estimate of the AR parameters $\{a_s\}$ whenever $n_1n_2\to\infty$, where n_1 and n_2 denote the horizontal and vertical sampling width respectively.

Introduction

We consider an AR model of two parameter spatial series $\{X_t, t\in N^2\}$, that is, $\{X_t, t\in N^2\}$ is homogeneous and satisfies the following difference equation

$$X_t - \sum_{s\in<0,p]} a_s X_{t-s} = W_t (t\in N^2), \tag{0.1}$$

where $\{W_t, t\in N^2\}$ is a two parameter white noise, $<0, p]$ denotes the set of two dimentional lattice points $\{(k_1, k_2): 0\leqslant k_1\leqslant p_1, 0\leqslant k_2\leqslant p_2$ but $(k_1, k_2)\neq 0\}$. The AR model is said to be stable if the two-variable polynomial

$$1-\sum_{(s_1,s_2)\in<0,p]} a_{(s_1,s_2)} Z_1^{s_1} Z_2^{s_2} \neq 0 \quad (|Z_1|\leqslant 1, \ |Z_2|\leqslant 1). \tag{0.2}$$

The probabilistic meaning of the stable AR model has been explained in [4], it exactly describes the finite order quadrant Markov property. [7] and [8] considered the problems of parameter estimation and order determination of a stable AR model. Under the hypothesis of $\{W_t\}$ being i. i. d, it is proved that the Y-W estimate of the parameters $\{a_s\}$ is weakly consistent provided $n_1n_2\to\infty$, where n_1

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and n_2 denote the horizontal and vertical sampling width respectively and are asymptoticly normal provided $n_1, n_2 \to \infty$. [8] also considered the following weaker conditions on $\{W_t\}$:

i) $\{W_t\}$ is strictly stationary with finite fourth moment;

ii) $E(W_t | \mathscr{F}\{W_s: s_1<t_1 \text{ or } s_2<t_2\})=0$ a. s.;

iii) $E(W_t^2 | \mathscr{F}\{W_s: s_1<t_1 \text{ or } s_2<t_2\})=\sigma_w^2$ a. s.;

and obtained strong consistency and asymptotic normality of the Y-W (LS) estimate of a_s whenever $n_1=n_2\to\infty$. It should be pointed out that the proof given in [7] and [8] seems to involve some flaws. However, it could be improved if we make some modifications on the definition of the sample autocovariances. Under the hypothesis of $\{W_t\}$ being i. i. d, [8] gives a weakly consistent estimate of the order p, using BIC criterion, but the proof also involves flaws.

In this paper, under the conditions weaker than i), ii), iii), we discuss the problems of parameter estimation of the stable AR model. The paper includes three parts: § 1 enumerates some ergodic theorems concerning strictly stationary spatial series which will be used latter; § 2 discusses strong consistency of the Y-W (LS) estimate $\hat{a}_s$ of the AR parameter a_s as $n_1n_2\to\infty$; § 3 discusses asymptotic normality of $\hat{a}_s$ as $n_1n_2\to\infty$ For simplicity, we consider 2-parameter case the results obtained can be easily generalized to q-parameter ($q>2$) cases.

In a subsequent paper, we will discuss uniform convergence rate of the sample autocovariances and give strongly consistent order determination method. An iterated logarithmic convergence rate of the AR parameter estimate will be also given.

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Notation

In this paper, N denotes the set of integers. For $t=(t_1, t_2)\in N^2$, $|t|$ denotes the product of absolute value of the nonzero components of t. The elements (0, 0) and (1, 1) will be also denoted by 0 and 1 and $|0|\triangleq 1$. For $s, t\in N^2$, $s\leqslant t$ iff $s_1\leqslant t_1$ and $s_2\leqslant t_2$; $s\leqslant\neq t (s\geqslant\neq t)$ iff $s\leqslant t (s\geqslant t)$ and $s\neq t$; $s^{(i)}\leqslant$ iff $s_i<t_i$ or $s_i=t_i$, $s_{3-i}\leqslant t_{3-i}$ ($i=1, 2$); $s^{(i)}<t$ iff $s^{(i)}\leqslant t$ and $s\neq t$. $t\to\infty$ means $t_1, t_2\to\infty$. For $a, b\in N^2$, $[a, b]=\{t\in N^2: a\leqslant t\leqslant b\}$, $<a, b]=\{t\in N^2: a\neq\leqslant t\leqslant b\}$, $[a, \infty)=\{t\in N^2: t\geqslant a\}$.

Let $\{\mathscr{F}_t, t\in N^2\}$ denote a family of σ-fields with $\mathscr{F}_t\in\mathscr{F}$, where $(\Omega, \mathscr{F}, P)$ is the basic probability space, and suppose $\mathscr{F}_s\subset\mathscr{F}_t$ if $s\leqslant t$. Define

$$\mathscr{F}^i_{t_i}=\bigvee_{s_i<t_i}\mathscr{F}_s,\quad \mathscr{F}_i(t)=\bigvee_{s\overset{(i)}{<}t}\mathscr{F}_s,\quad \mathscr{F}_i(t-)=\bigvee_{s\overset{(i)}{<}t}\mathscr{F}_s (i=1, 2),$$

$$\mathscr{F}_{t-}=\bigvee_{s\leqslant\neq t}\mathscr{F}_s,\ \mathscr{F}(t)=\mathscr{F}^1_{t_1}\vee\mathscr{F}^2_{t_2}.$$

$\xi\in\mathscr{F}_t$ means that ξ is measurable w. r. to $\mathscr{F}_t$. $\delta_{x,y}=1$ if $x=y$ and $\delta_{x,y}=0$ if $x\neq y$; $\operatorname{Log}\lambda=1$ if $\lambda<e$ and $\operatorname{Log}\lambda=\log\lambda$ if $\lambda\geqslant e$, where $\log(\cdot)$ is the usual logarithm function.

§ 1. Some Ergodic Theorems about Strictly Stationary Spatial Series

Theorem 1.1. *There exist constants A, $B>0$ such that for every strictly stationary spatial series $\{X_t,\ t\in N^2\}$ and every $\lambda>0$*

$$\lambda P\left(\sup_{n\geqslant 1}\frac{1}{|n|}\sum_{1\leqslant t\leqslant n}|X_t|>\lambda\right)\leqslant AE|X_0|\operatorname{Log}|X_0|+B. \tag{1.1}$$

The proof is contained in Appendix.

Let $\{X_t,\ t\in N^2\}$ be a strictly stationary spatial series and define the measure preserving transformations U, V on the probability space $(R^{N^2}, B^{N^2}, PX^{-1})$ by $(U_x)_t=x_{t+u}$, $(V_x)_t=x_{t+v}$, where $u=(1,\ 0)$, $v=(0,\ 1)$ and $x=(x_t)_{t\in N^2}$. Let $\bar\tau_U$, $\bar\tau_V$ denote the a.s. invariant σ-fields corresponding to U, V respectively. Let $\bar\tau=\bar\tau_U\cap\bar\tau_V$. $\{U,\ V\}$ is said to be ergodic if $\bar\tau=\overline{\{\phi,\ \Omega\}}$, it is said to be strongly ergodic if $\bar\tau_U=\bar\tau_V=\overline{\{\phi,\ \Omega\}}$. $\{X_t,\ t\in N^2\}$ is said to be ergodic if $X^{-1}(\bar\tau)=\overline{\{\phi,\ \Omega\}}$, it is said to be strongly ergodic if $X^{-1}(\bar\tau_U)=X^{-1}(\bar\tau_V)=\overline{\{\phi,\ \Omega\}}$.

Example 1.1. Consider the probability space $([0,\ 1],\ B_{[0,1]},\ P)$, where P is the lebesgue measure. Let λ, $\mu\geqslant 0$ and $U_x=x+\lambda$, $V_x=x+\mu\ (\mathrm{mod}\ 1)$. Then $\{U,\ V\}$ is ergodic iff one of the λ, μ is rational; $\{U,\ V\}$ is strongly ergodic iff both λ and μ are rational.

From Theorem 1.1 we get (see [2, 9])

Theorem 1.2. *Let $\{X_t,\ t\in N^2\}$ be strictly stationary and $E|X_0|\operatorname{Log}|X_0|<\infty$. Then as $n\to\infty$,*

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n}X_t\to E(X_0|X^{-1}(\bar\tau))\quad \text{a. s. \& } L^1. \tag{1.2}$$

If $X^{-1}(\bar\tau_U)=X^{-1}(\bar\tau_V)$, then (1.2) *holds as $|n|\to\infty$. Particularly, if $\{X_t,\ t\in N^2\}$ is ergodic, then as $n\to\infty$,*

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n}X_t\to EX_0 \text{ a. s. \&} L^1. \tag{1.3}$$

If $\{X_t,\ t\in N^2\}$ is strongly ergodic, then (1.3) *holds as $|n|\to\infty$.*

Finaly we point out that the moment condition $E|X_0|\operatorname{Log}|X_0|<\infty$ can not be arbitrarily weakened (see [5]).

§ 2. Consistency of the Parameter Estimate

Consider the following spatial linear process

$$X_t = \sum_{s \geq 0} b_s W_{t-s}, \tag{2.1}$$

where $\{W_t,\ t \in N^2\}$ is white noise and

$$\sum_{s \geq 0} |b_s| < \infty. \tag{2.2}$$

Some times we need $\{b_s,\ s \geq 0\}$ to have better convergence property and suppose

$$\sum_{s \geq 0} |b_s|\, |s+1|^{1/2} < \infty. \tag{2.3}$$

It is easy to see that a stable AR model is a special case of model (2.1), (2.2) or (2.1), (2.3) (see [7]). Because the model (2.1), (2.2) is homogeneous, we can define its autocovariances

$$r(u,\ v) = E(X_{t-u} X_{t-v}) \tag{2.4}$$

and sample autocovariances

$$\hat{r}(u,\ v) = \frac{1}{|n|} \sum_{1 \leq t \leq n} X_{t-u} X_{t-v}. \tag{2.5}$$

For a stable AR(p) model (0.1) we get the following Yule-Walker equation

$$\sum_{v \in <0, p]} a_v r(u,\ v) = r(u,\ 0) \tag{2.6}$$

and as in [7] we can write it in matrix form

$$Ra = r, \tag{2.7}$$

where the elements of

$$R = (r(u,\ v))_{u, v \in <0, p]},\ r = (r(u,\ 0))_{u \in <0, p]} \text{ and } a = (a_v)_{v \in <0, p]}$$

are ordered according to $\overset{(1)}{\leqslant}$. Analogically we have

$$\hat{R} = (\hat{r}(u,\ v))_{u, v \in <0, p]} \text{ and } \hat{r} = (\hat{r}(u,\ 0))_{u \in <0, p]}.$$

If $\hat{R}$ is reversible, we define the Yule-Walker estimate $\hat{a}$ of the AR parameter vector a to be the solution of the following equation

$$\hat{R} b = \hat{r}; \tag{2.8}$$

if $\hat{R}$ is not reversible, we fefine $\hat{a} = 0$.

Another type of estimate is the least-square (LS) estimate which can be defined by minimizing

$$\frac{1}{|n|} \sum_{1 \leq t \leq n} |X_t - \sum_{s \in <0, p]} c_s X_{t-s}|^2 \text{ among } c = (c_s)_{s \in <0, p]}.$$

We will mainly discuss this kind of estimate in the subsequent paper. As for the problems discussed here, the two kinds of estimate are equivalent in the sense that the results we get for one kind of estimate hold naturally for the other one, so we need only to discuss the former kind of estimate. For simplicity and without loss of generality we suppose

$$\sigma_W^2 = E W_0^2 = 1. \tag{2.9}$$

Lemma 2.1. *Let* $\{X_t,\ t \in N^2\}$ *satisfy* (2.1) *and* (2.2), *where* $\{W_t,\ t \in N^2\}$ *is strictly stationary and strongly ergodic and*

$$E(W_0^2 \operatorname{Log}|W_0|)<\infty. \tag{2.10}$$

Then

$$\hat{r}(u, v) \xrightarrow[|n|\to\infty]{a.\,s.} r(u, v) \quad (u, v \geqslant 0). \tag{2.11}$$

The proof can be straightly obtained from Theorem 1.2.

Theorem 2.1. *Let $\{X_t, t \in N^2\}$ be a stable $AR(p)$ model such that the innovation sequence $\{W_t, t\in N^2\}$ satisfies the conditions of Lemma* 2.1. *Then*

$$\hat{a} \xrightarrow[|n|\to\infty]{a.\,s.} a. \tag{2.12}$$

Lemma 2.2. *Let $\{X_t, t\in N^2\}$ satisfy* (2.1) *and* (2.2), *where $\{W_t, t\in N^2\}$ is strictly stationary and satisfies* (2.10), *and*

$$E(W_t|\mathscr{F}_i(t-))=0 \quad (i=1, 2), \tag{2.13}$$

$$\sum_{i=0}^{\infty}\{E|E(W^2_{(i,0)}|\mathscr{F}_{(-1,0)})-1|+E|E(W^2_{(0,i)}|\mathscr{F}_{(0,-1)})-1|\}<\infty, \tag{2.14}$$

where $\mathscr{F}_t=\sigma(W_s, s\leqslant t)$. Then (2.11) *holds.*

Proof By Theorem 1.2, we need only to show for every $t_{3-i}\geqslant 1$,

$$\frac{1}{|n_i|}\sum_{t_i=1}^{n_i} X_{t-u}X_{t-v} \xrightarrow[n_i\to\infty]{P} r(u, v) \quad (i=1, 2). \tag{2.15}$$

And it is enough to see $i=2$. (2.15) will follow provided we can show for $x, y\geqslant 0$

$$\frac{1}{n_2}\sum_{t_2=1}^{n_2} W_{(t_1-x_1, t_2-x_2)}W_{(t_1-y_1, t_2-y_2)} \xrightarrow[n_2\to\infty]{P} \delta_{x,y}. \tag{2.16}$$

First suppose $x\neq y$. Then one of the following cases will occur: (1) $x\geqslant\neq y$; (2) $x\leqslant\neq y$; (3) $x_1>y_1$, $x_2<y_2$; (4) $x_1<y_1$; $x_2>y_2$. By symmetry, it needs only to see (1) (3).

Let $\mathscr{G}_{t_2}=\mathscr{F}^2_{t_2-y_2}$ in (1) and $\mathscr{G}_{t_2}=\mathscr{F}^2_{(t_2-x_2)}$ in (3). It is easy to verify that $\{\xi_t= W_{(t_1-x_1, t_2-x_2)}W_{(t_1-y_1, t_2-y_2)}, \mathscr{G}_{t_2}, t_2\geqslant 1\}$ is a strictly stationary martingale difference sequence. So by Theorem 2.19 of [5] we get (2.16).

Now suppose $x\neq y$. Let

$$\xi_{t_2}=W^2_{t_1-x_1, t_2-x_2}-1, \ \mathscr{G}_{t_2}=\mathscr{F}_{(t_1-x_1, t_2-x_2)},$$

$$Z_{t_2}=\sum_{s_2=0}^{\infty}E(\xi_{t_2+s_2}|\mathscr{G}_{t_2-1}), \ Y_{t_2}=\sum_{s_2=0}^{\infty}\{E(\xi_{t_2+s_2}|\mathscr{G}_{t_2})-E(\xi_{t_2+s_2}|\mathscr{G}_{t_2-1})\}.$$

The result then easily follows from

$$\xi_{t_2}=Y_{t_2}+Z_{t_2}-Z_{t_2-1}. \tag{2.17}$$

Theorem 2.2. *Let $\{X_t, t\in N^2\}$ be a stable $AR(p)$ model such that the innovation sequence $\{W_t, t\in N^2\}$ satisfies the conditions of Lemma* 2.2. *Then* (2.12) *holds.*

Lemma 2.3. *Suppose for $\{M_k, 1\leqslant k\leqslant n\}$ there exist $\{\mathscr{G}^{(i)}_{k_i}, 1\leqslant k_i\leqslant n_i\}$ $(i=1, 2)$ such that for every $1\leqslant k_{3-i}\leqslant n_{3-i}$, $\{M_{(k_1,k_2)}, \mathscr{G}^{(i)}_{k_i}, 1\leqslant k_i\leqslant n_i\]$ is a mertingale $(i=1, 2)$. Then*

$$E(\max_{1\leqslant k\leqslant n}|M_k|)^2\leqslant 16EM_n^2. \tag{2.18}$$

The proof is easy. Let $\psi(\lambda)$ be a function defined on $(0, \infty)$ satisfying: 1) $\psi(\lambda)$

>0; 2) $\psi(\lambda)$ is nondecreasing; 3) $\lim\limits_{\lambda\to\infty}\psi(\lambda)=\infty$.

Lemma 2.4. *Let $G_t\geqslant 0$, $t\in N^2$ and*

$$\sum_{t\in N^2} G_t/\psi(|t|)<\infty. \tag{2.19}$$

Then

$$\lim_{|n|\to\infty}\frac{1}{\psi(|n|)}\sum_{|t|\leqslant|n|} G_t=0. \tag{2.20}$$

Now we introduce the following condition on $\{\mathscr{F}_t\}$: for every r. v. ξ

$$E(E(\xi|\mathscr{F}_t)|\mathscr{F}_i(t-))=E(\xi|\mathscr{F}_{t-})\quad (i=1,\ 2). \tag{2.21}$$

A stronger condition is: for every r. v. ξ

$$E(E(\xi|\mathscr{F}_t)|\mathscr{F}(t-1))=E(\xi|\mathscr{F}_{t-}). \tag{2.22}$$

Example 2.1. Let $\{\xi_t,\ t\in N^2\}$ be independent and $\mathscr{F}_t=\sigma(\xi_s,\ s\leqslant t)$. Then (2.22) holds.

Lemma 2.5. *Let $\{W_t,\ t\in N^2\}$ satisfy*

$$P(|W_t|>\lambda)\leqslant CP(|W|>\lambda), \tag{2.23}$$

where C is a constant and W is a r. v., and suppose

$$W_t\in\mathscr{F}_t,\ E(W_t|\mathscr{F}_{t-})=0, \tag{2.24}$$

where $\{\mathscr{F}_t,\ t\in N^2\}$ satisfies (2.21).

i) *If $EW^2\mathrm{Log}|W|<\infty$ and for every $x\geqslant 0$*

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} E(W^2_{t-x}|\mathscr{F}_{t-x-})\xrightarrow[|n|\to\infty]{P}1, \tag{2.25}$$

then for every $x,\ y\geqslant 0$

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} W_{t-x}W_{t-y}\xrightarrow[|n|\to\infty]{P}\delta_{x,y}. \tag{2.26}$$

ii) *If $E(W\,\mathrm{Log}|W|)^2<\infty$ and for every $x\geqslant 0$*

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} E(W^2_{x-t}|\mathscr{F}_{t-x-})\xrightarrow[|n|\to\infty]{a.\ s.}1, \tag{2.27}$$

then for every $x,\ y\geqslant 0$

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} W_{t-x}W_{t-y}\xrightarrow[|n|\to\infty]{a.\ s.}\delta_{x,y}. \tag{2.28}$$

Proof i) we have

$$\begin{aligned} W_t&=W_tI_{\{|W_t|\leqslant|t|^{1/8}\}}+W_tI_{\{|W_t|>|t|^{1/8}\}}\\ &=U_t+V_t=U_t-E(U_t|\mathscr{F}_{t-})+V_t-E(V_t|\mathscr{F}_{t-})=u_t+v_t. \end{aligned}\tag{2.29}$$

For $x,\ y\geqslant 0,\ x\neq y$, let $M_k=\sum\limits_{1\leqslant t\leqslant k} u_{t-x}u_{t-y}$. By Lemma 2.3, it is easy to show that

$$\begin{aligned} &\sum_{i\geqslant 0} P(\max_{1\leqslant k\leqslant 2^i}|M_k|\geqslant\epsilon|2^i|)\\ &\leqslant\left(\frac{16}{3\epsilon}\right)^2\sum_{t\geqslant 1}\frac{1}{|t|^2}Eu^2_{t-x}u^2_{t-y}<\infty. \end{aligned}$$

Thus by Borel-Cantelli's Lemma we have

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} u_{t-x}u_{t-y}\xrightarrow[|n|\to\infty]{a.s.}0. \tag{2.30}$$

On the other hand, by (2.23) it is easy to show that as $|n|\to\infty$,

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} u_{t-x}v_{t-y},\ \frac{1}{|n|}\sum_{1\leqslant t\leqslant n} v_{t-x}u_{t-y} \text{ and } \frac{1}{|n|}\sum_{1\leqslant t\leqslant n} v_{s-x}v_{t-y} \tag{2.31}$$

convergence in probability to zero. Thus we have (2.26) for $x\neq y$. In the case $x=y\geqslant 0$, one can similarly prove

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n}[W^2_{t-x}-E(W^2_{t-x}|\mathscr{F}_{t-x-})]\xrightarrow[|n|\to\infty]{P} 0. \tag{2.32}$$

Using (2.25), we get (2.26) for $x=y$.

ii) Using (2.23) and Lemma 2.4, one can show that the three terms in (2.31) convergence a.s. to zero. Thus (2.23) follows for $x\neq y$. The proof for the case $x=y$ is similar.

Lemma 2.6. I) *Let* $\{X_t,\ t\in N^2\}$ *satisfy* (2.1) *and* (2.2), *where* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 2.5 *i*). *Then*

$$\hat{r}(u,v)\xrightarrow[|n|\to\infty]{P} r(u,v)\quad (u,v\geqslant 0). \tag{2.33}$$

II) *Let* $\{X_t,\ t\in N^2\}$ *satisfy* (2.1) *and* (2.3), *where* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 2.5 *ii*), *and*

$$\sup_{n\geqslant 1}\frac{1}{|n|}\sum_{-n\leqslant t\leqslant n} E(W_t^2|\infty<\mathscr{F}_{t-})<\infty \ a.s. \tag{2.34}$$

Then (2.11) *holds.*

Proof We need only to see II). It is easy to verify

$$\left|\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} W_{t-u-k}W_{t-v-l}-\delta_{u+k,v+l}\right|$$
$$\leqslant 1+(|u+1||v+1||k+1||l+1|)^2\sup_{n\geqslant 1}\frac{1}{|n|}\sum_{-n\leqslant t\leqslant n}W_t^2. \tag{2.35}$$

Similar to Lemma 2.9, one can prove

$$\frac{1}{|n|}\sum_{-n\leqslant t\leqslant n}[W_t^2-E(W_t^2|\mathscr{F}_{t-})]\xrightarrow[|n|\to\infty]{a.s.} 0. \tag{2.36}$$

So the rosult easily follows by using rhe dominated convergence theorem.

Theorem 2.3. *Let* $\{X_t,\ t\in N^2\}$ *be a stable* $AR(p)$ *model.*

(I) *If the innovation sequence* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 2.6 I), *then*

$$\hat{a}\xrightarrow[|n|\to\infty]{P} a. \tag{2.37}$$

(II) *If the innovation sequence* $\{W_t,\ t\in N^2\}$ *satisfies the conditoins of Lemma* 2.6 II), *then* (2.12) *holds.*

§ 3. Asymptotic Normality

From (2.5) we get

$$\hat{r}(u,0)=\sum_{v\in<0,p]}\hat{r}(u,v)a_v+\hat{A}(u),\tag{3.1}$$

where $\hat{A}(u)=\frac{1}{|n|}\sum_{1\leqslant t\leqslant n}X_{t-u}W_t$, or writing in matrix from,

$$\hat{r}=\hat{R}a+\hat{A},\tag{3.2}$$

where $\hat{A}=(\hat{A}(u))_{u\in<0,p]}$. But when $\hat{R}$ is reversible we have

$$\hat{r}=\hat{R}\hat{a}.\tag{3.3}$$

Thus we get

$$\hat{R}(\hat{a}-a)=\hat{A}.\tag{3.4}$$

As in [7] and [8], the key step to get the main results is to prove the asymptotic normality of $\sqrt{|n|}\hat{A}$. First we give two lemmas which are multi-parameter analogy of Lemma 3.1 and Theorem 3.2 in [3], a special form of this kind of analogy has been considered in [8], but the proof given there for the latter lemma has place which is not clear.

Lemma 3.1. *Let* $\{\xi_{n,t},\ 1\leqslant t\leqslant n,\ n\geqslant 1\}$ *satisfy*

$$\max_{1\leqslant t\leqslant n}|\xi_{n,t}|\xrightarrow[|n|\to\infty]{P}0\tag{3.5}$$

$$\sum_{1\leqslant t\leqslant n}\xi^2_{n,t}\xrightarrow[|n|\to\infty]{P}\eta^2,\tag{3.6}$$

where η *is a bounded r.v., and for every real number* λ

$$\prod_{1\leqslant t\leqslant n}(1+i\lambda\xi_{n,t})\xrightarrow[|n|\to\infty]{}1\ (\textit{weakly in } L^1).\tag{3.7}$$

Then

$$J_n=\sum_{1\leqslant t\leqslant n}\xi_{n,t}\xrightarrow[|n|\to\infty]{d}Z\ (\textit{stably})\ (\textit{see }[1]),\tag{3.8}$$

where the r. v. Z *has characteristic function* $E\left\{\exp\left(-\frac{1}{2}\eta^2\lambda^2\right)\right\}$.

The proof is essentially the same as [3].

Lemma 3.2. *Suppose for* $\{\xi_{n,t},\ 1\leqslant t\leqslant n,\ n\geqslant 1\}$ *there exists* $\{\mathscr{F}_t,\ t\geqslant 1\}$ *such that*

$$\xi_{n,t}\in\mathscr{F}_t,\ E(\xi_{n,t}|\mathscr{F}_i(t-))=0\quad(i=1,2).\tag{3.9}$$

Let (3.5), (3.6) *be satisfied and there exists constant* M *such that for every* $n\geqslant 1$

$$E(\max_{1\leqslant t\leqslant n}\xi^2_{n,t})\leqslant M.\tag{3.10}$$

Then the conclusion of Lemma 3.1 *holds.*

Proof Suppose $\eta^2\leqslant C$. Let $\xi^{(1)}_{n,t}=\xi_{n,t}I\{\sum_{u\in S^{(1)}_{n,t}}\xi^2_{n,u}\leqslant 2C\}$, where $S^{(1)}_{n,t}=\{u\in[1,n]:u\overset{(1)}{<}t\}$, $J^{(1)}_n=\sum_{1\leqslant t\leqslant n}\xi^{(1)}_{n,t}$. Then for every $E\in\mathscr{F}$,

$$|E(\exp(i\lambda J_n)I_E)-E(\exp(i\lambda J^{(1)}_n)I_E)|\leqslant 2P(B_n),$$

where $B_n=\{\sum_{1\leqslant t\leqslant n}\xi^2_{n,t}>2C\}$. Thus we need only to show

$$\exp(i\lambda J^{(1)}_n)\xrightarrow[|n|\to\infty]{}\exp\left(-\frac{1}{2}\eta^2\lambda^2\right)\quad(\text{weakly in } L^1).\tag{3.11}$$

Note that $\{\xi^{(1)}_{n,t},\ 1\leqslant t\leqslant n,\ n\geqslant 1\}$ satisfies (3.5) and (3.6), so we only need to prove that for every real number λ

$$\prod_{1<t<n}(1+i\lambda\xi^{(1)}_{n,t})\xrightarrow[|n|\to\infty]{}1 \text{ (weakly in } L^1) \tag{3.12}$$

(see [3]). For this purpose we first prove that for every $E\in\mathscr{F}$,

$$E\{I_E\prod_{1<t<0}(1+i\lambda\xi^{(1)}_{n,t})\}\xrightarrow[n\to\infty]{}P(E). \tag{3.13}$$

Fix n and let $E\in\mathscr{F}_n$. Then for $n'\geqslant n$ we have

$$E\{I_E\prod_{1<t<n}(1+i\lambda\lambda\xi^{(1)}_{n',t})\}=E\{I_E\prod_{t\in L^{(1)}(n,n')}(1+i\lambda\xi^{(1)}_{n',t})\}, \tag{3.14}$$

where $L^{(1)}(n,n')=\{t\in[1,n']: t_1\leqslant n_1\}$. Furthermore, let t^* be the maximal element of the set $\{t\in[1,n']: \sum_{u\in S^{(1)}_{n',t}}\xi^{2'.u}_n\leqslant 2C\}$ under $\overset{(1)}{<}$. Then we have

$$\prod_{1<t<n'}(1+\lambda^2\xi^{(1)2}_{n',t})=\prod_{t\in S^{(1)}_{n',t*}}(1+\lambda^2\xi^{(1)2}_{n',t})(1+\lambda^2\xi^{(1)2}_{n',t^*})\leqslant\exp(2C\lambda^2)(1+\lambda^2\max_{1<t<n'}\xi^2_{n',t}). \tag{3.15}$$

For $t\in L^{(1)}(n,n')$, let $\xi^{(2)}_{n',t}=\xi_{n',t}I\{\sum_{u\in S^{(2)}_{n',t}}\xi^2_{n',u}\leqslant 2C\}$, where $S^{(2)}_{n',t}=\{u\in L^{(1)}(n,n'): u(2)<t\}$. Then similarly we have

$$E\{I_E\prod_{t\in L^{(1)}(n,n')}(1+i\lambda\,\xi^{(2)}_{n',t})\}=E\{I_E\prod_{1<t<n}(1+i\lambda\xi^{(2)}_{n',t})\} \tag{3.16}$$

and

$$\prod_{t\in L^{(1)}(n,n')}(1+\lambda^2\xi^{(2)2}_{n',t})\leqslant\exp(2C\lambda^2)(1+\lambda^2\max_{1<t<n'}\xi^2_{n',t}). \tag{3.17}$$

Thus we easily get

$$\begin{aligned}&|E\{I_E\prod_{1<t<n'}(1+i\lambda\xi^{(1)}_{n',t})\}-E\{I_E\prod_{1<t<n}(1+i\lambda\xi^{(2)}_{n',t})\}|\\&\leqslant 2\exp(C\lambda^2)\,(1+\lambda^2M)^{1/2}P(B_{n'})^{1/2}\xrightarrow[n'\to\infty]{}0.\end{aligned} \tag{3.18}$$

It is easy to see that

$$\prod_{1<t<n}(1+i\lambda\xi^{(2)}_{n',t})\xrightarrow[n'\to\infty]{L^1}1, \tag{3.19}$$

so (3.13) holds for $E\in\mathscr{F}_n$; from this one easily shows that (3.13) holds for $E\in\mathscr{F}$ (see [3]).

Now we consider the case $|n|\to\infty$. Reasoning exactly as before, one can show that for every fixed $n_{3-i}\geqslant 1$,

$$E\{I_E\prod_{1<t<(n_1,n_2)}(1+i\lambda\xi^{(1)}_{(n_1,n_2),t})\}\xrightarrow[n_j\to\infty]{}P(E)\quad(j=1,2). \tag{3.20}$$

By (3.13) and (3.20) it is easy to get (3.12).

Now let $d=(d_u)_{u\in<0,p]}$ be a fixed vector and

$$\xi_t=W_t\sum_{u\in<0,p]}d_uX_{t-u},\ \xi_{n,t}=\frac{1}{\sqrt{|n|}}\xi_t,\ J^*_n=d'\sqrt{|n|}\,\hat{A}=\sum_{1<t<n}\xi_{n,t}. \tag{3.21}$$

Lemma 3.3. *Let $\{X_t,\ t\in N^2\}$ satisfy (2.1) and (2.2), where $\{W_t,\ t\in N^2\}$ is strictly stationary and strongly ergodic and*

$$E(W_0^4\operatorname{Log}|W_0|)<\infty. \tag{3.22}$$

Moreover, let

$$W_t\in\mathscr{F}_t,\ E(W_t|\mathscr{F}_i(t-))=0\ (i=1,2). \tag{3.23}$$

Then

$$J_n^* \xrightarrow[|n|\to\infty]{d} N(0, d'\Sigma d), \tag{3.24}$$

where $\Sigma = (\sigma(u, v))_{u,v\in<0,p]}$, $\sigma(u, v) = E(W_0^2 X_{-u} X_{-v})$.

Proof We need only to prove that the $\xi_{n,t}$ defined by (3.21) satisfy the coditions of Lemma 3.2.

Lemma 3.4. *Let* $\{W_t,\ t\in N^2\}$ *be strictly stationary and satisfy* (3.22) *and* (3.23). *If for every* $x,\ y \geqslant\neq 0$ *and* $t_{3-i}\in N$,

$$\frac{1}{n_i}\left\{\left|\sum_{t_i=1}^{n_i} W_{t-x}W_{t-y}(E(W_t^2|\mathscr{F}_{t-})-1)\right| + \left|\sum_{t_i=1}^{n_i}(E(W_t^2|\mathscr{F}_{t-})-1)\right|\right\} \xrightarrow[n_i\to\infty]{P} 0 \tag{3.25}$$

$(i=1,\ 2)$, *then for every* $x,\ y\geqslant\neq 0$,

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n} W_t^2 W_{t-x}W_{t-y} \xrightarrow[|n|\to\infty]{a.\ s.} \delta_{x,y}. \tag{3.26}$$

Proof Fix $x,\ y\geqslant\neq 0$ and let $\eta_t = W_t^2 W_{t-x}W_{t-y}$. By Lemma 1.1 and Theorem 1.2, we need only to verify for every $t_{3-i}\geqslant 1$

$$\frac{1}{n_i}\sum_{t_i=1}^{n_i}\eta_{(t_1,t_2)} \xrightarrow[n_i\to\infty]{P} \delta_{x,y} \quad (i=1,\ 2). \tag{3.27}$$

By symmetry, it is enough to see $i=1,\ 2$. Let $\mathscr{G}_{(t_1,t_2)} = F_{(t_1,t_2+1)}$. Then by [3] Theorem 2.19 we have

$$\frac{1}{n_2}\sum_{t_2=1}^{n_2}[\eta_{(t_1,t_2)} - W_{(t_1-x_1,t_2-x_2)}W_{(t_1-y_1,t_2-y_2)}E(W_{(t_1,t_2)}^2|\mathscr{F}_{(t_1,t_2)-})]$$

$$= \frac{1}{n_2}\sum_{t_2=1}^{n_2}[\eta_{(t_1,\ t_2)} - E(\eta_{(t_1,t_2)}|\mathscr{G}_{(t_1,t_2-1)})] \xrightarrow[n_2\to\infty]{P} 0. \tag{3.28}$$

Thus by (3.25) we need only to show

$$\frac{1}{n_2}\sum_{t_2=1}^{n_2} W_{(t_1-x_1,t_2-x_2)}W_{(t_1-y_1,t_2-y_2)} \xrightarrow[n_2\to\infty]{P} \delta_{x,y}. \tag{3.29}$$

If $x\neq y$, then

$$E\left(\frac{1}{n_2}\sum_{t_2=1}^{n_2} W_{t-x}W_{t-y}\right)^2 = \frac{1}{n_2}E(W_{-x}^2 W_{-y}^2) \xrightarrow[n_2\to\infty]{} 0. \tag{3.30}$$

If $x=y$, then $W_{t-x}^2\in\mathscr{F}_{t-x}\subset\mathscr{F}_{(t_1-x_1,t_2-x_2+1)-} = \mathscr{H}_{t_2}$. Again using [3] Theorem 2.19 and (3.25) we get the results.

Lemma 3.5. *Let* $\{X_t,\ t\in N^2\}$ *satisfy* (2.1) *and* (2.2), *where* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 3.4. *Then*

$$J_n^* \xrightarrow[|n|\to\infty]{d} N(0, d'Rd), \tag{3.31}$$

where $R = (r(u, v))_{u,v\in<0,p]}$, $r(u, v) = E(X_u X_v)$.

Proof It is enough to verify

$$\sum_{1\leqslant t\leqslant n}\xi_{n,t}^2 = \frac{1}{n}\sum_{1\leqslant t\leqslant n}\xi_t^2 \xrightarrow[|n|\to\infty]{P} d'Rd, \tag{3.32}$$

but this follows directly from Lemma 3.4.

If we strengthen the conditions on the conditional variance of W_t, the moment condition in the above lemma can be reduced, that is, we have

Lemma 3.6. *Let $\{X_t, t\in N^2\}$ satisfy* (2.1) *and* (2.2), *where $\{W_t, t\in N^2\}$ is strictly stationary and satisfies* (3.23) *and $E(W_t^2|\mathscr{F}(t-1))\overset{a.s.}{=\!=}1$. Then* (3.31) *holds.*

Proof As in the proof of Lemma 3.5, it is enough to verify (3.32). Using the following facts:

1) $Z_n \xrightarrow[|n|\to\infty]{P} Z$ iff for every $\{n_k\}\subset[1, \infty)$ satisfying $|n_k| \xrightarrow[k\to\infty]{} \infty$ there exist $\{m_k\}\subset\{n_k\}$ such that $Z_{m_k}\xrightarrow[k\to\infty]{P} Z$;

2) for every $\{n_k\}\subset[1, \infty)$ satisfying $|n_k|\xrightarrow[k\to\infty]{} \infty$, there exist $\{m_k\}\subset\{n_k\}$ such that $\{m_k\}$ is nondecreasing (with respect to '$\leqslant$') and $|m_k|\xrightarrow[k\to\infty]{} \infty$, we need only to show that if $\{n_k\}$ is a nondecreasing sequence of lattice points and $|n_k|\xrightarrow[k\to\infty]{}\infty$, then

$$\frac{1}{|n_k|}\sum_{1\leqslant t\leqslant n_k}\xi_t^2\xrightarrow[k\to\infty]{P} d'Rd. \tag{3.33}$$

Order the points in $[1, \infty)$ by $t_1, t_2, \cdots$ as follows: first order the points in $[1, n_1]$ according to $\overset{(1)}{\leqslant}$, then order the points in $[1, n_2]\setminus[1, n_1]$ according to $\overset{(1)}{\leqslant}$, etc.

Now let

$$\mathscr{G}_i=\begin{cases}\mathscr{F}(0), & i=0;\\ \mathscr{F}(0)\vee(\bigvee\limits_{s\in[1,n_1],s\overset{(1)}{<}t_i}\mathscr{F}_t), & 1\leqslant i\leqslant|n_1|;\\ \mathscr{F}(0)\vee\mathscr{F}_{n_{k-1}}\vee(\bigvee\limits_{s\in[1,n_k]\setminus[1,n_{k-1}],s\overset{(1)}{<}t_i}\mathscr{F}_s), & |n_{k-1}|<i\leqslant|n_k|,\ k\geqslant2.\end{cases}$$

It is not hard to verify

i) $\mathscr{G}_i$ is nondecreasing; ii) $\eta_i=\xi_{t_i}^2\in\mathscr{G}_i$; iii) $\mathscr{F}_{t_i-}\subset\mathscr{G}_{i-1}\subset\mathscr{F}(t_i-1)$.

So using Theorem 2.19 in [3], one can prove

$$\frac{1}{|n_k|}\sum_{1\leqslant t\leqslant n_k}[\xi_t^2-(\sum_{u\in<0,p]}d_uX_{t-u})^2]\xrightarrow[k\to\infty]{P}0. \tag{3.34}$$

The result then follows provided we can prove $\frac{1}{|n_k|}\sum\limits_{1\leqslant t\leqslant n_k}W_{t-x}W_{t-y}\xrightarrow[k\to\infty]{P}\delta_{x,y}$, but this can be done by calculating the second moments in case $x\neq y$ and by an ordering analogous to the above one in case $x=y$.

Lemma 3.7. *Let $\{W_t, t\in N^2\}$ satisfy* (2.23), (2.24) *and* (2.21) *and*

$$E(W^4\,\mathrm{Log}|W|)<\infty. \tag{3.35}$$

Moreover, suppose for every $x, y\geqslant\neq0$ we have

$$\frac{1}{|n|}\Big\{\Big|\sum_{1\leqslant t\leqslant n}W_{t-x}W_{t-y}(E(W_t^2|\mathscr{F}_{t-})-1)\Big|+\Big|\sum_{1\leqslant t\leqslant n}(E(W_{t-x}^2|\mathscr{F}_{t-x-})-1)\Big|\Big\}\xrightarrow[|n|\to\infty]{P}0. \tag{3.36}$$

Then for every $x, y\geqslant\neq0$ we have

$$\frac{1}{|n|}\sum_{1\leqslant t\leqslant n}W_t^2W_{t-x}W_{t-y}\xrightarrow[|n|\to\infty]{P}\delta_{x,y}. \tag{3.37}$$

The proof can be easily given, as in the proof of Lemma 2.5.

Lemma 3.8. *Let* $\{X_t,\ t\in N^2\}$ *satisfy* (2.1) *and* (2.2), *where* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 3.7. *Then* (3.31) *holds.*

Now we can give the main theorems of this section.

Theorem 3.1. *Let* $\{X_t,\ t\in N^2\}$ *be a stable AR* (p) *model such that the innovation sequence* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 3.3. *Then*

$$\sqrt{|n|}\,(\hat{a}-a)\xrightarrow[|n|\to\infty]{d} N(0,\ R^{-1}\Sigma R^{-1}). \tag{3.38}$$

Proof By Lemma 3.3 and the arbitrariness of d in (3.21), one easily gets the results, using Cramer–Wold's lemma. Note in this case we obviously have

$$\hat{R}\xrightarrow[|n|\to\infty]{P} R.$$

Theorem 3.2. *Let* $\{X_t,\ t\in N^2\}$ *be a stable AR*(p) *model such that the innovation sequence* $\{W_t,\ t\in N^2\}$ *satisfies the conditions of Lemma* 3.5 *or the conditions of Lemma* 3.6 *or the conditons of Lemma* 3.8. *Then*

$$\sqrt{|n|}\,(\hat{a}-a)\xrightarrow[|n|\to\infty]{d} N(0,\ R^{-1}). \tag{3.39}$$

Appendix

The proof of Lemma 1.1:

Lemma. *Let* ξ, η *be nonnegative random variables satisfying for every* $\lambda>0$

$$\lambda P(\xi>\lambda)\leqslant E\eta I_{\{\xi<\lambda\}}. \tag{A-1}$$

Then for $\beta=0, 1, 2, \cdots$, *we have*

$$E(\xi(\operatorname{Log}\xi)^{\beta})\leqslant A_\beta E(\eta(\operatorname{Log}\eta)^{\beta+1})+5e, \tag{A-2}$$

where $A_\beta=2e\left\{2^{\beta+1}+\left(\dfrac{5}{2e-5}\right)^{\beta+1}\right\}$.

Let $M_0=|x|$, $M_1=\sup\limits_{m\geqslant1}\dfrac{1}{m}\sum\limits_{i=0}^{m-1}M_0U^i$, $M_2=\sup\limits_{n\geqslant1}\dfrac{1}{n}\sum\limits_{j=0}^{n-1}M_1V^j$. Then

$$M_2\geqslant\sup_{m,n\geqslant1}\frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|x|U^iV^j. \tag{A-3}$$

By the one parameter maximal ergodic theorem (see [6] P_{175}) we have for every $\lambda>0$

$$\lambda P(M_k>\lambda)\leqslant EM_{k-1}I_{\{M_k>\lambda\}}\quad (k=1,\ 2). \tag{A-4}$$

The conclusion now follows from (A-3), (A-4) and the above lemma.

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