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A RESULT ON THE BIRATIONAL MAP BETWEEN TWO NORMAL 3-FOLDS

CHEN MENG (陈 猛)*

Abstract

This paper tries to study crepant morphism of algebraic 3-folds. For certain birational map f between normal 3-folds, when is f an isomorphism in codimension one? The author gives one sufficient condition.

§ 1. Definition and Introduction

The research in algebraic 3-fold is very brisk recent years. Here, all the terminologies and results remain the same as that in [1], [2] and [3].

The main object considered here is normal algebraic 3-fold. Let X denote a normal 3-fold. Generally speaking, Weil divisor and Cartier divisor on X need not be the same thing. You can know in [4].

1. Canonical divisor on X

Suppose $X=\sum n_\Gamma \Gamma$, a Weil divisor on X. We define asheaf $O_X(D)\subset K(X)$, $\Gamma(U, O_X(D))=\{f\in K(X)\,|\,v_\Gamma(f)\geqslant -n_\Gamma$ for all prime divisors$\in U\}$.

Definition 1.1. *Let L be a sheaf on X. If it satisfies* $L=L^{**}$, $L^*=\mathrm{Hom}(L, O_X)$, *then we say L is a divisorial sheaf.*

According to[1], $L=L^{**}$ is equivalent to the following condition: If $X^0\subset X$ is a non-singular open subvariety such that $X\setminus X^0$ has codimension$\geqslant 2$, then $L|_{X^0}$ is invertible and $L=j_*(L|_{X^0})$, where j denotes the inclusion.

Actually the sheaf $O_X(D)$ is a divisorial sheaf. Futhermore, there exists a bijective correspondence $D\mapsto O_X(D)$:

$$\{\text{Weil divisors on } X\}\overset{1-1}{\longleftrightarrow}\{\text{divisorial subsheaves } L\subset K(X)\}/\Gamma(X, O_X)^*.$$

For $X\subset P^N$, define $\omega_X=\mathcal{E}xt_{P^N}^{N-3}(O_X, \omega_{P^N})$. Let X^0 be non-singular locus on X. Because of the normality of X, we have codim $X\setminus X^0\geqslant 2$, $\omega_X|_{X^0}=\Omega^3_{X^0}$, ω_X is a divisorial sheaf. Hence ω_X corresponds to a Weil divisor K_X and then $O_X(K_X)=(\Omega^3_X)^{**}=j_*\omega_{X^0}$, j: $X^0\to X$.

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* Department of Basic Course, Shanghai Institute of Building Material, Shanghai, China.

2. Partial Resolution

Definition 1.2. *Let X, Y be normal varieties. If there exists a birational morphism $f: Y \mapsto X$, we say f is a partial resolution.*

We always suppose X to be a variety with slightly good singularities (terminal or canonical).

Definition 1.3. *X is said to have only terminal (resp. canonical) singularities, if the following two conditions are satisfied:*

(1) *$\exists$ positive integer r, $\ni rK_X$ is a Cartier divisor;*

(2) *$\exists$ a resolution of singularities $f: Y \mapsto X$, $\ni rK_Y = f^*(rK_X) + \Delta$, $\Delta = \sum r_i E_i$ is a Cartier divisor with $r_i > 0$ (resp. $r_i \geqslant 0$) where E_i run hrough all the prime divisors on Y contracted by f.*

Remark. (1) Condition (2) of Definition 1.3 is equivalent to saying $rK_Y = f^*(rK_X) + \Delta$ for every partial resolution $f: Y \mapsto X$. [3] has accounted for it.

(2) X has canonical singularitios $\Rightarrow X$ is Cohen-Macaulay. The case of dimension 3 has been solved by Shephered Barron. High dimensional cases have also been solved.

Definition 1.5. *Suppose X have only canonical singularities. If $f: Y \mapsto X$ is a partial resolution, then $K_Y = f^*K_X + \Delta$, $\Delta \geqslant 0$ is made up of some divisors. Δ is said to be the discrepancy of f.*

We say f is crepant if $\Delta = 0$.

Definition 1.6. *f: $X_1 \dashrightarrow X_2$ is an isomorphism in codimension one, which means that f is a birational map and by deleting subsets of X_1 and X_2 of codimension $\geqslant 2$, f becomes an isomorphism.*

§ 2. Use of Discrepancy

At first, we introduce two results of Miles Reid.

Theorem 1. (Weak Index Theorem) *Let $f: Y \mapsto X$ be a proper birational morphism between normal varieties, of which Y is supposed to be Cohen-Macaulay. Let Δ be a Cartier divisor made up of exceptional prime divisors. Then there exists a component F of Δ such that*

(1) *$C\Delta \leqslant 0$ for all curves $C \subset F$ contracted by f and not lying in a proper closed subvariety of F;*

(2) *the curves $C \subset F$ contracted by f, such that $C\Delta < 0$, fill up a dense subset of F.*

Theorem 2. *Let $P \in X$ be a canonical singularity. Let $f: Y \mapsto X$ be a crepant morphism. Then Y has canonical singularities on $f^{-1}P$. If X has only terminal singularities, then Y has terminal singularities too.*

M. Reid thinks it is probably true that a birational map $f: X \dashrightarrow Y$ between two

3-folds with terminal singularities, such that K_X and K_Y are both nef, is necessarilly an isomorphism in codimension one. He did not give the proof.

I do not want to answer this problem directly. Now I give one sufficient condition in below. Let us see the following simple corollary.

Corollary 1. *Assume that X has only canonical singularities. Let $f: Y\mapsto X$ be a partial resolution. Then f is crepant if and only if Y is Cohen-Macaulay and K_Y is relatively nef (that means $K_Y C\geqslant 0$ for every curve $C\subset Y$ contracted by f).*

Proof Theorem 2 implys that Y has canonical singularities. We know from Remark (2) that Y is Cohen-Macauly. We can write $K_Y=f^*K_X+\Delta$, Δ is Q-Cartier. Because f is crepant, $\Delta=0$, $K_Y=f^*K_X$. For any curve contracted by f, we have $CK_Y=Cf^*K_X=0$. K_Y is certainly relatively nef.

Conversely, Y is Cohen-Macauly, K_Y is relatively nef. We have $K_Y=f^*K_X+\Delta$. K_Y is relatively nef $\Rightarrow \Delta=K_Y-f^*K_X$ is relatively nef too. Theorem 1 implies $\Delta=0$, and hence f is crepant.

Theorem 3. *Let X, Y be normal 3-folds. Let $f: X\dashrightarrow Y$ be a birational map. X has only canonical singularities. Y has only terminal singularities. K_Y is nef. If there exists partial resolutions g, h such that $h=fg$ and h is crepant, then g is also crepant.*

Proof $g: Z\mapsto X$, $h: Z\mapsto Y$, they are both partial resolutions. Because X and Y both have canonical singularities, we have

$$K_Z=g^*K_X+\Delta(g),$$
$$K_Z=h^*K_Y+\Delta(h)=h^*K_Y,$$

$\Delta(h)=0$ because n is crepant.

From Theorem 2, we know that Y has only terminal singularities $\Rightarrow Z$ has only canonical singularities $\Rightarrow Z$ is Cohen-Macaulay. If g is not crepant, that is to say $\Delta(g)\neq 0$, then according to Theorem 1, there exists a component $F\subset\Delta(g)$, such that the curves $C\subset F$ contracted by f and satisfying $C\Delta(g)<0$ fill up a dense subset of F.

$$C\Delta(g)=C(g^*K_X+\Delta(g))=CK_Z=Ch^*K_Y<0.$$

This means that C can not be contracted by h. Because Y has terminal singularities and h is crepant, you can see obviously that h is an isomorphism in codimension one.

Now F is a two dimensional prime divisor, there exists an irreducible curve $\bar{C}\subset F$ and g contracts $\bar{C}$. Let $C'=h(\bar{C})$, C' is an irreducible curve in Y.

We see that C is birationally equivalent to C'. $h: Z\mapsto Y$ is an isomorphism except for a part whose dimension $\leqslant 1$. K_Y is nef and according to projective formula,

$$\bar{C}h^*K_Y=C'K_Y\geqslant 0,$$

which is contradictory to $\bar{C}h^*K_Y<0$. Hence g is crepant.

Corollary 2. *Under the above conditions, if X has terminal singularities, then $f: X \to Y$ is an isomorphism in codimension one.*

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References

[1] Reid, M., Canonical 3-folds, Algebraic Geometry Angers, 1971, edited by Arnaud Beauville.

[2] Reid, M., Minimal models of canonical 3-folds, Advanced studies in pure Mathematics 1, 1983, Algebraic Varieties and Analysis Varieties, 131—180.

[3] Kawamata, Y., Elementary contractions of algebraic 3-folds, *Annals of Mathematics*, **119** (1984), 95—110.

[4] Hartshorne, Robin, Algebraic Geometry, GTM52, Springer-Verlag.

[5] Fulton, W., Introduction to intersection theory in algebraic geometry, Conference board of the Mathematical Sciences, number 54.