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CHARACTERIZATION OF THE UPPER SUBDERIVATIVE AND ITS CONSEQUENCES IN NONSMOOTH ANALYSIS

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Abstract

It is proved that the upper subderivative of a lower semicontinuous function on a Banach space is upper semicontinuous for the first variable as $x'\to_f x$, i. e., $x'-x$, $f(x')-f(x)$. By taking account of the work of Treiman. it is further shown that the upper subderivative of a l. s. c. function is the upper limit of the contingent directtional derivatives around the concorned point. This new characterization of the upper subberivative allows simple derivations and natural extension of many results in nonsmooth analysis.

§ 1. Introduction

One of the objectives of this paper is to propose a characterization of the upper subderivative $f^+(x, y)$ of a l. s. c. function f on a Banach space in terms of the contingent directional derivatives $f^\#(x, y)$ of f, that is,

$$f^\uparrow(x, y)=\limsup_{x'\to_f x} f^\#(x', y), \tag{1}$$

and we shall use this formula to generalize in a very simple way some important results in nonsmooth analysis.

We shall develop our result from a very geometric point of view by using the recent characterization of the Clarke's tangent cone due to Treiman [15].

Before we establish the announced characterization (1), Let us take a few words to review some related results in the carly days of nonsmooth analysis.

We could date back to Clarke's original generalized notion of differentiability for the class of locally Lipschitzian functions on R^n. The starting point of his definition was Rademacher's theorem, which asserts that these functions are differentiable almost everywhere. This fact permits him to define the generalized gradient of a locally Lipschitz function f: $R^n\to R$ at a point $a\in R^n$ as a compact convex set in the following way:

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$$\partial f(a):=\mathrm{co}\{\lim \nabla f(x_i)\,|\,x_i(\in \mathrm{dom}\,\nabla f)\to a\} \tag{2}$$

and he characterized the support function to this set., i. e.,

$$f^\circ(a, v)=\limsup_{x(\in \mathrm{dom}\nabla f)\to a}\langle \nabla f(x), v\rangle \tag{3}$$

as

$$f^\circ(a, v)=\limsup_{\substack{x\to a\\ t\to 0^+}}\frac{f(x+tv)-f(x)}{t} \tag{4}$$

which is now known as the Clarke generalized directional derivative of f at a in the direction $v\in R^n$.

When

$$f'(x, v)=\lim_{t\to 0+}\frac{f(x+tv)-f(x)}{t}$$

exists for all $x, v\in R^n$, from (3) we obtain the characterization

$$f^\circ(a, v)=\limsup_{x\to a} f'(x, v). \tag{5}$$

Several authors extended the Clarke generalized directional derivetive in the similar spirit as (4) to real functions defined on a general topological vector space. Meanwhile, fimilar characterizations of (5) were proposed.

Recently, Correa in his Ph. D thesis [3] obtained characterization of the type as (5), which was shown to be valid for a l. s. c. function on a general topological vector space. Namely, he proved that $f^\circ(x, v)$ is the upper semicontinuous regularization of any Dini directional derivatives $Df(\cdot, v)$ of the function f, that is

$$f^\circ(a, v)=\limsup_{x\to a} Df(x, v), \tag{6}$$

where $Df(x, v)$ denotes one of the following four Dini directional derivatives:

$$D^+f(a, v)=\limsup_{t\to 0+}\frac{f(a+tv)-f(a)}{t} \tag{7}$$

$$D_+f(a, v)=\liminf_{t\to 0+}\frac{f(a+tv)-f(a)}{t}, \tag{8}$$

$$D^-f(a, v)=\limsup_{t\to 0-}\frac{f(a+tv)-f(a)}{t}, \tag{9}$$

$$D_-f(a, v)=\liminf_{t\to 0-}\frac{f(a+tv)-f(a)}{t}. \tag{10}$$

The main tool he used is the following Dini mean value theorem [8]: For a l. s. c. f: $X\to \overline{R}$ and for all $x, y\in X$, there exists $a\in [x, y)$ such that

$$f(y)-f(x)\leqslant D_+f(a, y-x). \tag{11}$$

When X is finite-dimensional, Rockafellar [8] and Ioffe [5] have already proposed the followiug result

$$f^\uparrow(x, y)=\limsup_{x'\to_f x}\inf_{y'\to y} f^\#(x', y'), \tag{12}$$

where

$$f^{\uparrow}(x,\ y)=\limsup_{\substack{x'\to_f x\\ t\to 0+}}\inf_{y'\to y}\frac{f(x'+ty')-f(x')}{t}, \tag{13}$$

is known as the upper subderivative of f at x in the direction y, which is introduced by Rockafellar [7] and

$$f^{\#}(x,\ y)=\liminf_{\substack{y'\to y\\ t\to 0+}}\frac{f(x+ty')-f(x)}{t} \tag{14}$$

is called the contingent directional derivative of f at x in the direction y [7] or the lower Hadamard derivative.

In fact, formula (12) is an analytic consequence of (indeed. equivalent to) the following importat geometric relation between the Clarke tangent cone [1] and the contingent cones [1,2]:

$$T_C(x)=\liminf_{x'\to_C x}K_C(x'). \tag{15}$$

In what follows, we shall discuss the relationships between various generalized directional derivatives and their corresponding geometric counterpart, tangent cones from the point of view of the epigarph. We shall use Treimans new characterization of the Clarke tangent cone [15] and prove that for a l. s. c. function on a Banach space the following characterization holds:

$$f^{\uparrow}(x,\ y)=\limsup_{x_i\to_f x}f^{\#}(x',\ y). \tag{16}$$

We devote the rest of the paper to developing some extensions of important results in nonsmooth analysis.

§ 2. Tangent Cones and Their Properties

Assume that X is a Banach space, C is a closed set of X, and $x\in C$, Denote by $N(x)$ the set of all neighborhoods of x. As we known. the following tangent cones (to C at x) are frequently used in nonsmooth analysis.

Definition 2. 1. (a) *Clarke tangent cone*[1,2]

$$T_C(x)=\{y\mid\forall N(y)\in\mathcal{N}(y),\ \exists N(x)\in\mathcal{N}(x),\ \lambda>0.\ \forall x'\in N(x)\cap C,\ t\in(0,\ \lambda),\ \exists y'\in N(y)\ \text{such that}\ x'+ty'\in C\}. \tag{17}$$

(b) *(Bouligand) Contingent tangent cone*

$$K_C(x)=\{y\mid\forall N(y)\in\mathcal{N}(y),\ \lambda>0,\ \exists t\in(0,\ \lambda).\ y'\in N(y)\ \text{such that}\ x+ty'\in C\}. \tag{18}$$

(c) $A_C(x)=\{y\mid\forall\lambda>0,\ \exists t\in(0,\ \lambda)\ \text{such that}\ x+ty\in C\}$.

(d) $E_C(x)=\{y\mid\exists\lambda>0,\ \forall t\in(0,\ \lambda)\ \text{such that}\ x+ty\notin C\}$.

(e)

$$T_C^{*}(x)=\{y\mid\forall N(y)\in\mathcal{N}(x),\ \exists N(x)\in\mathcal{N}(x),\ \forall\lambda>0,\ \forall x'\in N(x)\cap C,\ \exists t\in(0,\ \lambda),\ y'\in N(y)\ \text{such that}\ x'+ty'\in C\}. \tag{19}$$

(f)

$$H_C(x)=\{y\mid \exists N(x)\in\mathcal{N}(x),\ \lambda>0 \text{ such that } \forall x'\in N(x)\cap C,\ t\in(0,\lambda),\ x'+ty\in C\}. \tag{20}$$

(g)

$$H_C^*(x)=\{y\mid \exists N(x)\in\mathcal{N}(x),\ \forall\lambda>0,\ \forall x'\in N(x)\cap C.\ \exists t\in(0,\lambda) \text{ such that } x'+ty\in C\}. \tag{21}$$

Since X is a Banach space, according to Ioffe [4], we have an interpretation of 'the limit of sequences' to these tangent cones as follows:

$$T_C(x)=T_{\forall\forall\exists}(C,x),\ K_C(x)=T_{\cdot\exists\exists}(C,x),$$
$$A_C(x)=T_{\cdot\exists\cdot}(C,x),\ E_C(x)=T_{\cdot\forall\cdot}(C,x),$$
$$T_C^*(x)=T_{\forall\exists\exists}(C,x),\ H_C(x)=T_{\forall\forall\cdot}(C,x),$$
$$H_C^*(x)=T_{\forall\exists\cdot}(C,x),$$

where

$$T_{QRS}(C,x)=\{v\mid Qx_n\to_C x.\ Rt_n\to 0+\ Sv_n\to v \text{ such that } x_n+t_nv_n\in C\}, \tag{23}$$
$$Q=\{\forall,\exists,.\},\ R=\{\forall,\exists\},\ S=\{\forall,\exists,.\}$$

and "." means $x_n\equiv x$ or $v_n\equiv v$.

As an obvious consequence of the definition. we get the following

Proposition 2. 1.

$$H_C(x)\subset E_C(x)\subset A_C(x)\subset K_C(x),$$
$$H_C(x)\subset T_C(x)\subset T_C^*(x)\subset K_C(x), \tag{24}$$
$$H_C(x)\subset H_C^*(x).$$

According to Treiman [25] and Ward [28] we have further

Proposition 2. 2. *If X is a Banach space, C is a closed subset of X and $x\in C$, then*

$$T_C(x)=T_C^*(x).\ H_C(x)=H_C^*(x). \tag{25}$$

These two facts are crucial to our argument in the next section.

By the way, the following important facts about the tangent cones characterize the finite-dimensional space and reflexive space.

Proposition 2.3.[16] *A Banach space X is finite-dimensional if and only if for any closed $C\subset X$ and any $x\in C$,*

$$\liminf_{x'\to_C x} K_C(x')=T_C(x). \tag{26}$$

A Banach space X is reflexive if and only if for any closed $C\subset X$ and any $x\in C$,

$$\liminf_{x'\to_C x} WK_C(x')=T_C(x), \tag{27}$$

where $WK_C(x)$ is the weak contingent cone, which is the set of all weak limits of sequences $\{t_n^{-1}[c_n-x]\}$ with $s_n\to 0+$ and $c_n\in C$.

In general, for any closed set C in a Banach space X and $x\in C$, we have only[15]

$$\liminf_{x'\to_C x} K_C(x') \subset T(x). \tag{28}$$

If C is epi-Lipschitzian or even compactly epi-Lipschitzian at x[16], we get

$$\liminf_{x'\to_C x} K_C(x') = T_C(x). \tag{29}$$

For more details about these properties and applications, we are referred to [9, 15, 18].

In the next seton, we shall translate the above geometric facts, by using the epigraph of a function, into various genoralized directional derivatives and obtain our new echaracterization by proving a very useful property about the upper semicontinuity of the upper subderivative.

§ 3. Generalized Directional Derivatives and Their Properties

Let $f: X\to\bar{R}$ be l. s. c. Denote the epigraph of f by

$$\operatorname{epi} f := \{(x, r)\in X\times R \mid f(x)\leqslant r\} \tag{30}$$

which is closed.

Let $x\in X$ be a point at which f is finite and let $y\in X$. The upper subderivative of f at x in the direction y is defined by [7,17]

$$f^{\uparrow}(x, y) := \inf\{r\in R \mid (y, r)\in_{\operatorname{epi} f}(x. f(x))\}, \tag{31}$$

which is equivalent to

$$\operatorname{epi} f^{\uparrow}(x, y) = T_{\operatorname{epi} f}(x, f(x)). \tag{32}$$

Since $T_{\operatorname{epi} f}(x, f(x))$ is a closed convex cone, $f^{+}(x, y)$ is a l. s. c. positive homogeneous convex function. We have the following definition.

Definition 3. 1. *Let $f: X\to\bar{R}$ and $x\in X$ such that $f(x)$ is finite. The subgradient of of f at x is the set*

$$\partial f(x) := \{x^* \in X^* \mid \langle y, x^*\rangle \leqslant f^{+}(x, y), \ \forall y\in X\}. \tag{33}$$

Rockafellar [7] provided the following direct characterization of $f^{+}(x, y)$:

$$f^{+}(x, y) = \limsup_{\substack{x'\to_f x\\ t\to 0+}} \inf_{y'\to y} \frac{f(x'+ty')-f(x')}{t} \tag{34}$$

$$= \sup_{N(y)\in\mathcal{N}(y)} \inf_{\substack{N(x)\in\mathcal{N}(x)\\ \lambda>0}} \sup_{\substack{x'\in N(x)\\ t\in(0,\lambda)\\ f(x')\leqslant f(x)+\lambda}} \inf_{y'\in N(y)} \frac{f(x'+ty')-f(x')}{t}.$$

In the same way, for the tangent ones $K_C(x)$, $A_C(x)$, $E_C(x)$, $H_C(x)$, $H_C^*(x)$ and $T_C^*(x)$, we can define

$$f^{K}(x, y) = f^{\#}(x, y) := \inf\{r \mid (y, r)\in K_{\operatorname{epi} f}(x, f(x))\}, \tag{35}$$

$$f^{E}(x, y) = f_{\forall\exists}(x, y) := \inf\{r \mid (y, r)\in E_{\operatorname{epi} f}(x, f(x))\}, \tag{36}$$

$$f^{A}(x, y) = f_{\cdot\exists\cdot}(x, y) := \inf\{r \mid (y, r)\in A_{\operatorname{epi} f}(x, f(x))\}, \tag{37}$$

$$f^{H}(x, y) = f^{\circ}(x, y) := \inf\{r \mid (y, r)\in H_{\operatorname{epi} f}(x, f(x))\}, \tag{38}$$

$$f^{H*}(x, y)=f_{\forall\exists^*}(x, y):=\inf\{r \mid (y, r)\in H^*_{\mathrm{epi}f}(x, f(x))\}, \tag{39}$$

$$f^{T*}(x, y)=f_{\forall\exists\exists}(x, y):=\inf\{r \mid (y, r)\in T^*_{\mathrm{epi}f}(x, f(x))\}. \tag{40}$$

From the definitions, we have exactly

$$\mathrm{epi}\, f^{\#}(x, y)=K_{\mathrm{epi}f}(x, f(x)), \tag{41}$$

$$\mathrm{epi}\, f^{E}(x, y)=\mathrm{cl}E_{\mathrm{epi}f}(x, f(x)), \tag{42}$$

$$\mathrm{epi}\, f^{A}(x, y)=\mathrm{cl}A_{\mathrm{epi}f}(x, f(x)), \tag{43}$$

$$\mathrm{epi}\, f^{H}(x, y)=\mathrm{cl}H_{\mathrm{epi}f}(x, f(x)), \tag{44}$$

$$\mathrm{epi}\, f^{H*}(x, y)=\mathrm{cl}H^*_{\mathrm{epi}f}(x, f(x)), \tag{45}$$

$$\mathrm{epi}\, f^{T*}(x, y)=T^*_{\mathrm{epi}f}(x, f(x)). \tag{46}$$

There are also direct characterizations of these generalized directional derivatives, i. e.,

$$f^{\#}(x, y)=\liminf_{\substack{t\to 0+\\ y'\to y}}\frac{f(x+ty')-f(x)}{t}, \tag{47}$$

$$f^{E}(x, y)=\limsup_{t\to 0+}\frac{f(x+ty)-f(x)}{t}, \tag{48}$$

$$f^{A}(x, y)=\liminf_{t\to 0+}\frac{f(x+ty)-f(x)}{t}, \tag{49}$$

$$f^{H}(x, y)=f^{\circ}(x, y)=\limsup_{\substack{x'\to_f x\\ t\to 0+}}\frac{f(x'+ty)-f(x')}{t}, \tag{50}$$

$$f^{H*}(x, y)=\limsup_{x'\to_f x}\inf_{t\to 0+}\frac{f(x'+ty)-f(x')}{t}, \tag{51}$$

$$f^{T*}(x, y)=\limsup_{x'\to_f x}\inf_{\substack{t\to 0+\\ y'\to y}}\frac{f(x'+ty')-f(x')}{t}. \tag{52}$$

Recall the relation (25) of Proposition 2.2 and (50), (51), (52), and then we have the following lemma.

Lemma 3. 1. *Let $f: X\to\bar{R}$ be l. s. c.. Then*

$$f^{\uparrow}(x, y)=f^{T*}(x, y)=\limsup_{\substack{x'\to_f x\\ t\to 0+}}\inf_{y'\to y}\frac{f(x'+ty')-f(x')}{t}, \tag{53}$$

$$=\limsup_{x'\to_f x}\inf_{\substack{t\to 0+\\ y'\to y}}\frac{f(x'+ty')-f(x')}{t}$$

and

$$f^{\circ}(x, y)=f^{H}(x, y)=f^{H*}(x, y)=\limsup_{\substack{x'\to_f x\\ t\to 0+}}\frac{f(x'+ty)-f(x')}{t}$$

$$=\limsup_{x'\to_f x}\inf_{t\to 0+}\frac{f(x'+ty)-f(x')}{t}. \tag{54}$$

The above mixed limit operations are associated with the epi-limit convergence of l. s. c. functions, which allows us to transpose our results about sequences of closed sets and of set-valued maps to sequences of l. s. c. functions. We recall the followingsRockafellar and Wets's result[10].

Lemma 3. 2. *Suppose that $\{f_v, v\in(N, \mathscr{X})\}$ is a filtered family of l. s. c.*

functions on X. *Then*

$$\mathrm{epi}(\mathrm{epi}\,f - \limsup_{v\in N} f_v) = \liminf_{v\in N} \mathrm{epi}\, f_v \tag{55}$$

and

$$\mathrm{epi}(\mathrm{epi}\,f - \liminf_{v\in N} f_v) = \limsup_{v\in N} \mathrm{epi}\, f_v \tag{56}$$

where

$$(\mathrm{epi} - \limsup_{v\in N} f_v)(x) := \sup_{V\in N(x)} \limsup_{v\in N} \inf_{x'\in V} f_v(x')$$

and

$$(\mathrm{epi} - \liminf_{v\in N} f_v)(x) := \sup_{V\in N(x)} \liminf_{v\in N} \inf_{x'\in V} f_v(x').$$

In view of (28), (31), (32), (35) and of the above lemma we get the following proposition:

Proposition 3. 1. *Let* $f: X\to\bar{R}$ *be l. s. c. and* $x\in X$ *such that* $f(x)$ *is finite. Then*

$$\limsup_{x'\to_f x} \inf_{y'\to y} f^{\#}(x', y') \geqslant f^{\uparrow}(x, y). \tag{57}$$

If either X *is finite-dimensional or* f *is directionally Lipschitz at* x[9]*, we have further*

$$\limsup_{x'\to_f x} \inf_{y'\to y} f^{\#}(x', y') = f^{\uparrow}(x, y). \tag{58}$$

We shall strengthen the above results after a number of lemmas are proved.

Lemma 3. 3. *Let* $f: X\to\bar{R}$ *be l. s. c.. Then*

$$\limsup_{x'\to_f x} f^{\uparrow}(x', y) \leqslant f^{\uparrow}(x, y), \tag{59}$$

i. e., $f^{\uparrow}(\cdot, y)$ *is upper semicontinuous for the first variable as* $x'\to_f x$.

Proof From the very definition, we have only to prove that for all $\varepsilon>0$, there exists

$$V_f(x) = \{x'\in X \mid \|x'-x\|<\delta,\ |f(x')-f(x)|<\delta\} \tag{60}$$

such that

$$\forall x'\in V_f(x).\ f^{\uparrow}(x', y) < f^{\uparrow}(x, y) + \varepsilon.$$

Set

$$g(x'', t, y') = \frac{f(x''+ty') - f(x'')}{t},$$

Since

$$f^{\uparrow}(x, y) = \limsup_{\substack{x''\to_f x\\ t\to 0+}} \inf_{y'\to y} g(x'', t, y') = \sup_{N(y)\in \mathcal{N}(y)} \inf_{V_f(x)\in N_f(x)} \sup_{\substack{x''\in V_f(x)\\ t\in(0,\lambda)}} \inf_{y'\in N(y)} g(x'', t, y')$$

where $N_f(x)$ is the set of neighborhoods of the same type as (60), which defines the topology induced by the convergence $x'\to_f x$, we see that for all $N(y)\in\mathcal{N}(y)$.

$$\inf_{\substack{V_f(x)\in\mathcal{N}_f(x)\\ \lambda>0}} \sup_{\substack{x''\in V_f(x)\\ t\in(0,\lambda)}} \inf_{y'\in N(y)} g(x'', t, y') < f^{\uparrow}(x, y) + \varepsilon$$

and then there exists $V_f(x)\in\mathcal{N}_f(x)$ and $\lambda>0$ such that

$$\sup_{\substack{x''\in V_f(x)\\ t\in(0,\lambda)}} \inf_{y'\in N(y)} g(x'', t, y') < f^{\uparrow}(x,y) + \varepsilon.$$

For any $x'\in V_f(x)\in\mathcal{N}_f(x)$, we can find a $V(x')\in\mathcal{N}_f(x')$ such that

$$V_f(x')\subset V_f(x).$$

Hence,

$$\sup_{\substack{x''\in V_f(x')\\ t\in(0,\lambda)}}\ \inf_{y'\in N(y)} g(x'',t,y')<f^+(x,y)+\varepsilon$$

and

$$\inf_{\substack{V_f(x')\in\mathcal{N}_f(x')\\ \lambda>0}}\ \sup_{\substack{x''\in V_f(x')\\ t\in(0,\lambda)}}\ \inf_{y'\in\mathcal{N}(y)} g(x'',t,y')<f^+(x,y)+\varepsilon.$$

Therefore,

$$\sup_{N(y)\in\mathcal{N}(y)}\ \inf_{\substack{V_f(x')\in\mathcal{N}_f(x')\\ \lambda>0}}\ \sup_{\substack{x''\in V_f(x')\\ t\in(0,\lambda)}}\ \inf_{y'\in N(y)} g(x'',t,y')<f^+(x,y)+\varepsilon.$$

It follows that

$$\forall x'\in V(x),\ f^+(x',y)<f^+(x,y)+\varepsilon,$$

and then

$$\limsup_{x'\to_f x} f^+(x',y)\leqslant f^+(x,y)$$

holds.

As an easy consequence, the following corollary is interesting in its own right.

Corollary 3. 1. *Let $f\colon X\to\bar{R}$ be l. s.c.. Then the set-valued map $\partial f(\cdot)$ from X to X^* has a closed graph for the norm topology of X and the weakstar topology of X^*.*

Proof Let $y_\alpha\in\partial f(x_\alpha)$, $\alpha\in A$, which is a directed set, $x_\alpha\to x$ and $y_\alpha\xrightarrow{w^*}y$. By the definition we have

$$\forall v\in X,\ \alpha\in A,\ \langle y_\alpha, v\rangle\leqslant f^\uparrow(x_\alpha, v)$$

and so,

$$\lim_\alpha\langle y_\alpha, v\rangle\leqslant\limsup_\alpha f^\uparrow(x_\alpha,v)\leqslant f^\uparrow(x,v).$$

Therefore,

$$\forall v\in X,\ \langle y,v\rangle\leqslant f^\uparrow(x,v),$$

and thus $y\in\partial f(x)$. The corollary is proved.

In order to include Correa's characterization about the Clarke directional derivative in terms of Dini directional derivatives, we state the following lemma, which is obvious from the definition.

Lemma 3. 4. *Let $f\colon X\to\bar{R}$ be l. s. c.. Then the Clarke directional derivative $f^\circ(x,y)$ is upper semicontinuous for the first variable as $x'\to_f x$, i. e.,*

$$\limsup_{x'\to_f x} f^\circ(x',y)\leqslant f^\circ(x,y). \tag{61}$$

Now we can give the main results of our paper as follows:

Theorem 3. 1. *Let $f\colon X\to\bar{R}$ be l. s. c.. Then*

$$\limsup_{x'\to_f x} f^\#(x',y)=f^\uparrow(x,y). \tag{62}$$

Proof By Lemma 3.1, we have

$$f^{\uparrow}(x, y)=\limsup_{x'\to_f x}\inf_{\substack{t\to 0+\\ y'\to y}}\frac{f(x+ty')-f(x)}{t}$$
$$\leqslant\limsup_{x'\to_f x}\Big(\liminf_{\substack{t\to 0+\\ y'\to y}}\frac{f(t+ty')-f(x)}{t}\Big)$$
$$=\limsup_{x\to_f x'} f^{\#}(x', y),$$

where we use lim sup inf ⩽lim sup (lim inf).

On the other hand, since

$$\forall x, y\in X, f^{\#}(x, y)\leqslant f^{\uparrow}(x, y),$$

we have

$$\limsup_{x'\to_f x} f^{\#}(x', y)\leqslant\limsup_{x'\to_f x} f^{\uparrow}((x', y)\leqslant f^{\uparrow}(x' y)\ \text{(Lemma 3.3)},$$

and then we concludce the theorem.

Theorem 3.2.[3] *Let $f: X\to\bar{R}$ be l. s... Then*

$$f^{\circ}(x, y)=\limsup_{x'\to_f x} Df(x', y), \tag{63}$$

where $Df(x, y)=f^{A}(x, y)$ or $f^{E}(x, y)$, i. e., the Dini upper or lower directional derivative.

Proof By Lemma 3.1 and the definition we have

$$f^{\circ}(x, y)=\limsup_{x\to_f x}\inf_{t\to 0+}\frac{f(x'+ty)-f(x)}{t}$$
$$\leqslant\limsup_{x'\to_f x}\Big(\liminf_{t\to 0+}\frac{f(x'+ty)-f(x')}{t}$$
$$=\limsup_{x'\to_f x} f^{A}(x', y)\leqslant\limsup_{x'\to_f x} f^{E}(x', y).$$

Since

$$\forall x, y\in X, f^{E}(x, y)\leqslant f^{\circ}(x, y).$$

we have

$$\limsup_{x'\to_f x} f^{E}(x', y)\leqslant\limsup_{x'\to_f x} f^{\circ}(x', y)\leqslant f^{\circ}(x, y)\ \text{(Lemma 3. 4)}.$$

Therefore, (63) follows.

The above two characterizations allow simple derivations and extensions of many important results in nonsmoothanalysis. These topics are discussed in the next section.

§ 4. Applications of the Characterization of the Upper Subderivative

4.1 On the Generalized Gradient of a Nonsmooth Function on a Banach space

The purpose of this part is to generalize the characterization of the generalized gradient of a locally Lipschitz function in terms of gradient limits into a wider class of functions. The breakthrough along this line was due to Shih Shu-Chung[12], which provided necessary and sufficient conditions for the characterization of this type.

Let us come back to the introduction of our paper. As a beginning of nonsmooth analysis, Clarke gave the definition of generalized gradient of a locally Lipschitz function defined on R^n in the form

$$\partial f(x) = \text{co}\{\lim_{y\to x} \nabla f(y) \mid y \in \text{dom} \nabla f\}. \tag{64}$$

He characterized it with its support function

$$f^\circ(x, v) = \limsup_{\substack{x'\to x \\ t\to 0+}} \frac{f(x'+tv) - f(x')}{t} \tag{65}$$

in the following relations:

$$\partial f(x) = \{\xi \in R^n \mid \langle \xi, v\rangle \leqslant f(x, v),\ \forall v \in R^n\} \tag{66}$$

and

$$f^\circ(x, v) = \max\{\langle \xi, v\rangle \mid \xi \in \partial f(x)\}. \tag{67}$$

If we replace R^n by a general topological vector space, X, definition (64) is meaningless because Rademacher's theorem is not valid in general. On the other hand. the definition (65) of the Clarke generalized directional derivative of a locally Lipsbhitz function is still available. Therefore, we can define the generalized gradient by formula (66) and this technique has been proved to be fundamental in the development of nonsmooth analysis.

Rockafellar has further gone by introducing the upper subderivative of a l. s. c. function on a locally convex topological vector space and defined the subgradient in the same spirit as (66). He showed that if f is locally Lipschitz, $\partial f(x)$ coincides with the Clark generalized gradient, which in turn coincides with the ordinary subgradient for a convex function if f is convex [7].

The generalization of(64), when f is defined on an infinite-dimensional space, previously requires a generalization of Rademacher's theorem. In this sense Christensen [19] proved that if X is a separable Banach space and if f is a locally Lipschitz function defined on X, then f is Gateaux-differentiable except on a Harr-null set. Other similar results have been obtained by Aronszajn [20] and Mignot[21]. By taking (66) as the definition of the generalized gradient and using the result of Christensen, Thibault [22] obtained the characterization

$$\partial f(x) = \text{co}\{w^* - \lim \nabla f(y_1) \mid y_1 \in \text{dom} \nabla f\}. \tag{68}$$

In what follows, we shall investingate what occurs to the characterization (68) when f is merely continuous and defined on a Banach space X.

Theorem 4.1. *Let f: $X \to \bar{R}$ be l. s. c., $D \subset X$ is a subset of X where f is Hadamard differentiable, i. e., $\forall x \in D$, $\exists \nabla f(x) \in X^*$. $\forall y \in X$ such that*

$$\lim_{\substack{t\to 0+\\ y'\to y}} \frac{f(x+ty)-f(x)}{t} = \langle \nabla f(x), y\rangle,$$

Then

$$\forall x\in X,\ \partial f(x) = \mathrm{co}\{w^*-\lim_{y_i\to x} \nabla f(y_1)\,|\,y_1\in D\} \tag{69}$$

holds if and only if

$$\forall x, y\in X.\ f^{\uparrow}(x, y) = \limsup_{x'\xrightarrow[D]{}_f x} \langle \nabla f(x'), y\rangle \tag{70}$$

is valid where $x'\xrightarrow[D]{}_f x$ means that $x'(\in D)\to x$, and $f(x')\to f(x)$.

Proof The "only if" part is obvious. Now we prove the "if" part. From the definition and Corollary 3. 1 we have

$$\partial f(x) \supset \mathrm{co}\{w^*-\lim_{y_i\to x} \nabla f\ (y_i)\,|\,y_i\in D\}.$$

For the inclusion "$\subset$", we have only to prove that

$$f^{\uparrow}(x, y) \leqslant \limsup_{x'\to D^x} \langle \nabla f(x'), y\rangle.$$

By assumption,

$$f^{\uparrow}(x, y) = \limsup_{x'\xrightarrow[D]{}_f x} \langle \nabla f(x'), y\rangle \leqslant \limsup_{x'\to D^x} \langle \nabla f(x'). y\rangle,$$

and thus the theorem is proved.

Theorem 4. 2. [12] *Let f: $X\to R$ be continuous and $D\subset X$ is the same as in Theorem 4. 1. Then*

$$\forall x\in X,\ \partial f(x) = \mathrm{co}\{w^*-\lim_{y\to x} \nabla f(y)\,|\,y\in D\}$$

holds if and only if

$$\forall x, h\in X,\ D_+f(x((h) := \liminf_{t\to 0+} \frac{f(x+th)-f(x)}{t} = \limsup_{y\to D^x} \langle \nabla f(y), h\rangle. \tag{71}$$

Proof The necessity is obvious. We prove the sufficiency. If (71) holds, then

$$D_+f(x)(h) \leqslant \limsup_{y\to D^x} f^{\#}(y, h) \leqslant f(x, h).$$

Since

$$\limsup_{x'\to_f x} D_+f(x')(h) = f^{\circ}(x, h)\ \ (\text{Theorem 3. 2}),$$

we have

$$f^{\circ}(x, h) \leqslant f^{\uparrow}(x, h).$$

But we have always

$$f^{\uparrow}(x, h) \leqslant f^{\circ}(x, h).$$

Hence,

$$\forall h\in X,\ f^{\uparrow}(x, h) = f^{\circ}(x, h),$$

and then

$$f^{\uparrow}(x, h) = \limsup_{x'\to D^x} f^{\#}(x', h) = \limsup_{x'\to D^x} \langle \nabla f(x'), h\rangle.$$

By Theorem 4.1 we prove the theorem.

Remarks. We only assume that f is continuous instead of locally Lipschitz property as in [12]. But X is a Banach space here other than a general topological vector space in [12] due to the fact that in this case the Hadamard differentiability can be defined in the form [8]

$$\lim_{\substack{t\to 0+\\ h'\to y}} \frac{f(x+ty')-f(x)}{t} = \langle \nabla f(x), y\rangle.$$

4.2 Characterization of Subdifferentiably Regular Functions

Definition 4. 2. 1.[7] *The function* $f: X\to \bar{R}$ *is said to be subdifferentiably regular at* $x\in \operatorname{dom} f$ *if*

$$\forall y\in X.\ f^{\#}(x, y)=f^{\uparrow}(x, y)$$

and f *is subdifferentiably regular if it is so at every* $x\in X$.

The following theorem gives a characterization of subdifferentiably regular functions in terms of the continuity property of $f^{\#}(x, y)$.

Theorem 4. 3. *Let* $f: X\to R$ *be continous. Then* f *is subdifferentiably regular at* x *if and only if the function* $f^{\#}(\cdot, y)$ *is upper semicontinuous (u. s. c.) for all* $y\in X$.

Proof By Theorem 3. 1 we have

$$f^{\uparrow}(x, y)=\limsup_{x'\to_f x} f^{\#}(x', y)=\limsup_{x'\to x} f^{\#}(x', y).$$

If $f^{\#}(\cdot, y)$ is u. s. c. at x, then we get

$$\forall y\in X.\ f^{\uparrow}(x, y)=f^{\#}(x, y),$$

and so f is subdifferentiably regular at x.

On the other hand, if f is subdifferentiably regular at x, we easily know that $f^{\#}(\cdot, y)$ is u. s. c. at x. Hence, the theorem is proved.

Definition 4. 2. 2.[2] *A loally Lipschitz function* $f: X\to R$ *is said to be subregular at* x *provided* $f'(x, y)$ *exists and* $f^{\circ}(x, y)=f'(x, y)$ *for all* $y\in X$ *and* f *is subregular if it is so at every* $x\in X$.

Theorem 4. 4. *A locally Lipschitz function* $f: X\to R$ *such that* $f'(x, y)$ *exists for all* $x, y\in X$ *is subregular if and only if the function* $f'(\cdot, y)$ *is u. s. c. for all* $y\in X$.

Proof Since

$$f^{\circ}(x, y)=\limsup_{x'\to_f x} Df(x', y)=\limsup_{x'\to x} f'(x', y),$$

tbe conclusion is obvious.

Subregular functions or subdifferentiably regular functions enjoy many good properties in nonsmooth analysis. For more details we are referred to [3].

4.3 Characterization of Tangent Cones

In this last part we shall generalize Treiman's Theorem 3. 3 in [15] by using our characterization of the subderivative. We shall see that our poof seems to be simple and clear.

As we know from the introduction, if C is a closed subset of X, $x\in C$, then we have

$$T_C(x) \supset \liminf_{x' \to_C x} K_C(x'). \tag{72}$$

A counter-example was furnished in [15] to show that (15) is not always true in a Banach space. But the following definition and theorem give an important case where it is valid.

Definition 4. 3. 1. [7] *A set C is called to be epi-Lipschitz at $x \in C$ if for some $y \in X$, there exist $N(y) \in \mathcal{N}(y)$, $N(x) \in \mathcal{N}(x)$ and $\lambda > 0$ such that*

$$\forall x' \in N(x) \cap C,\ x' + [0, \lambda].\ N(y) \subset C. \tag{73}$$

Rockafellar[7] has shown that in a finite-dimensional space the condition (73) is equivalent to C being the epigraph of a Lipschitz function in a neighborhood of x.

Theorem 4. 5.[16] *Let $C \subset X$ be epi-Lipschitz at $x \in C$. Then*

$$T_C(x) = \liminf_{x' \to_C x} K_C(x'). \tag{74}$$

Theorem 4. 6.[15] *Let f be a function on a Banach space X and Lipschitz on a neighborhood of x. Then*

$$T_{\mathrm{epi}f}(x, f(x)) = \liminf_{\substack{(x', \alpha) \to (x, f(x)) \\ \alpha \geqslant f(x')}} K_{\mathrm{epi}f}(x', f(x')) = \liminf_{x' \to x} K_{\mathrm{epi}f}(x', f(x')). \tag{75}$$

Now we show that the above theorems are still valid if f is l. s. c. on X, i. e., if a closed set can be expressed as the epigraph of a l. s. c. function, then the Clarke tangent cone is the regulation of the contingent cones at the neighboring points.

Theorem 4. 7. *Let f: $X \to \overline{R}$ be l. s. c. and $x \in \operatorname{dom} f$. Then*

$$T_{\mathrm{epi}f}(x, f(x)) = \liminf_{x' \to_f x} K_{\mathrm{epi}f}(x', f(x')). \tag{76}$$

Proof. From (72) it is obvious that

$$\liminf_{x' \to_f x} K_{\mathrm{epi}f}(x', f(x')) \subset T_{\mathrm{epi}f}(x, f(x)).$$

Now we show that

$$T_{\mathrm{epi}f}(x, f(x)) \subset \liminf_{x' \to_f x} K_{\mathrm{epi}f}(x', f(x')).$$

It is easy to know from Lemma 3. 2 in the last section that the above inclusion is equivalent to

$$f^{\uparrow}(x, y) \geqslant \limsup_{x' \to_f x} \inf_{y' \to y} f^{\#}(x', y').$$

By Theorem 3. 1 in the last section, we observe that

$$f^{\uparrow}(x, y) = \limsup_{x' \to_f x} f^{\#}(x', y) \geqslant \limsup_{x' \to_f x} \inf_{y' \to y} f^{\#}(x', y'),$$

and then the theorem is proved.

§ 5. Concluding Remarks

We have established a new characterization for the upper subderivative and derived an upper semicontinuity property for the upper subderivative of a l. s. c. function on a Banach space.

The technique we use is very geometric, i. e,. we start from the epigraph of a function and we frequently come to its analytic counterparts. This technique could be traced back to the functional analysis method of Bishop and Phelps [23-25], which has been proved to be powerful in nonsmooth analysis from the Ekeland variational principle [26-27] to the new characterization of the Clarke tagent cone of Treiman [15]. It has been fully explored in recent years[28-30].

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