Chin. Ann. of Math.
12B:4(1991), 454—463.

ON CURVATURE PINCHING FOR MINIMAL AND KAEHLER SUBMANIFOLDS WITH ISOTROPIC SECOND FUNDAMENTAL FORM**

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Abstract.

An isometrically immersed submanifold is said to have isotropic second fundamental form if the length of the second fundamental form related ito any normal vector is the same one. In this note, some curvature pinching theorems for compact minimal (resp. Kaehler) submanifolds in $S^{n+p}(c)$ (resp. $CP^{n+p}(c)$) with isotropic second fundamental form are given.

Introduction

Let M be an n-dimensional Riemannian manifold isometrically immersed in an $(n+p)$-dimensiona Riemannianlmanifold $\overline{M}$. We denote by $\langle,\rangle$ the metric of $\overline{M}$ as well as that induced on M. If σ is the second fundamental form of the immersion and A_ξ the Weingarten endomorphism associated to a normal vector ξ, we define T: $T_p^\perp M\times T_p^\perp M\to \mathbf{R}$ by the expression $T(\xi, \eta)=\text{trace } A_\xi A_\eta$, where $T_p^\perp M$ is the normal space to M at $p\in M$ and $\xi, \eta\in T_p^\perp M$. In [7, 9] it was proposed to study a special class of submanifolds for which $T=k\langle,\rangle$. By taking the trace we have $k=\|\sigma\|^2/p$. If $\sigma_\xi=\langle\sigma, \xi\rangle\xi$ is the second fundamental form with respect to a normal vector ξ, then $T=k\langle,\rangle$ if and only if $\|\sigma_\xi\|^2=\|\sigma\|^2/p$, which is independent of ξ. So, we give the following

Definition. *A submanifold M is said to have isotropic second fundamental form if $T=\|\sigma\|^2\langle,\rangle/p$, i. e., the length of the second fundamental form with respect to any normal vector is the same one.*

Obviously, hypersurfaces (codimension=1) are trivial. In a Euclidean sphere $S^{n+p}(c)$ of constant curvature c, a remarkable class of submanifolds with isotropic second fundamental form is of order $\{u_1, u_2\}$ for some natural numbers $u_1, u_2\geqslant 1$, in which case submanifolds are Einstein (cf. [7]). In particular, compact homogeneoms irreducible spaces and strongly harmonic manifolds all are submanifolds with

Manuscript received April 11, 1989.

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** Projects supported by the National Natural Science Fund & K. C. Wang's Educational Fund.

isotropic second fundamental form[7]. In the complex projective space $CP^{n+p}(c)$ of constant holomorphic sectional curvature c, a Kaehler submanifold of order $\{k_1, k_2\}$ for some natural numbers k_2 and k_2 is one of this type submanifolds (cf. [9]). Particularly, seven compact Kaehler submanifolds with parallel second fundamental form in $CP^{n+p}(c)$ are remarkble, which were exposed by H. Nakagawa-R. Takagi ([4]), M. Takeuchi ([12]) and A. Ros([8]), respectively.

In this paper, by using the idea introduced by A. Ros, etc. in [3, 9], some curvature pinching theorems for compact minimal (resp. Kaehler) submanifolds in $S^{n+p}(c)$ (resp. $CP^{n+p}(c)$) with isotropic second fundamental form are given. In the case of real minimal submanifolds of $S^{n+p}(c)$ our pinching condition characterizes the compact symmetric spaces of rank one (Theorem 1), of which the immersions are planar geodesic ([10]). In the case of Kaehler submanifolds of $CP^{n+p}(c)$ a characterization of seven compact Kaehler submanifolds by Ricci curvature pinching is shown (Theorem 2), which was studied already by A. Ros in [8, 9].

Unless otherwise stated, manifolds are assumed to be connected and of real dimension$\geqslant$2.

§ 1. Preliminaries

We begin with some notations and formulas following closely the expressions in [3]. Let M be an n-dimensional compact Riemannian manifold. We denote by UM the unit tangent bundle over M and by UM_p its fiber at $p\in M$. For any continuous function f: $UM\to\mathbf{R}$, we have

$$\int_{UM} f dv=\int_M\left\{\int_{UM_p} f dv_p\right\}dp,$$

where dp, dv_p and dv stand for, the canonical measures on M, UM_p and UM respectively.

Suppose now that M is minimally immersed in an $(n+p)$-dimensional Euclidean sphere $S^{n+p}(c)$ of constant curvature c. Let σ be the second fundamental form of the minimal immersion and A_ξ the Weingarten endomophism for a normal vector ξ. If $T_p M$ and $T_p^\perp M$ denote the tangent and normal spaces to M at p, one can define

$$L: T_pM\to T_pM \text{ and } T: T_p^\perp M\times T_p^\perp M\to\mathbf{R}$$

by the expressions

$$Lv=\sum_{i=1}^n A_{\sigma(v,e_i)}e_i \text{ and } T(\xi, \eta)=\sum_{i=1}^n \langle A_\xi e_i, A_\eta e_i\rangle, \tag{1.1}$$

where $\{e_i\}_{1<i<n}$ is an orthonormal basis of $T_p M$. By a modified version of Simons' formula given in [3], we have

$$0=\frac{n-4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv+(n+4)\int_{UM}|A_{\sigma(v,v)}v|^2dv-4\int_{UM}\langle Lv,\ A_{\sigma(v,v)}v\rangle dv$$
$$-2\int_{UM}T(\sigma(v,\ v),\ \sigma(v,\ v))dv+2c\int_{UM}(\langle Lv,\ v\rangle-|\sigma(v,\ v)|^2)dv. \tag{1.2}$$

Let S and ρ be the Ricci tensor and the scalar curvature of M. It follows from the Gauss equation that

$$S(u,\ v)=(n-1)c\langle u,\ v\rangle-\langle Lu,\ v\rangle \tag{1.3}$$

and

$$\rho=n(n-1)c-\|\sigma\|^2, \tag{1.4}$$

where $\|\sigma\|^2$ is the length square of σ.

Some useful formulas given in [3] are as follows ([3], Lemma 1):

$$\int_{UM_p}\langle Lv,\ A_{\sigma(v,v)}v\rangle\,dv_p=\frac{2}{n+2}\int_{UM_p}|Lv|^2dv_p, \tag{1.5}$$

$$\int_{UM_p}\langle Lv,\ \ v\rangle dv_p=\frac{1}{n}\int_{UM_p}\|\sigma\|^2dv_p, \tag{1.6}$$

$$\int_{UM^p}|\sigma(v,\ \ v)|^2dv_p=\frac{2}{n(n+2)}\int_{UM_p}\|\sigma\|^2dv_p. \tag{1.7}$$

Now, at any point $p\in M$, let $f(v)=A_{\sigma(v,v)}v$ for $v\in UM_p$. By considering UM_p as a unit $(n-1)$-sphere, we have (cf. [3], p. 543)

$$-\int_{UM_p}\langle\Delta_v f,\ f\rangle dv_p\geqslant(n-1)\int_{UM_p}|f|^2dv_p$$

and

$$(\Delta_v f)(v)=-3(n+1)f(v)+4Lv,$$

where Δ_v stands for the Laplacian of UM_p. Hence, it follows that

$$\int_{UM_p}|A_{\sigma(v,v)}v|^2dv_p\geqslant\frac{2}{n+2}\int_{UM_p}\langle Tv,\ A_{\sigma(v,v)}v\rangle dv_p, \tag{1.8}$$

where the equality holds if and only if f is the first eigenfunction of Δ_v, i. e.,

$$A_{\sigma(v,v)}v=\frac{2}{n+2}\,Lv.$$

For details for the geometry of Kaehler submanifolds see [5].

§2. Minimal Submanifolds in a Sphere

In this section, we prove the following

Theorem 1. *Let M be an n-dimensional compact minimal submanifold immersed in $S^{n+p}(c)$ with isotropic second fundamental form. If the Ricci curvature Ric (M) of M satisfies*

$$\text{Ric}\ (M)\geqslant\left\{n-1-\frac{p(n+2)}{2(n+p+2)}\right\}c, \tag{1.2}$$

then M is either totally geodesic or a compact symmetric space of rank one.

Proof Let Q be the function which assigns to each point of M the infimum of the Ricci curvature of M at that point. Then, from (1.3) it follows that

$$0\leqslant\langle Lv,\ v\rangle\leqslant(n-1)c-Q \tag{2.2}$$

for all $v\in UM$. If $e_1, \cdots, e_n$ is an orthonormal basis of T_pM, $p\in M$, such that $Le_i=\lambda_i e_i$, we have $\lambda_i=\langle Le_i,\ e_i\rangle\geqslant 0$ and

$$|Lv|^2=\sum_{i=1}^{n}\lambda_i^2\langle v,\ e_i\rangle^2\leqslant[(n-1)c-Q]\sum_{i=1}^{n}\lambda_i\langle v,\ e_i\rangle^2=[(n-1)c-Q]\langle Lv,\ v\rangle, \tag{2.3}$$

where the equality implies that $\lambda_i=(n-1)c-Q$ for all $i=1, \cdots, n$, i. e., the Ricci curvature of M is equal to Q at p.

By (1.5), (1.6), (1.8) and (2.3) we have

$$\begin{aligned}&(n+4)\int_{UM}|A_{\sigma(v,v)}v|^2dv-4\int_{UM}\langle Lv,\ A_{\sigma(v,v)}v\rangle dv\\&\geqslant-\frac{2n}{n+2}\int_{UM}\langle Lv,\ A_{\sigma(v,v)}v\rangle dv=-\frac{4n}{(n+2)^2}\int_{UM}|Lv|^2dv\\&\geqslant\frac{4}{(n+2)^2}\int_{UM}[Q-(n-1)c]\,\|\sigma\|^2dv,\end{aligned} \tag{2.4}$$

where the equality implies that the Ricci curvature of M is isotropic almost everywhere, so that M is Einsteinian because of the connectedness and compactness of M.

Since M has isotropic second fundamental form, one can see easily that

$$\begin{aligned}\int_{UM}T(\sigma(v,\ v),\ \sigma(v,\ v))dv&=\int_M\frac{\|\sigma\|^2}{p}\left\{\int_{UM_p}|\sigma(v,v)|^2dv_p\right\}dp\\&=\frac{2}{np(n+2)}\int_{UM}\|\sigma\|^4dv\end{aligned} \tag{2.5}$$

according to (1.7).

Now, introducing (2.4) and (2.5) to (1.2) and using (1.6) and (1.7), we can get

$$\begin{aligned}0&\geqslant\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv+\frac{2}{(n+2)^2}\int_{UM}[2Q-(n-4)c]\,\|\sigma\|^2dv\\&\quad-\frac{4}{pn(n+2)}\int_{UM}\|\sigma\|^4dv\\&=\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv+\frac{2}{n+2}\int_{UM}\|\sigma\|^2\left\{\frac{2Q-(n-4)c}{n+2}-\frac{2}{pn}\|\sigma\|^2\right\}dv.\end{aligned} \tag{2.6}$$

On the other hand, from (1.4) we have

$$\|\sigma\|^2=n(n-1)c-\rho\leqslant n(n-1)c-nQ, \tag{2.7}$$

which together with (2.6) yields

$$\begin{aligned}0&\geqslant\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv\\&\quad+\frac{4(n+p+2)}{p(n+2)^2}\int_{UM}\|\sigma\|^2\left\{Q-\left(n-1-\frac{p(n+2)}{2(n+p+2)}\right)c\right\}dv.\end{aligned} \tag{2.8}$$

Thus, unedr the hypothesis (2.1), (2.8) must be an equality, which implies that

(2.6) and (2.4) are equalities. Hence, M is Einsteinian and the equality in (2.7) holds, from which and (2.8) it follows that either $\|\sigma\|^2=0$, i. e., M is totally geodesec, or

$$\|\sigma\|^2=\frac{np(n+2)}{2(n+p+2)}\, c. \tag{2.9}$$

Since M is Einsteinian, from (1.3) and (1.4) it follows that

$$Lv=\|\sigma\|^2 v/n \text{ for all } v\in UM. \tag{2.10}$$

Substituting (2.5) and (2.10) into (1.2) and using (1.6) and (1.7), we find

$$0=\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv+(n+4)\int_{UM}|A_{\sigma(v,v)}v|^2dv-\frac{4(2p+n)}{n^2p(n+2)}\int_{UM}\|\sigma\|^4dv$$
$$+\frac{2c}{n+2}\int_{UM}\|\sigma\|^2dv. \tag{2.11}$$

Now, fixed any point $p\in M$, let α_v be the 1-form on $UM_p(\cong S^{n-1})$ given by

$$\alpha_v(e)=|\sigma(v,\ v)|^2\langle\sigma(v,\ v),\ \sigma(v,\ e)\rangle$$

with $v\in UM_p$ and $e\in T_v(UM_p)$. If $e_1, \cdots, e_{n-1}$ is an orthonormal basis of $T_v(UM_p)$, then by a straightforward calculation one can easily see that the codifferential $\delta\alpha_v$ of α_v is

$$\delta\alpha_v=-\sum_{i=1}^{n-1}\tilde{\nabla}_{e_i}\alpha_v(e_i)=(n+6)|\sigma(v,\ v)|^4-4|A_{\sigma(v,v)}v|^2-2|\sigma(v,\ v)|^2\cdot\|\sigma\|^2/n,$$

where $\tilde{\nabla}$ stands for the canonical connection on S^{n-1} and $e_1, \cdots, e_{n-1}, e_n=v$ is an orthonormal basis of T_pM. Integrating over UM_p and using

$$\int_{UM_p}\delta\alpha_v dv_p=0,$$

we have

$$(n+6)\int_{UM_p}|\sigma(v,\ v)|^4dv_p=4\int_{UM_p}|A_{\sigma(v,v)}v|^2dv_p+\frac{4}{n^2(n+2)}\int_{UM_p}\|\sigma\|^4dv_p. \tag{2.12}$$

By the Schwarz inequality we have

$$|\sigma(v,\ v)|^4=\langle A_{\sigma(v,v)}v,\ v\rangle^2\leqslant|A_{\sigma(v,v)}v|^2,$$

where the equality holds if and only if $A_{\sigma(v,v)}v=\lambda v$, i. e., M is isotropic at p (cf. [6]). Thus, (2.12) can be rewritten as

$$\int_{UM_p}|A_{\sigma(v,v)}v|^2dv_p\geqslant\frac{4}{n^2(n+2)^2}\int_{UM_p}\|\sigma\|^4dv_p. \tag{2.13}$$

Introducing (2.13) into (2.11), we obtain finally

$$0\geqslant\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv+\frac{2}{n+2}\int_{UM}\|\sigma\|^2\left\{c-\frac{2(n+p+2)}{np(n+2)}\|\sigma\|^2\right\}dv. \tag{2.14}$$

In the case of (2.9) it is obvious that (2.14) becomes an equality, which implies that M is isotropic and $\nabla\sigma=0$, i. e., M has parallel second fundamental form. Using [10] we conclude that M is a compact symmetric space of rank one. Thus, Theorem 1 is proved completely.

Remark 1. If M is Einsteinian, the condition (2.1) is equivalent to

$$\|\sigma\|^2 < \frac{np(n+2)}{2(n+p+2)} c,$$

which was shown in [3]. So, Theorem 1 may be regarded as a generalization of Lemma 5 in [3].

Remark 2. For $p \geqslant 2(1+2/n)$ the pinching constant of (2.1) is not larger than $(n-2)c$ obtained by N. Ejiri in [2]. Of course, we have imposed an additional condition that M has isotropic second fundamental form.

Remark 3. As is well known, the only compact symmetric 2-spaces of rank one are 2-spheres of constant Gauss curvature. Then, Theorem 1 together with the result of J. L. M. Barbosa in [1] yields the following

Corollary 1.1 *Let M^2 be a compact minimal surface in $S^{2+p}(c)$ with isotropic second fundamental form. If the Gauss curvature of M^2 is not less than $(4-p)c/(4+p)$, then M^2 is either totally geodesic or an $S^2(K)$ in $S^{2m}(c)$ with $K=2c/m(m+1)$ for some natural number $m>1$.*

§3. Kaehler Submanifolds in a Complex Projective Space

Let $CP^{n+p}(c)$ be a complex projective space, of complex dimension $n+p$, with the Fubini-Study metric of constant holomorphic sectional curvature c. Suppose that M^n is a compact Kaehler submanifold, of complex dimension n, immersed in $CP^{n+p}(c)$. We denote by J the complex structure of $CP^{n+p}(c)$ as well as that induced on M^n. Choose a local field of orthonormal frames $e_1, \cdots, e_{n+p}, e_{1*}=Je_1, \cdots, e_{(n+p)*}=Je_{n+p}$ in $CP^{n+p}(c)$ such that, restricted to M^n, $e_1, \cdots, e_n, e_{1*}, \cdots, e_{n*}$ are tangent to M^n. Throughout this section we use the following convention on the range of indices

$$\alpha, \beta=n+1, \cdots, n+p; \lambda, \mu=n+1, \cdots, n+p, (n+1)^*, \cdots, (n+p)^*.$$

If we write A_λ for the Weingarten endomorphism A_{e_λ} and S for the Ricci tensor of M^n, then (cf. [5])

$$S(u, v)=\frac{1}{2}(n+1)c\langle u, v\rangle - 2\sum_\alpha \langle A_\alpha^2 v, u\rangle \tag{3.1}$$

for $u, v \in UM$. It follows that the scalar curvature ρ of M is

$$\rho=n(n+1)c-\|\sigma\|^2, \tag{3.2}$$

where

$$\|\sigma\|^2=\sum_\lambda \mathrm{tr} A_\lambda^2=2\sum_\alpha \mathrm{tr}\, A_\alpha^2. \tag{3.3}$$

We now prove the following

Theorem 2. *Let M^n be a compact Kaehler submanifold in $CP^{n+p}(1)$ with* *isotropic second fundamental form. If the Ricci curvature Ric(M) of M^n satisfies*

$$\mathrm{Ric}(M)\geqslant n(n+p+1)/2(2p+n), \tag{3.4}$$

then M^n is either totally geodesic or an imbedded submanifold congruent to the standard imbedding of one of the following submanifolds:

Submanifolds	n	p	Ric(M)
$M_1=CP^n(1/2)$	n	$n(n+1)/2$	$(n+1)/4$
$M_2=Q^n$	n	1	$n/2$
$M_3=CP^s(1)\times CP^s(1)$	$2s$	s	$(s+1)/2$
$M_4=U(s+2)/U(s)\times U(2)$, $s\geqslant 3$	$2s$	$s(s-1)/2$	$(s+2)/2$
$M_5=SO(10)/U(5)$	10	5	4
$M_6=E_6/\mathrm{Spin}(10)\times T$	16	10	6

where n is the complex dimension, p the full complex codimension and Ric (M) the Ricci curvature of M^n.

Proof Let Δ be the Laplacian on M^n. In [5] a formula of Simons' type for Kaehler submanifolds of $CP^{n+p}(1)$ says that

$$\frac{1}{2}\Delta\|\sigma\|^2=\|\nabla\|\sigma^2-8\mathrm{tr}(\sum_\alpha A_\alpha^2)^2-\sum_{\lambda,\mu}(\mathrm{tr}\ A_\lambda A_\mu)^2+\frac{1}{2}(n+2)\|\sigma\|^2. \tag{3.5}$$

Since M^n has isotropic second fundamental form, we have

$$\sum_{\lambda,\mu}(\mathrm{tr}\ A_\lambda A_\mu)^2=\sum_{\lambda,\mu}\|T(e_\lambda,\ e_\mu)\|^2=\|\sigma\|^4/2p,$$

from which (3.5) may be reduced to

$$\frac{1}{2}\ \Delta\|\sigma\|^2=\|\nabla\sigma\|^2-8\ \mathrm{tr}(\sum_n A_\alpha^2)^2-\frac{1}{2p}\|\sigma\|^4+\frac{n+2}{2}\|\sigma\|^2. \tag{3.6}$$

On putting

$$c_R=n(n+p+1)/2(2p+n), \tag{3.7}$$

one can see from (3.1) and (3.4) that

$$\frac{1}{2}(n+1)\langle u,\ v\rangle-2\langle\sum_\alpha A_\alpha^2u,\ v\rangle\geqslant c_R\langle u,\ v\rangle. \tag{3.8}$$

Thus, $\frac{1}{2}(n+1)I-2\sum_\alpha A_\alpha^2$ is positively definite, where I stands for the identity. Moreover, since A_α's are symmetric linear transformations, $\sum_\alpha A_\alpha^2$ is positively semi-definite. So, $\sum_\alpha A_\alpha^2$ and $\frac{1}{2}(n+1)I-2\sum_\alpha A_\alpha^2$ can be transformed simultaneously by an orthogonal matrix into diagonal forms at each dointof M^n. Hence, (3.8) implies that

$$\mathrm{tr}\Big\{\Big(\frac{1}{2}(n+1)I-2\sum_\alpha A_\alpha^2\Big)(\sum_\alpha A_\alpha^2)\Big\}\geqslant c_R\cdot(\mathrm{tr}\sum_\alpha A_\alpha^2),$$

i. e., by (3.3),

$$\Big(\frac{n+1}{2}-c_R\Big)\|\sigma\|^2\geqslant 4\,\mathrm{tr}(\sum_\alpha A_\alpha^2)^2, \tag{3.9}$$

where the equality holds if and only if the Ricci curvatuae of M^n is equal to c_R at that point.

Substituting (3.7) and (3.9) into (3.6), we have

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant \|\nabla\sigma\|^2 + \frac{1}{2}\|\sigma\|^2\left\{\frac{2n(n+p+1)}{2p+n} - n - \frac{1}{p}\|\sigma\|^2\right\},$$

which together with the compactness of M^n yields

$$0 \geqslant \int_M \|\nabla\sigma\|^2 dp + \frac{1}{2}\int_M \left\{\frac{n(n+2)}{2p+n} - \frac{1}{p}\|\sigma\|^2\right\} dp. \qquad (3.10)$$

From (3.2) and (3.4) one sees easily that

$$-\|\sigma\|^2 = \rho - n(n+1) \geqslant 2nc_R - n(n+1) = -\frac{pn(n+2)}{2p+n}, \qquad (3.11)$$

which implies that the right-hand side of (3.10) is nonnegative. Hence, it follows from (3.10) and (3.11) that $\nabla\sigma = 0$ and M^n is either totally geodesic or an Einstein-Kaehler submanifold with Ricci curvature c_R. Now, the conclusion of Theorem 2 follows directly from the classification of Kaehler submanifolds in $CP^{n+p}(1)$ with parallel second fundamental form given in [4] (see also Table 1 in [9]). Theorem 2 is proved.

Remark 4. If M^n is Einstein-Kaehler, the condition (3.4) is equivalent to $\|\sigma\|^2 \leqslant np(n+2)/(2p+n)$, which was shown in [9, Lemma 4.3].

Remark 5. Obviously, the pinching constant of (3.4) is not larger than $n/2$, which was given by K. Ogiue in [5] without the additional hypothesis that M^n has isotropic second fundamental form.

§ 4. On Sectional Cuvature and Scalar Curvature

We return to the case of minimal submanifolds in $S^{n+p}(c)$.

By the equation (1.4) of the minimal immersion, usually, one trasforms the study of the scalar curvatuse to that of the length square of second fundamental form, for which we have

Theorem 3. *Let M be an n-dimensional compact minimal submanifold immersed in $S^{n+p}(c)$ with isotropic second fundamental form. If*

$$\|\sigma\|^2 \leqslant np(n+2)c/2(np+n+2) \qquad (4.1)$$

everywhere, then M is totally geodesic.

Proof As has been done in the proof of Theorem 1, let $e_1, \cdots, e_n$ be an orthonormal basis of T_pM for a point $p\in M$ such that $Le_i = \lambda_i e_i$. Since $\lambda_i \geqslant 0$, by using (1.3) and (1.4) we have

$$|Lv|^2 = \sum_{i=1}^n \lambda_i^2 \langle v, e_i\rangle^2 \leqslant (\sum_k \lambda_k)(\sum_i \lambda_i \langle v, e_i\rangle^2) = \|\sigma\|^2 \langle Lv, v\rangle, \qquad (4.2)$$

where the equality holds if and only if all $\lambda_i=0$ because of $n\geqslant 2$.

From (1.5), (1.6), (1.8) and (4.2) it follows that

$$(n+4)\int_{UM}|A_{\sigma(v,v)}v|^2dv-4\int_{UM}\langle Lv,\ A_{\sigma(v,v)}v\rangle dv\geqslant-\frac{4n}{(n+2)^2}\int_{UM}\|\sigma\|^2\langle Lv,\ v\rangle dv$$
$$=-\frac{4n}{n(n+2)^2}\int_{UM}\|\sigma\|^4dv. \tag{4.3}$$

Now, introducing (4.3) and (2.5) into (1.2)and using(1.6)and (1.7), we have

$$0\geqslant\frac{n+4}{3}\int_{UM}|(\nabla\sigma)(v,\ v,\ v)|^2dv+\frac{2}{n+2}\int_{UM}\|\sigma\|^2\left\{c-\frac{2(np+n+2)}{np(n+2)}\|\sigma\|^2\right\}dv, \tag{4.4}$$

where the equaliy implies that (4.2) becomes an equality, which yields $\|\sigma\|^2=0$, i. e., M is totally geodesic. This proves Theorem 3.

Remark 6. When $p\geqslant 3(n+2)/4$, the pinching constant of (4.1) is not less than the well-known Simons' constant $n/(2-1/p)$.

Furthermore, employing(4.4), we prove the following

Theorem 4. *Let M be an n-dimensional compact minimal submanifold immersed in $S^{n+p}(c)$ with isotropic second fundamental form. If the sectional curvature of M is not less than*

$$(2np+n+2)c/4(np+n+2) \tag{4.5}$$

everywhere, then M must be totally geodesic.

Proof Let K_M be the function which assigns to each point of M the infimum of the sectional curvature of M at that point. Then, the Yau's integral formula(10.1) in [12] says that

$$0\geqslant\int_{UM}\|\sigma\|^2\{2nK_M+\|\sigma\|^2-npc\}dv. \tag{4.6}$$

From (4.4) it follows that

$$0\geqslant\int_{UM}\|\sigma\|^2\left\{\frac{np(n+2)}{2(np+n+2)}c-\|\sigma\|^2\right\}dv, \tag{4.7}$$

which together with(4.6)yields

$$0\geqslant\int_{UM}\|\sigma\|^2\left\{K_M-\frac{2np+n+2}{4(np+n+2)}c\right\}dv, \tag{4.8}$$

where the equality implies that either $\|\sigma\|^2=0$ or $K_M=(2np+n+2)c/4(np+n-2)$. In latter case, (4.6) becomes

$$0\geqslant\int_{UM}\|\sigma\|^2\left\{\|\sigma\|^2-\frac{np(n+2)}{2(np+n+2)}c\right\}dv,$$

which makes (4.7)an equality as well as (4.4). This proves Theorem 4.

Remark 7. When $p\geqslant 3(n+2)/4$, our pinching constant (4.5) is not larger than the Yau's constant $(p-1)/(2p-1)$.[12].

For Kaehler submanifolds in $CP^{n+p}(c)$ there are analogous results, which we omit here.

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