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COMPACT HYPERSURFACES IN ELLIPSOID WITH PRESCRIBED MEAN CURVATURE

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Abstract

$S^n(1)$ and $S^1(1)\times S^1(1)$ can be embedded into special ellipsoid with suitable mean curvature function.

Introduction

In 1982, S. T. Yau raised a nonlinear global problem[6].

H is a function in R^{n+1}, can you find a suitable condition on H such that there is a closed hypersurface in R^{n+1} which is homeomorphic to $S^n(1)$ and whose mean curvature is given by H?

Bakelman and Kantor, Treibergs and Wei proved a theorem[1,5].

Similarly, we considered $S^{n+1}(1)$ in place of R^{n+1} and obtained an existence theorem[2].

In § 1, we discuss a compact hypersurface in an $n+1$ dimensional ellipsoid and we have a result which is more general than our theorem in [2].

In § 2, we discuss torus $S^1(1)\times S^1(1)$ which is imbedded into special three-dimensional ellipsoid with suitable mean curvature.

§ 1. Compact Hypersurface

In this section, we consider the following $n+1$ dimensional ellipsoid:

$$M_{n+1}=\left\{(x_1,\ x_2,\ \cdots,\ x_{n+1},\ x_{n+2})\in R^{n+2}\,\middle|\,\frac{x_1^2+x_2^2+\cdots+x_{n+1}^2}{a^2}+\frac{x_{n+2}^2}{b^2}=1\right\},\tag{1.1}$$

where a, b are positive constants. π^* is a hyperplane $x_{n+2}=0$ in R^{n+2}. $\pi^*\cap M_{n+1}$ is an n-dimensional sphere.

$$S^n(a)=\{(ax_1,\ ax_2,\ \cdots,\ ax_{n+1},\ 0)\in R^{n+2}\,|\,x_1^2+x_2^2+\cdots+x_{n+1}^2=1\}.\tag{1.2}$$

In the hyperplane π^*, we consider a variety of n-dimensional spheres

$$S^n(\lambda a)=\{(\lambda ax_1,\ \lambda ax_2,\ \cdots,\ \lambda ax_{n+1},\ 0)\in R^{n+2}\,|\,x_1^2+x_2^2+\cdots+x_{n+1}^2=1\},\tag{1.3}$$

where λ is a positive constant.

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For the fixed point $A(0, 0, \cdots, 0, b)$ in M_{n+1} and a point $B(\lambda a x_1, \lambda a x_2, \cdots, \lambda a x_{n+1}, 0)$ in $S^n(\lambda a)$, by a calculation. We can see that there is a unique point

$$C\left(\frac{2\lambda a x_1}{\lambda^2+1}, \frac{2\lambda a x_2}{\lambda^2+1}, \cdots, \frac{2\lambda a x_{n+1}}{\lambda^2+1}, \frac{b(\lambda^2-1)}{\lambda^2+1}\right)$$

in M_{n+1}, such that three points A, B, C lie in a straight line.

So we can set up a compact hypersurface M_n in M_{n+1}, M_n is homeomorphic to n-dimensional sphere. Its position vector field

$$\boldsymbol{X}=\frac{2ae^u}{1+e^{2u}}\boldsymbol{r}+\frac{b(e^{2u}-1)}{1+e^{2u}}N, \tag{1.4}$$

where $N=(0, 0, \cdots, 0, 1)$, $\boldsymbol{r}=(x_1, x_2, \cdots, x_{n+1}, 0)$ is the position vector field of n-dimensional unit sphere $S^n(1)$ in the hyperplane π^*. u is a differentiable function on the $S^n(1)$. $M_n=X(S^n(1))$.

At a fixed point $p\in S^n(1)\subset\pi^*$, we can choose a local orthogonal frame field $\{e_1, e_2, \cdots, e_{n+1}\}$ such that, restricted to $S^n(1)$, $e_1, e_2, \cdots, e_n$ are tangent to $S^n(1)$ and e_{n+1} is the radical direction of $S^n(1)$. $\boldsymbol{r}_i(p)=e_i(p)$, $\langle \boldsymbol{r}_i(p), \boldsymbol{r}_j(p)\rangle=\delta_{ij}$. In this section, $1\leqslant i, j, k, \cdots, \leqslant n$, where subscript $i(1\leqslant i\leqslant n)$ expresses the covariant derivative along e_i. $\langle\cdot, \cdot\rangle$ denotes the inner product of two vectors in R^{n+2}. $\{\omega_1, \omega_2, \cdots, \omega_{n+1}\}$ is a dual frame field.

At point $X(p)$ of M_n, we can see

$$\boldsymbol{X}_i=\frac{2e^u}{1+e^{2u}}\left[\frac{a(1-e^{2u})}{1+e^{2u}}u_i\boldsymbol{r}+a\boldsymbol{r}_i+\frac{2be^u}{1+e^{2u}}u_iN\right]. \tag{1.5}$$

The metric tensor of M_n is

$$g_{ij}=\langle \boldsymbol{X}_i, \boldsymbol{X}_j\rangle=\frac{4a^2e^{2u}}{(1+e^{2u})^2}\left[\delta_{ij}+\frac{1}{A(x, u)}u_iu_j\right], \tag{1.6}$$

where

$$A(x, u)=\frac{a^2(1+e^{2u})^2}{a^2(1-e^{2u})^2+4b^2e^{2u}}.$$

The inverse metric tensor of M_n satisfies

$$g^{ij}=\frac{(1+e^{2u})^2}{4a^2e^{2u}}\left[\delta_{ij}-\frac{u_iu_j}{|\nabla u|^2+A(x, u)}\right], \tag{1.7}$$

where $|\nabla u|^2=\sum_i u_i^2$. $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_n$ is a basis of the tangent space at point $X(p)$ of M_n in M_{n+1}. We know a normal vector $\boldsymbol{n}^*$ at point $X(p)$ of M_{n+1} in R^{n+2} is

$$\boldsymbol{n}^*=\frac{2e^u}{a(1+e^{2u})}\boldsymbol{r}+\frac{e^{2u}-1}{b(1+e^{2u})}N. \tag{1.8}$$

At point $X(p)$, the unit normal vector $\boldsymbol{n}$ of M_n in M_{n+1} satisfies

$$\langle \boldsymbol{n}, \boldsymbol{X}_i\rangle=0, \ \langle \boldsymbol{n}, \boldsymbol{n}^*\rangle=0. \tag{1.9}$$

By a calculation, we can see

$$\boldsymbol{n}=\frac{1}{\sqrt{A(x, u)+|\nabla u|^2}}\left[\frac{1-e^{2u}}{1+e^{2u}}A(x, u)\boldsymbol{r}-\sum_i u_i\boldsymbol{r}_i+\frac{2be^u}{a(1+e^{2u})}A(x, u)N\right]. \tag{1.10}$$

Using Weingarten formula, we see that the mean curvature H along n in M_{n+1} satisfies

$$nH(X) = -\sum_{ij} g^{ij}\langle dn(e_i), X_j\rangle. \tag{1.11}$$

By a straight calculation, we obtain

$$dn = -\frac{1}{2}\, d\ln[A(x, u) + |\nabla u|^2]n + \frac{1}{\sqrt{A(x, u) + |\nabla u|^2}}\Big\{\Big[\Big[1 - \frac{4e^{2u}}{(1+e^{2u})^2}A(x, u)\Big]du + \frac{1-e^{2u}}{1+e^{2u}}\, dA(x, u)\Big] r + \frac{1-e^{2u}}{1+e^{2u}}A(x, u)dr - \sum_{ij} u_{ij}\omega_j r_i + \frac{2be^u}{a(1+e^{2u})}\Big[dA(x, u) + \frac{1-e^{2u}}{1+e^{2u}} du A(x, u)\Big]N\Big\}. \tag{1.12}$$

Then

$$\langle dn(e_i), X_j\rangle = \frac{2ae^u}{1+e^{2u}}\frac{1}{\sqrt{A(x, u)+|\nabla u|^2}}\Big\{-u_{ij} + \frac{1-e^{2u}}{1+e^{2u}}A(x, u)\delta_{ij} + \frac{1-e^{2u}}{1+e^{2u}}u_i u_j\Big[1 + \frac{4\left(1-\frac{b^2}{a^2}\right)e^{2u}}{(1+e^{2u})^2}A(x, u)\Big]\Big\}. \tag{1.13}$$

Inserting (1.7) and (1.13) into (1.11) we can see

$$[A(x, u) + |\nabla u|^2]\sum_j u_{jj} - \sum_{ij} u_i u_j u_{ij} = B(x, u, \nabla u), \tag{1.14}$$

where

$$B(x, u, \nabla u) = \frac{n(1-e^{2u})}{1+e^{2u}}A(x, u)[A(x, u) + |\nabla u|^2] + \frac{4\left(1-\frac{b^2}{a^2}\right)e^{2u}(1-e^{2u})}{(1+e^{2u})^3} \times A^2(x, u)|\nabla u|^2 + \frac{2nae^u}{1+e^{2u}}[A(x, u) + |\nabla u|^2]^{3/2}H\left(\frac{2ae^u}{1+e^{2u}}r + \frac{b(e^{2u}-1)}{1+e^{2u}}\right)N. \tag{1.15}$$

Set

$$D = \{X = a\sqrt{1-s^2}\, r + bsN \in M_{n+1} \mid -1 < S < 1\}. \tag{1.16}$$

In other words, $D = M_{n+1} - \{(0, 0, \cdots, 0, b) \cup (0, 0, \cdots, 0, -b)\}$.

In this section, $H(X) \in C^{\bar{k},\alpha}(D)$, where $\bar{k} \geq 1$ is an integer, $0 < \alpha < 1$. (1.15) is a quasilinear elliptic equation.

Elliptic equation theory tells us: If there is a constant C_1, which is independent of $t \in (0, 1]$, such that $\|u\|_{C^1(S^n(1))} \leq C_1$ for all u satisfying

$$[A(x, u) + |\nabla u|^2]\sum_j u_{jj} - \sum_{ij} u_i u_j u_{ij} = tB(x, u, \nabla u) + (1-t)u, \tag{1.17}$$

where $t \in (0, 1]$, then Equation (1.15) has a differentiable solution.

Now we shall set up a theorem.

Theorem 1.

$$M_{n+1} = \left\{(x_1, x_2, \cdots, x_{n+1}, x_{n+2}) \in R^{n+2} \,\middle|\, \frac{x_1^2 + x_2^2 + \cdots + x_{n+1}^2}{a^2} + \frac{x_{n+2}^2}{b^2} = 1\right\}$$

is an $n+1$ dimensoonal ellipsoid in R^{n+2}.

$$D = \{a\sqrt{1-s^2}\, r + bsN \in M_{n+1} \mid -1 < S < 1\},$$

is an open subset in M_{n+1}, where $N=(0, 0, \cdots, 0, 1)$ and r is the position vector field of the n-dimensional unit sphere $S^n(1)$ in the hyperplane $X_{n+2}=0$. Given function $H(X)\in C^{k,\alpha}(D)$, $k\geqslant 1$, an integer, $0<\alpha<1$. Suppose $H(X)$ satisfies the following two conditions: (1) There are two constants r_1, r_2, $r_2\geqslant 1\geqslant r_1>0$. When $s>\frac{r_2^2-1}{r_2^2+1}$,

$$H(a\sqrt{1-s^2}\,r+bsN)>H_L\,[a\sqrt{1-s^2}\,r+bsN],$$

where

$$H_L[a\sqrt{1-s^2}\,r+bsN]=\frac{s}{\sqrt{(1-s^2)\,[a^2s^2+b^2(1-s^2)]}},$$

$H_L\,[a\sqrt{1-s^2}\,r+bsN]$ is a constant mean curvature of the latitude hypersurface

$$X=a\sqrt{1-s^2}\,r+bsN \text{ in } M_{n+1}\ (s \text{ is a constant}).$$

When $s<\frac{r_1^2-1}{r_1^2+1}$,

$$H[a\sqrt{1-s^2}\,r+bsN]<H_L\,[a\sqrt{1-s^2}\,r+bsN].$$

(2) Set

$$G=\left\{a\sqrt{1-s^2}\,r+bsN\in D\;\middle|\;\frac{r_1^2-1}{r_1^2+1}\leqslant s\leqslant\frac{r_2^2-1}{r_2^2+1}\right\},$$

$$\frac{\partial H[a\sqrt{1-s^2}\,r+bsN]}{\partial s}\geqslant\frac{s}{1-s^2}\,H[a\sqrt{1-s^2}\,r+bsN],$$

in G. Then, there is a compact hypersurface in M_{n+1}, which is homeomorphic to S^n (1) and whose mean curvature is given by $H(X)$.

Proof In the following, u denotes a differentiable solution of (1.17) for some $t\in(0, 1]$,

$$u(\bar{x}_1)=\max_{x\in S^n(1)} u(x). \tag{1.18}$$

The left side of (1.17) at point $\bar{x}_1$ is less than zero or equal to zero. If $u(\bar{x}_1)>\ln r_2\geqslant 0$, the right side of (1.17) at point $\bar{x}_1$,

$$\begin{aligned}&[tB(x, u, \nabla u)+(1-t)u]\,(\bar{x}_1)\\&=\left\{\frac{2tnae^{2u}}{1+e^{2u}}A^{\frac{3}{2}}(x, u)\left[\frac{(1-e^{2u})}{2ae^u}A^{\frac{1}{2}}(x, u)+H\left(\frac{2ae^u}{1+e^{2u}}\,r\right.\right.\right.\\&\left.\left.\left.+\frac{b(e^{2u}-1)}{1+e^{2u}}\,N\right)\right]+(1-t)u\right\}(\bar{x}_1)\end{aligned} \tag{1.19}$$

Obviously

$$\frac{e^{2u}-1}{1+e^{2u}}(\bar{x}_1)>\frac{r_2^2-1}{r_2^2+1}, \tag{1.20}$$

$$\begin{aligned}&\left[H\left(\frac{2ae^u}{1+e^{2u}}\,r+\frac{b(e^{2u}-1)}{1+e^{2u}}\,N\right)\right](\bar{x}_1)>H_L\left[\left(\frac{2ae^u}{1+e^{2u}}\,r+\frac{b(e^{2u}-1)}{1+e^{2u}}\,N\right)(\bar{x}_1)\right]\\&=\left[\frac{(e^{2u}-1)}{2ae^u}A^{\frac{1}{2}}(x, u)\right](\bar{x}_1).\end{aligned} \tag{1.21}$$

The right side of (1.19) $>(1-t)u(\bar{x}_1)\geqslant 0$. It is impossible.

Then, we have

$$u(\bar{x}_1)\leqslant\ln r_2. \tag{1.22}$$

Similarly, we can see

$$u(\bar{x}_2)=\min_{x\in S^n(1)} u(x)\geqslant \ln r_1 \tag{1.23}$$

Secondly, we estimate $|\nabla u|^2$. Set function[3,4]

$$\Psi=e^{2u}\ln(|\nabla u|^2+1). \tag{1.24}$$

By successive differentiation

$$\Psi_i=2e^{2u}\Big[u_i\ln(|\nabla u|^2+1)+\frac{\sum_k u_k u_{ki}}{|\nabla u|^2+1}\Big], \tag{1.25}$$

$$\begin{aligned}\Psi_{ij}=&4e^{2u}u_j\Big[u_i\ln(|\nabla u|^2+1)+\frac{\sum_k u_k u_{ki}}{|\nabla u|^2+1}\Big]\\&+2e^{2u}\Big[u_{ij}\ln(|\nabla u|^2+1)+\frac{2u_i}{|\nabla u|^2+1}\sum_k u_k u_{kj}\\&+\frac{1}{|\nabla u|^2+1}\big(\sum_k u_{kj}u_{ki}+\sum_k u_k u_{kij}\big)-\frac{2}{(|\nabla u|^2+1)^2}\sum_{kl}u_k u_{ki}u_l u_{lj}\Big].\end{aligned} \tag{1.26}$$

Suppose that at the point $x_0\in S^n(1)$, function Ψ attains its maximum

$$\Psi_i(x_0)=0,\quad \big(\sum_{ij}\{[A(x,u)+|\nabla u|^2]\delta_{ij}-u_iu_j\}\Psi_{ij}\big)(x_0)\leqslant 0. \tag{1.27}$$

By (1.25), at point x_0, we can see

$$\sum_k u_k u_{ki}=-u_i(|\nabla u|^2+1)\ln(|\nabla u|^2+1). \tag{1.28}$$

All of the following items are valued at point x_0. From (1.28), we have

$$\begin{aligned}&\sum_{ik}u_k u_{ki}u_i=-|\nabla u|^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1),\\&\sum_{ijk}u_k u_{ki}u_j u_{ji}=|\nabla u|^2(|\nabla u|^2+1)^2[\ln(|\nabla u|^2+1)]^2.\end{aligned} \tag{1.29}$$

Using (1.26), (1.29) and the second formula of (1.27), we obtain

$$\begin{aligned}0\geqslant&[A(x,u)+|\nabla u|^2]\ln(|\nabla u|^2+1)\sum_j u_{jj}+\frac{A(x,u)+|\nabla u|^2}{|\nabla u|^2+1}\sum_{ij}u_{ij}^2\\&+\frac{1}{|\nabla u|^2+1}\{[A(x,u)+|\nabla u|^2]\sum_{ij}u_iu_{ijj}-\sum_{ijk}u_iu_ju_ku_{ijk}\}\\&-2A(x,u)|\nabla u|^2\ln(|\nabla u|^2+1)[1+\ln(|\nabla u|^2+1)].\end{aligned} \tag{1.30}$$

Without loss of generality, we assume $|\nabla u|(x_0)>1$. For $|\nabla u|(x_0)\leqslant 1$, at once, we have $\forall x\in S^n(1)$

$$|\nabla u|^2(x)\leqslant e^{c_2}-1, \tag{1.31}$$

where

$$c_2=\frac{r_2^2}{r_2^2}\ln 2.$$

We differentiate (1.17) and use (1.29). We can see

$$\begin{aligned}&[A(x,u)+|\nabla v|^2]\sum_{ij}u_{jji}u_i-\sum_{ijk}u_iu_ju_ku_{ijk}=t\sum_k[B(x,u,\nabla u)]_ku_k+(1-t)|\nabla u|^2\\&-[\sum_k(A(x,u))_ku_k-2|\nabla u|^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)]\sum_j u_{jj}+2|\nabla u|^2(|\nabla u|^2+1)^2\\&\times[\ln(|\nabla u|^2+1)]^2.\end{aligned} \tag{1.32}$$

Using the Ricci formula of $S^n(1)$, we can see

$$\sum_{ij} u_i u_{ijj} = \sum_{ij} u_i u_{jji} + (n-1)|\nabla u|^2. \tag{1.33}$$

Using (1.17) and (1.29) we obtain

$$\sum_j u_{jj} = \frac{1}{A(x, u)+|\nabla u|^2}\{tB(x, u, \nabla u)+(1-t)u-|\nabla u|^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)\}. \tag{1.34}$$

By the Cauchy inequality, we can see

$$\sum_{ij} u_{ij}^2 = \frac{1}{|\nabla u|^2}\sum_k u_k^2 \sum_{ij} u_{ij}^2 \geqslant \frac{1}{|\nabla u|^2}\sum_i\Big(\sum_k u_k u_{ki}\Big)^2 = (|\nabla u|+1)^2[\ln(|\nabla u|^2+1)]^2. \tag{1.35}$$

By use of (1.32), (1.33), (1.34) and (1.35), (1.30) can be reduced to the following

$$0 \geqslant A(x, u)|\nabla u|^2\ln(|\nabla u|^2+1)+3tB(x, u, \nabla u) + \frac{t}{(|\nabla u|^2+1)\ln(|\nabla u|^2+1)}\sum_k[B(x, u, \nabla u)]_k u_k - C_3|\nabla u|^2, \tag{1.36}$$

where C_3 is a positive constant and it is independent of t, so is a constant $C_\alpha(\alpha \geqslant 4)$ next.

From the expression of $B(x, u, \nabla u)$, we obtain

$$3B(x, u, \nabla u) \geqslant \frac{6nae^u}{1+e^{2u}}[A(x, u)+|\nabla u|^2]^{3/2}H - C_4|\nabla u|^2, \tag{1.37}$$

where

$$H = H\Big(\frac{2ae^u}{1+e^{2u}}\boldsymbol{r} + \frac{b(e^{2u}-1)}{1+e^{2u}}N\Big),$$

$$\begin{aligned}&\sum_q [B(x, u, \nabla u)]_k u_k \\ &\geqslant \frac{2nae^u}{1+e^{2u}}\Big\{\frac{1-e^{2u}}{1+e^{2u}}|\nabla u|^2[A(x, u)+|\nabla u|^2]^{3/2}H \\ &\quad -3|\nabla u|^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)[A(x, u)+|\nabla u|^2]^{1/2}H + [A(x, u) \\ &\quad +|\nabla u|^2]^{3/2}\sum_k H_k u_k\Big\} - C_5|\nabla u|^2(\nabla u|^2+1)\ln(|\nabla u|^2+1).\end{aligned} \tag{1.38}$$

Inserting (1.37) and (1.38) into (1.36) we have

$$\begin{aligned}0 \geqslant\ & A(x, u)\ |\nabla u|^2\ln(|\nabla u|^2+1) + \frac{2\,tnae^u}{1+e^{2u}}\Big\{\frac{1-e^{2u}}{1+e^{2u}}|\nabla u|^2H + \sum_n H_k u_k\Big\} \\ &\times \frac{[A(x, u)+|\nabla u|^2]^{3/2}}{(|\nabla u|^2+1)\ln(|\nabla u|^2+1)} - C_6|\nabla u|^2.\end{aligned} \tag{1.39}$$

By a straight calculation, we can see

$$\sum_k H_k u_k \geqslant \frac{4e^{2u}}{(1+e^{2u})^2}|\nabla u|^2 \frac{\partial H(a\sqrt{1-s^2}\,\boldsymbol{r}+bsN)}{\partial s}\Big|_{s=(e^{2u}-1)/(1+e^{2u})} - C_7|\nabla u| \tag{1.40}$$

$$\begin{aligned}&\frac{1-e^{2u}}{1+e^{2u}}|\nabla u|^2H + \sum_k H_k u_k \\ &\geqslant \frac{4e^{2u}}{(1+e^{2u})^2}|\nabla u|^2\Big[\frac{-s}{1-s^2}H(a\sqrt{1-t^2}\,\boldsymbol{r}+bsN) \\ &\quad + \frac{\partial H(a\sqrt{1-s^2}\,\boldsymbol{r}+bsN)}{\partial s}\Big]\Big|_{s=(e^{2u}-1)/(1+e^{2u})} - C_7|\nabla u| \geqslant -C_7|\nabla u|,\end{aligned} \tag{1.41}$$

where we use the condition (2) in the theorem.

Inserting (1.40) and (1.41) into (1.39), at once, we can see

$$|\nabla u|^2(x_0)\leqslant C_8. \tag{1.42}$$

Then, $\forall x\in S^n(1)$

$$|\nabla u|^2(x)\leqslant e^{C_9}-1, \tag{1.43}$$

where

$$C_9=\frac{r_2^2}{r_1^2}\ln(C_8+1).$$

The Theorem 1 is proved.

§ 2. Compact Surface

In this section, M_3 is a special three-dimensional ellipsoid in R^4. Its position vector field is

$$Y=\left\{(x_1, x_2, x_3, x_4)\in R^4\left|\frac{x_1^2+x_2^2}{a^2}+\frac{x_3^2+x_4^2}{b^2}=1\right.\right\}, \tag{2.1}$$

where a, b are positive constants.

We consider campact surface $M_2\subset M_3$. Its position vector field is

$$X=\frac{ae^u}{\sqrt{1+e^{2u}}}\boldsymbol{r}+\frac{b}{\sqrt{1+e^{2u}}}\rho, \tag{2.2}$$

where $R^4=R^2\times R^2$, $\boldsymbol{r}$ is the position vector field of a circle $S^1(1)\subset R^2\subset R^4$ (the first R^2). ρ is the position vector field of another unit circle $S^1(1)\subset R^2\subset R^4$ (the second R^2). $\langle\cdot, \cdot\rangle$ denotes the inner product of two vectors in R^4. At any fixed point, $\langle\boldsymbol{r}, \rho\rangle=0$. u is a differentiable function on $S^1(1)\times S^1(1)$, M_2 is homeomorphic to $S^1(1)\times S^1(1)$.

We adapt a local orthogonal frame $\{e_1, e_2, e_3, e_4\}$ in R^4, such that restricted to the first R^2, e_1 is tangent to the first $S^1(1)$, e_3 is the radical direction on the first $S^1(1)$. And, restricted to the second R^2, e_2 is tangent to the second $S^1(1)$, e_4 is the radical direction on the second $S^1(1)$. $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is dual frame. The indices i, j, $k\cdots$ run over 1, 2. $M_2=X(S^1(1)\times S^1(1))$. At a fixed point $x(p)$ of M_2, wher $p\in S^1(1)\times S^1(1)$, we choose the normal coordinate. Then

$$\boldsymbol{r}_1(p)=e_1(p),\ \boldsymbol{r}_2(p)=0,$$
$$\rho_1(p)=0,\ \rho_2(p)=e_2(p), \tag{2.3}$$

where the subscripts 1, 2 express the covariant derivatives with respect to directions e_1, e_2, respectively. At point $x(p)$

$$X_1=e^u(1+e^{2u})^{-3/2}[au_1\boldsymbol{r}+a(1+e^{2u})\boldsymbol{r}_1-be^uu_1\rho], \tag{2.4}$$

$$X_2=(1+e^{2u})^{-3/2}[e^uu_2(a\boldsymbol{r}-e^ub\rho)+b(1+e^{2u})\rho_2]. \tag{2.5}$$

The metric tensor of M_2 is

$$g_{11}=\langle X_1, X_1\rangle=e^{2u}(1+e^{2u})^{-3}[a^2(1+e^{2u})^2+(a^2+b^2e^{2u})u_1^2], \tag{2.6}$$

$$g_{12}=\langle X_1, X_2\rangle=e^{2u}(1+e^{2u})^{-3}(a^2+b^2e^{2u})u_1u_2, \tag{2.7}$$

$$g_{22}=\langle X_2, X_2\rangle=(1+e^{2u})^{-3}[b^2(1+e^{2u})^2+(a^2+b^2e^{2u})e^{2u}u_2^2]. \tag{2.8}$$

At point $x(p)$, the inverse metric tensor of M_2 satisfies

$$g^{11}=e^{-2u}\{a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)\}^{-1} \times[b^2(1+e^{2u})^2+(a^2+b^2e^{2u})e^{2u}u_2^2](1+e^{2u}), \tag{2.9}$$

$$g^{12}=-\{a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)\}^{-1} \times(1+e^{2u})(a^2+b^2e^{2u})u_1u_2, \tag{2.10}$$

$$g^{22}=\{a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)\}^{-1} \times(1+e^{2u})[a^2(1+e^{2u})^2+(a^2+b^2e^{2u})u_1^2]. \tag{2.11}$$

The normal vector N of ellipsoid M_3 in R^4 at point $x(p)$ may be written as

$$N=\frac{e^u}{a\sqrt{1+e^{2u}}}\,\boldsymbol{r}+\frac{1}{b\sqrt{1+e^{2u}}}\,\boldsymbol{\rho}. \tag{2.12}$$

The unit normal vector $\boldsymbol{n}$ of M_2 in ellipsoid M_3 satisfies

$$\langle \boldsymbol{n}, X_1\rangle=0,\ \langle \boldsymbol{n}, X_2\rangle=0,\ \langle \boldsymbol{n}, N\rangle=0. \tag{2.13}$$

By a calculation, we can see

$$\boldsymbol{n}=\{a^2b^2(1+e^{2u})^2(a^2+b^2e^{2u})+(a^2+b^2e^{2u})^2(b^2u_1^2+a^2e^{2u}u_2^2)\}^{-1/2} \cdot\{ab(1+e^{2u})(-a\boldsymbol{r}+be^u\boldsymbol{\rho})+(a^2+b^2e^{2u})(bu_1\boldsymbol{r}_1+ae^uu_2\boldsymbol{\rho}_2)\}. \tag{2.14}$$

We calculate the mean curvature H of M_2 at point $x(p)$ in ellipsoid M_3. By definition and the Weingarten formula

$$2H=-\sum_{ij}g^{ij}\langle d\boldsymbol{n}(e_i), X_j\rangle. \tag{2.15}$$

Using a straight calculation, we can see at point $x(p)$

$$\begin{aligned} d\boldsymbol{n}=&-\frac{1}{2}\,d\ln[a^2b^2(1+e^{2u})^2(a^2+b^2e^{2u})+(a^2+b^2e^{2u})^2(b^2u_1^2+a^2e^{2u}u_2^2)]\boldsymbol{n}\\ &+\{a^2b^2(1+e^{2u})^2(a^2+b^2e^{2u})+(a^2+b^2e^{2u})^2(b^2u_1^2+a^2e^{2u}u_2^2)\}^{-1/2}\\ &\times\{-[2a^2be^{2u}du+(a^2+b^2e^{2u})bu_1\omega_1]\boldsymbol{r}+[ab^2(1+3e^{2u})e^udu\\ &-a(a^2+b^2e^{2u})e^uu_2\omega_2]\boldsymbol{\rho}+[-a^2b(1+e^{2u})\omega_1+2b^3e^{2u}duu_1\\ &+(a^2+b^2e^{2u})bdu_1]\boldsymbol{r}_1+[ab^2(1+e^{2u})e^u\omega_2+a(a^2+3b^2e^{2u})e^uu_2du\\ &+a(a^2+b^2e^{2u})e^udu_2]\boldsymbol{\rho}_2\}. \end{aligned} \tag{2.16}$$

So, we have $\langle d\boldsymbol{n}(e_2), X_1\rangle=\langle d\boldsymbol{n}(e_1), X_2\rangle$, and

$$\begin{aligned}\langle d\boldsymbol{n}(e_1), X_1\rangle=&[a^2b^2(1+e^{2u})^2(a^2+b^2e^{2u})\\ &+(a^2+b^2e^{2u})^2(b^2u_1^2+a^2e^{2u}u_2^2)]^{-1/2}abe^u(1+e^{2u})^{-3/2}\\ &[(1+e^{2u})(a^2+b^2e^{2u})u_{11}-a^2(1+e^{2u})^2\\ &-(b^2e^{4u}+2a^2e^{2u}+a^2)u_1^2],\end{aligned} \tag{2.17}$$

$$\begin{aligned}\langle d\boldsymbol{n}(e_1), X_2\rangle=&[a^2b^2(1+e^{2u})^2(a^2+b^2e^{2u})+(a^2+b^2e^{2u})^2(b^2u_1^2+a^2e^{2u}u_2^2)]^{-1/2}\\ &\times abe^u(1+e^{2u})^{-3/2}[(1+e^{2u})(a^2+b^2e^{2u})u_{12}+e^{2u}(b^2-a^2)u_1u_2],\end{aligned} \tag{2.18}$$

$$\begin{aligned}\langle d\boldsymbol{n}(e_2), X_2\rangle=&[a^2b^2(1+e^{2u})^2(a^2+b^2e^{2u})+(a^2+b^2e^{2u})^2(b^2u_1^2+a^2e^{2u}u_2^2)]^{-1/2}\\ &\times abe^u(1+e^{2u})^{-3/2}[(1+e^{2u})(a^2+b^2e^{2u})u_{22}+b^2(1+e^{2u})^2\end{aligned}$$

$$+(a^2+2b^2e^{2u}+b^2e^{4u})u_2^2]. \tag{2.19}$$

Substituting (2.9), (2.10), (2.11), (2.17), (2.18) and (2.19) into (2.15), we can see

$$[b^2e^{-2u}(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}+u_2^2]u_{11}-2u_1u_2u_{12}+[a^2(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}+u_1^2]u_{22}$$
$$=B(x,\ u,\ \nabla u), \tag{2.20}$$

where

$$B(x,\ u,\ \nabla u)=a^2b^2(1+e^{2u})^3(a^2+b^2e^{2u})^{-2}(e^{-2u}-1)+a^2b^2e^{-2u}(1+e^{2u})^2$$
$$\times(a^2+b^2e^{2u})^{-2}u_1^2-a^2b^2e^{2u}(1+e^{2u})^2(a^2+b^2e^{2u})^{-2}u_2^2-2(ab)^{-1}e^{-u}$$
$$\times(1+e^{2u})^{-1/2}(a^2+b^2e^{2u})^{-3/2}\{a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})$$
$$\times(b^2u_1^2+a^2e^{2u}u_2^2)\}^{3/2}H\left(\frac{ae^u}{\sqrt{1+e^{2u}}}\boldsymbol{r}+\frac{b}{\sqrt{1+e^{2u}}}\boldsymbol{\rho}\right). \tag{2.21}$$

Because p is an arbitrary point on $S^1(1)\times S^1(1)$, (2.20) is valid on $S^1(1)\times S^1(1)$. Equation (2.20) is a quasilinear elliptic equation on $S^1(1)\times S^1(1)$, where H is a given function.

Let

$$N=\{\boldsymbol{X}=as\boldsymbol{r}+b\sqrt{1-s^2}\,\boldsymbol{\rho}\,|\,0<s<1\}. \tag{2.22}$$

$\boldsymbol{N}$ is an open subset in ellipsoid M_3. In this section, we give function $H(\boldsymbol{X})\in C^{\bar{k},\alpha}(N)$, where $\bar{k}\geq 1$ is an integer, $0<\alpha<1$.

If we can prove that there exists a constant C_1 such that

$$\|u\|_{C^1(S^1(1)\times S^1(1))}\leq C_1 \text{ for all } u \text{ satisfying } [b^2e^{-2u}(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}+u_2^2]u_{11}$$
$$-2u_1u_2u_{12}+[a^2(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}+u_1^2]u_{22}$$
$$=tB(x,\ u,\ \nabla u)+(1-t)u, \tag{2.23}$$

where $0<t\leq 1$, C_1 is independent of t, then there is a differentiable solution u of (2.20).

Now, we shall set up an existence theorem.

Theorem 2. *$R^4=R^2\times R^2$, $\boldsymbol{r}$ is the position vector field of the unit circle $S^1(1)$ in the first $R^2\subset R^4$, $\boldsymbol{\rho}$ is the position vector field of the second circle $S^1(1)$ in the second $R^2\subset R^4$. $N=\{as\boldsymbol{r}+b\sqrt{1-s^2}\,\boldsymbol{\rho}\,|\,0<s<1\}$ is an open subset of three-dimensional ellipsoid M_3. Suppose that function $H(\boldsymbol{X})\in C^{\bar{k},\alpha}(N)$ ($\bar{k}\geq 1$, an integer, $0<\alpha<1$) satisfies the following two conditions:*

(1) *There are two constants s_1, s_2 where*

$$1>s_1\geq\sqrt{\frac{1}{2}}\geq s_2>0,$$

when $1>s>s_1$,

$$H(as\boldsymbol{r}+b\sqrt{1-s^2}\boldsymbol{\rho})<\frac{1-2s^2}{2s\sqrt{1-s^2}}\,\frac{1}{\sqrt{a^2+(b^2-a^2)s^2}}$$

when $s_2>s>0$,

$$H(as\boldsymbol{r}+b\sqrt{1-s^2}\boldsymbol{\rho})>\frac{1-2s^2}{2s\sqrt{1-s^2}}\,\frac{1}{\sqrt{a^2+(b^2-a^2)s^2}}.$$

(2) *Set* $P=\{X=as\boldsymbol{r}+b\sqrt{1-s^2}\,\boldsymbol{\rho}\,|\,s_2\leqslant s\leqslant s_1\}$

$$-\frac{\partial H(as\boldsymbol{r}+b\sqrt{1-s^2}\,\boldsymbol{\rho})}{\partial s}\geqslant\frac{2}{s(1-s^2)}\,|H(as\boldsymbol{r}+b\sqrt{1-s^2}\,\boldsymbol{\rho})|\ \text{in } P$$

Then, there is a compact surface in the ellipsoid M_3 which is homeomorphic to torus $S^1(1)\times S^1(1)$ and whose mean curvature is given by $H(X)$.

Proof. In the following, u is a solution of Equation (2.23) for some $t\in(0,1]$,

$$u(\bar{x}_1)=\max_{x\in S^1(1)\times S^1(1)}u(x). \tag{2.24}$$

The left-hand side of (2.23) at point $\bar{x}_1$ is less than zero or equal to zero.

$$\begin{aligned}&[tB(x,u,\nabla u)+(1-t)u](\bar{x}_1)\\&=\Big\{ta^2b^2(1+e^{2u})^3(a^2+b^2e^{2u})^{-2}\Big[e^{-2u}-1-2e^{-u}\\&\quad\times(1+e^{2u})^{-1/2}(a^2+b^2e^{2u})^{1/2}H\Big(\frac{ae^u}{\sqrt{1+e^{2u}}}\boldsymbol{r}+\frac{b}{\sqrt{1+e^{2u}}}\boldsymbol{\rho}\Big)\Big]+(1-t)u\Big\}(\bar{x}_1).\end{aligned} \tag{2.25}$$

If $u(\bar{x}_1)>\ln\frac{s_1}{\sqrt{1-s_1^2}}\geqslant 0$, by condition (1) in Theorem 2, we can see

$$[tB(x,u,\nabla u)+(1-t)u](\bar{x}_1)>(1-t)u(\bar{x}_1)\geqslant 0. \tag{2.26}$$

It is a contradiction. Then

$$u(\bar{x}_1)\leqslant\ln\frac{s_1}{\sqrt{1-s_1^2}}.$$

Similarly, we have

$$u(\bar{x}_2)=\min_{x\in S^1(1)\times S^1(1)}u(x)\geqslant\ln\frac{s_2}{\sqrt{1-s_2^2}}. \tag{2.27}$$

We now shall estimate $|\nabla u|^2=u_1^2+u_2^2$. Using § 1, we introduce function

$$\psi=e^{2u}\ln(|\nabla u|^2+1). \tag{2.28}$$

Suppose at point $x_0\in S^1(1)\times S^1(1)$, ψ attains its maximum. All of the following items are valued at point x_0. From [3], we have

$$\begin{aligned}u_1u_{11}+u_2u_{21}&=-u_1(|\nabla u|^2+1)\ln(|\nabla u|^2+1),\\u_1u_{12}+u_2u_{22}&=-u_2(|\nabla u|^2+1)\ln(|\nabla u|^2+1).\end{aligned} \tag{2.29}$$

Without loss of generality, we assume $|\nabla u|(x_0)\geqslant 1$. Because inequality (2.39) in [3] is valid on any compact two dimensional manifold, we have

$$\begin{aligned}0\geqslant{}&3tB(x,u,\nabla u)+(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}(b^2e^{-2u}u_1^2+a^2u_2^2)\ln(|\nabla u|^2+1)\\&+t[(|\nabla u|^2+1)\ln(|\nabla u|^2+1)]^{-1}\sum_k[B(x,u,\nabla u)]_k u_k-C_2|\nabla u|^2.\end{aligned} \tag{2.30}$$

Using (2.21) and (2.29), we can see

$$\begin{aligned}3B(x,u,\nabla u)\geqslant{}&-6(ab)^{-1}e^{-u}(1+e^{2u})^{-1/2}(a^2+b^2e^{2u})^{-3/2}\{a^2b^2(1+e^{2u})^2\\&+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)\}^{3/2}H-C_3|\nabla u|^2,\end{aligned} \tag{2.31}$$

where C_2 in (2.30) and C_3 in (2.31) are positive constants and are independent of $t\in(0,1]$, so is $C_\alpha(\alpha>3)$ next.

$$H=H\left(\frac{ae^u}{\sqrt{1+e^{2u}}}\boldsymbol{r}+\frac{b}{\sqrt{1+e^{2u}}}\rho.\right)$$

By a calculation, we can see

$$\sum_k[B(x,u,\nabla u)]_k u_k\geqslant 2(ab)^{-1}\Big\{e^{-u}(a^2+b^2e^{2u})^{-3/2}(1+e^{2u})^{-1/2}[a^2b^2(1+e^{2u})^2$$
$$+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)]^{3/2}|\nabla u|^2H[(1+e^{2u})^{-1}(1+2e^{2u})$$
$$+3b^2e^{2u}(a^2+b^2e^{2u})^{-1}]-3e^{-u}(1+e^{2u})^{-1/2}(a^2+b^2e^{2u})^{-3/2}$$
$$\times[a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)]^{1/2}$$
$$\times[b^2e^{2u}(b^2u_1^2+a^2e^{2u}u_2^2)|\nabla u|^2+(a^2+b^2e^{2u})(a^2e^{2u}u_2^2|\nabla u|^2-b^2u_1^2$$
$$\times(|\nabla u|^2+1)\ln(|\nabla u|^2+1)-a^2e^{2u}u_2^2(|\nabla u|^2+1)\ln(|\nabla u|^2+1)]H$$
$$-(1+e^{2u})^{-2}(a^2+b^2e^{2u})^{-3/2}[a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})$$
$$(b^2u_1^2+a^2e^{2u}u_2^2)]^{3/2}\frac{\partial H(as\boldsymbol{r}+b\sqrt{1-s^2}\rho)}{\partial s}\Big|_{s=e^u/\sqrt{1+e^{2u}}}|\nabla u|^2\Big\}$$
$$-C_4|\nabla u|^4\ln(|\nabla u|^2+1). \tag{2.32}$$

Inserting (2.31) and (2.32) into (2.30), we can see

$$0\geqslant(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}(b^2e^{-2u}u_1^2+a^2u_2^2)\ln(|\nabla u|^2+1)+2t(ab)^{-1}[\ln(|\nabla u|^2+1)]^{-1}e^{-u}$$
$$\times(1+e^{2u})^{-1/2}(a^2+b^2e^{2u})^{-3/2}[a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})$$
$$\times(b^2u_1^2+a^2e^{2u}u_2^2)]^{3/2}\Big\{[(1+e^{2u})^{-1}(1+2e^{2u})+3b^2e^{2u}(a^2+b^2e^{2u})^{-1}]H$$
$$-3[a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)]^{-1}[b^2e^{2u}(b^2u_1^2+a^2e^{2u}u_2^2)$$
$$+a^2e^{2u}(a^2+b^2e^{2u})u_2^2]H-e^u(1+e^{2u})^{-3/2}\frac{\partial H(as\boldsymbol{r}+b\sqrt{1-s^2}\rho)}{\partial s}\Big|_{s=e^u/\sqrt{1+e^{2u}}}\Big\}$$
$$-C_5|\nabla u|^2, \tag{2.33}$$

when $H\geqslant 0$,

$$-[a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)]^{-1}[b^2e^{2u}(b^2u_1^2+a^2e^{2u}u_2^2)$$
$$+a^2e^{2u}(a^2+b^2e^{2u})u_2^2]H$$
$$\geqslant-(a^2+b^2e^{2u})^{-1}(a^2+2b^2e^{2u})H, \tag{2.34}$$

when $H<0$,

$$-[a^2b^2(1+e^{2u})^2+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)]^{-1}[b^2e^{2u}(b^2u_1^2+a^2e^{2u}u_2^2)$$
$$+a^2e^{2u}(a^2+b^2u^{2u})u_2^2]H$$
$$>-(a^2+b^2e^{2u})^{-1}b^2e^{2u}H-C_6|\nabla u|^{-2}. \tag{2.35}$$

Inserting (2.34) and (2.35) into (2.33), and using condition (2) in Theorem 2, we can see

$$0\geqslant(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}(b^2e^{-2u}u_1^2+a^2u_2^2)\ln(|\nabla u|^2+1)-2t[abe^u(1+e^{2u})^{1/2}$$
$$\times(a^2+b^2e^{2u})^{3/2}\ln(|\nabla u|^2+1)]^{-1}[a^2b^2(1+e^{2u})^2$$
$$+(a^2+b^2e^{2u})(b^2u_1^2+a^2e^{2u}u_2^2)]^{3/2}\Big\{\Big[1+\max\Big(\frac{1}{1+e^{2u}},\ \frac{e^{2u}}{1+e^{2u}}\Big)\Big]|H|$$
$$+e^u(1+e^{2u})^{-3/2}\frac{\partial H(as\boldsymbol{r}+b\sqrt{1-s^2}\rho)}{\partial s}\Big|_{s=e^u/\sqrt{1+e^{2u}}}\Big\}-C_7|\nabla u|^2$$
$$\geqslant(1+e^{2u})^2(a^2+b^2e^{2u})^{-1}(b^2e^{-2u}u_1^2+a^2u_2^2)\ln(|\nabla u|^2+1)-C_7|\nabla u|^2. \tag{2.36}$$

So, we obtain

$$|\nabla u|^2(x_0) \leqslant C_8. \tag{2.37}$$

Then, $\forall x \in S^1(1) \times S^1(1)$,

$$|\nabla u|^2(x) \leqslant e^{C_9} - 1, \tag{2.38}$$

where

$$C_9 = \frac{s_1^2(1-s_2^2)}{s_2^2(1-s_1^2)} \ln(C_8+1).$$

The theorem is proved.

Remark 1. When $a=b=1$, $M_3=S^3(1)$. The above theorem is our last result[4].

Remark 2. There are many functions satisfying the conditions of Theorem 2.

For example, set

$$H(as\boldsymbol{r}+b\sqrt{1-s^2}\boldsymbol{\rho}) = \frac{1-2s^2}{2s^{1+k}(1-s^2)^{1/2(k+1)}} + f(\boldsymbol{r}, \boldsymbol{\rho}), \tag{2.39}$$

where $f(\boldsymbol{r}, \boldsymbol{\rho})$ is a smooth function on $S^1(1) \times S^1(1)$, and constant

$$k>1, \ |f(\boldsymbol{r}, \boldsymbol{\rho})| \leqslant \left(1-\frac{1}{k}\right)2^{k-1}.$$

Then, $\forall 0<s<1$, we can see

$$-s(1-s^2)\frac{\partial}{\partial s} H(as\boldsymbol{r}+b\sqrt{1-s^2}\boldsymbol{\rho}) - 2|H(as\boldsymbol{r}+b\sqrt{1-s^2}\boldsymbol{\rho})| \geqslant \frac{1}{2^{1+k}(1-s^2)^{1/2(k+1)}}$$
$$[k(1-2s^2)^2 - 2|1-2s^2|+1] - 2|f(\boldsymbol{r}, \boldsymbol{\rho})|$$
$$\geqslant \frac{1}{2s^{1+k}(1-s^2)^{1/2(k+1)}}\left(1-\frac{1}{k}\right) - 2|f(\boldsymbol{r}, \boldsymbol{\rho})|$$
$$\geqslant \left(1-\frac{1}{k}\right)2^k - 2|f(\boldsymbol{r}, \boldsymbol{\rho})| \geqslant 0.$$

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