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GLOBAL SHOCK SOLUTIONS TO A CLASS OF PISTON PROBLEMS FOR THE SYSTEM OF ONE DIMENSIONAL ISENTROPIC FLOW

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Abstract

The autors apply the result obtained in [1] to consider a class of discontinuous piston problems for the system of one dimensional isentropic flow and prove that this problem admits a unique global classical discontinuous solution only containing one shock.

The system of isentropic flow can be written in Lagrangian representation as

$$\begin{cases} \dfrac{\partial \tau}{\partial t} - \dfrac{\partial u}{\partial x} = 0, \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(\tau)}{\partial x} = 0, \end{cases} \tag{1}$$

where τ is the specific volume, u is the velocity and $p=p(\tau)$ is the pressure. For polytropic gases

$$p=p(\tau)=A\tau^{-\gamma}, \tag{2}$$

where $\gamma>1$ is the adiabatic exponent and A is a positive constant.

Introducing the Riemann invariants

$$\begin{cases} r=\dfrac{1}{2}\Big(u-\displaystyle\int_\tau^\infty \sqrt{-p'(\eta)}\,d\eta\Big), \\ s=\dfrac{1}{2}\Big(u+\displaystyle\int_\tau^\infty \sqrt{-p'(\eta)}\,d\eta\Big) \end{cases} \tag{3}$$

as new unknown functions, system (1) can be reduced to be of the form

$$\begin{cases} \dfrac{\partial r}{\partial t} + \lambda(r, s)\dfrac{\partial r}{\partial x} = 0, \\ \dfrac{\partial s}{\partial t} + \mu(r, s)\dfrac{\partial s}{\partial x} = 0 \end{cases} \tag{4}$$

with

$$-\lambda(r, s)=\mu(r, s)=\sqrt{-p'(\tau(s-r))}=a(s-r)^{(\gamma+1)/(\gamma-1)}, \tag{5}$$

where a is a positive constant.

In terms of the Riemann invariants, the Rankine-Hugoniot condition and the entropy condition on a forward shock $x=x_2(t)$ can be written as

$$(r+s)-(r_++s_+)=\sqrt{-(p(\tau(s-r))-p(\tau(s_+-r_+)))(\tau(s-r)-\tau(s-r_+))}, \tag{6}$$

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$$\frac{dx_2}{dt}=\sqrt{-\frac{p(\tau(s-r))-p(\tau(s_+-r_+))}{\tau(s-r)-\tau(s_+-r_+)}}\tag{7}$$

and

$$s-r>s_+-r_+>0,\tag{8}$$

where (r_+, s_+) denotes the state just on the right side of the shock $x=x_2(t)$ and (r, s), as a state on the left side, can be connected with (r_+, s_+) by this forward shock.

It follows from (6)—(8) that on a forward shock $x=x_2(t)$, we have

$$\begin{cases}\dfrac{dx_2}{dt}>\mu(r_+, s_+)>\lambda(r_+, s_+),\\ \mu(r, s)>\dfrac{dx_2}{dt}>\lambda(r, s)\end{cases}\tag{9}$$

and

$$0\leqslant\frac{dr}{ds}<1,\tag{10}$$

moreover, the sign of equality in (10) holds if and only if $(r, s)=(r_+, s_+)$, namely, no discontinuity (cf. [3]).

Rewirte conditron (6)—(7) on a forward shock $x=x_2(t)$ as follows:

$$r=g(r_+, s_+, s),\tag{11}$$

$$\frac{dx_2}{dt}=G(r_+, s_+, r, s).\tag{12}$$

By (10) we have

$$0\leqslant\frac{\partial g}{\partial s}<1\tag{13}$$

and the sign of equality holds if and only if $(r, s)=(r_+, s_+)$.

We turn now to the piston problem. Suppose that a piston originally located at the origin at $t=0$ moves with the speed $u=\varphi(t)$ in a tube, we want to determine the state of the gas on the right side of this piston. In Lagrangian representation this problem asks us to solve the following mixed initial-boundary value problem for system (1) with the conditions:

$$t=0:\ u=u_0^+(x),\ r=r_0^+(x),\ x\geqslant 0\tag{14}$$

$$x=0:\ u=\varphi(t),\ t\geqslant 0.\tag{15}$$

Suppose that

$$\varphi(0)>u_0^+(0),\tag{16}$$

then the motion of the piston must produce a forward shock $x=x_2(t)$ passing through the origin at least for a short time.

Using the Riemann invariants, the preceding problem reduces equivalently to the following mixed initial-boundary value problem for system (4) (together with (5)):

$$t=0:\ r=r_0^+(x),\ s=s_0^+(x),\ x\geqslant 0,\tag{17}$$

$$x=0:\ s=-r+\varphi(t),\ t\geqslant 0,\tag{18}$$

moreover, (16) becomes

$$\varphi(0)>r_{+}+s_{+}, \tag{19}$$

where

$$r_{+}=r_0^{+}(0),\ s_{+}=s_0^{+}(0) \tag{20}$$

and we suppose that

$$s_{+}-r_{+}>0. \tag{21}$$

By (13), it is easy to see that the system

$$\begin{cases} r=g(r_{+},\ s_{+},\ s), \\ s=-r+\varphi(0) \end{cases} \tag{22}$$

possesses a unique solution $(r,\ s)=(r_0,\ s_0)$ which as a left state, can be connected with $(r_{+},\ s_{+})$ by a forward shock, then we have

$$s_0-r_0>s_{+}-r_{+}>0. \tag{23}$$

Remark 1. In the special case

$$r_0^{+}(x)\equiv r_{+},\ s_0^{+}(x)\equiv s_{+},\ \varphi(t)\equiv\varphi(0), \tag{24}$$

the solution of the previous problem is the following forward typical shock (cf. [4])

$$(r,\ s)=\begin{cases} (r_0,\ s_0),\ 0\leqslant x\leqslant Vt, \\ (r_{+},\ s_{+}),\ x\geqslant Vt, \end{cases} \tag{25}$$

where V is the speed of propagation of the shock:

$$V=G(r_{+},\ s_{+},\ r_0,\ s_0) \tag{26}$$

and by (9) it holds that

$$\begin{cases} V>\mu(r_{+},\ s_{+})>\lambda(r_{+},\ s_{+}), \\ \mu(r_0,\ s_0)>V>\lambda(r_0,\ s_0). \end{cases} \tag{27}$$

The piston problem under consideration can be regarded as a perturbation of the simplest problem in Remark 1 and we have

Theorem 1. *Suppose that* $r_0^{+}(x)$, $s_0^{+}(x)$ *and* $\varphi(t)\in C^1$ *and* (19)—(21) *hold. If it holds for suitably small* $\varepsilon>0$ *and* $\eta>0$ *that*

$$|r_0^{+}(x)-r_{+}|,\ |s_0^{+}(x)-s_{+}|\leqslant\varepsilon,\ \forall x\geqslant0, \tag{28}$$

$$|\varphi(t)-\varphi(0)|\leqslant\varepsilon,\ \forall t\geqslant0, \tag{29}$$

$$|r_0^{+\prime}(x)|,\ |s_0^{+\prime}(x)|\leqslant\frac{\eta}{x},\ \forall x>0, \tag{30}$$

$$|\varphi'(t)|\leqslant\frac{\eta}{t},\ \forall t>0, \tag{31}$$

then in a class of piecewise continuous and piecewise smooth functions, the piston problem (4), (17)—(18) *admits a unique globally defined discontinuous solution* $(r(t,\ x),\ s(t,\ x))$ *on the domain*

$$R_0=\{(t,\ x)\,|\,t\geqslant0,\ x\geqslant0\}. \tag{32}$$

This solution contains only one forward shock $x=x_2(t)$ *and satisfies the following estimates:*

on the domain

$$R_+=\{(t,x)\mid t\geqslant 0,\ x\geqslant x_2(t)\},\tag{33}$$

we have

$$|r(t,x)-r_+|,\ |s(t,x)-s_+|\leqslant \varepsilon,\tag{34}$$

$$\left|\frac{\partial r}{\partial x}(t,x)\right|,\ \left|\frac{\partial r}{\partial t}(t,x)\right|,\ \left|\frac{\partial s}{\partial x}(t,x)\right|,\ \left|\frac{\partial s}{\partial t}(t,x)\right|\leqslant\frac{K\eta}{t},\ t>0;\tag{35}$$

on the domain

$$R=\{(t,x)\mid t\geqslant 0,\ 0\leqslant x\leqslant x_2(t)\},\tag{36}$$

we have

$$|r(t,x)-r_0|,\ |s(t,x)-s_0|\geqslant K_0\varepsilon,\tag{37}$$

$$\left|\frac{\partial r}{\partial x}(t,x)\right|,\ \left|\frac{\partial r}{\partial t}(t,x)\right|,\ \left|\frac{\partial s}{\partial x}(t,x)\right|,\ \left|\frac{\partial s}{\partial t}(t,x)\right|\leqslant\frac{K_1\eta}{t},\ t>0;\tag{38}$$

besides, we have

$$|x_2'(t)-V|\leqslant K_2\varepsilon,\ \forall t\geqslant 0\tag{39}$$

$$|x_2''(t)|\leqslant\frac{K_3\eta}{t},\ \forall t>0,\tag{40}$$

where K and $K_i(i=0,1,2,3)$ are positive constants. Moreover, on the whole existence domain

$$s(t,x)-r(t,x)>0,$$

that is, there never exists any vacuum tate.

Proof Let

$$\xi_+=\frac{V+\mu(r_+,s_+)}{2},\tag{42}$$

by (27) we have

$$V>\xi_+>\mu(r_+,s_+),\tag{43}$$

We now need the following Lemma, the proof of which can be found in [5].

Lemma 1. *Suppose that (28) and (30) hold for some suitably small $\varepsilon>0$ and $\eta>0$, then the Cauchy problem for system (4) with the initial data $(r_0^+(x),\ s_0^+(x))$ on $x>0$ admits a unique global C^1 solution $(r_+(t,x),\ s_+(t,x))$ on the domain*

$$\hat{R}_+=\{(t,x)\mid t\geqslant 0,\ x\geqslant \xi_+t\}.\tag{44}$$

Moreover, we have

$$s_+(t,x)-r_+(t,x)>0,\ \forall(t,x)\in\hat{R}_+,\tag{45}$$

$$|r_+(t,x)-r_+|,\ |s_+(t,x)-s_+|\leqslant\varepsilon,\ \forall(t,x)\in\hat{R}_+,\tag{46}$$

$$\left|\frac{\partial r_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial r_+}{\partial t}(t,x)\right|,\ \left|\frac{\partial s_+}{\partial x}(t,x)\right|,\ \left|\frac{\partial s_+}{\partial t}(t,x)\right|\leqslant\frac{K\eta}{t},\quad \forall((t,x)\in\hat{R}_+,\ t>0,\tag{47}$$

where K is a positive constant.

Now we prove Theorem 1.

According to the local existence of discontinuous solutions in a class of piecewise continuous and piecewise smooth functions (cf. [2]), the piston problem (4), (17)—(18) admits a discontinuous solution only containing a forward shock $x=x_2(t)$ passing through the origin at least on a local domain

$$R_0(\delta)=\{(t,\ x)\,|\,0\leqslant t\leqslant\delta,\ x\geqslant 0\} \tag{48}$$

where $\delta>0$ is sufficiently small. By (9) and (43), $x=x_2(t)$ must lie in the interior of the domain $\hat{R}_+$ and then the solution on the right side of $x=x_2(t)$ should be furnished by $(r_+(t, x),\ s_+(t, x))$. Thus, noticing Lemma 1, in order to construct a globally defined discontinuous solution containing only a forward snock, it is only necessary to solve the following typical free boundary problem for system (4):

$$\text{on } x=0,\ s=-r+\varphi(t). \tag{49}$$

on $x=x_2(t)$,

$$r=g(r_+(t,\ x),\ s_+(t,\ x),\ s). \tag{50}$$

$$\frac{dx_2}{dt}=G(r_+(t,\ x),\ s_+(t,\ x),\ r,\ s). \tag{51}$$

Moreover, according to the entropy condition, the solution should be asked to satisfy the following property:

$$s-r>s_+(t,\ x)-r_+(t,\ x)>0 \text{ on } x=x_2(t); \tag{52}$$

and $x=x_2(t)$ should always lie in the interior of the domain $\hat{R}_+$.

In this typical free boundary problem, $x=x_2(t)$ is a free boundary while $x=0$ is a fixed boundary. Since a given boundary can be considered as a special case of free boundaries, all the results in § 3 of [1] are still valid in this case, Thus, using Theorem 3.1 and Remark 3.3 of [1] and noting Lemma 1, it is easy to see that this typical free boundary problem (4), (49)—(51) admits a unique global C^1 solution $(r(t, x),\ s(t, x))$ on the domain R and (37)—(40) hold. Hence by (23) and noting Lemma 1, we can choose $\varepsilon>0$ so small that (52) and (41) hold and, by (43), $x=x_2(t)$ always lies in the interior of the domain $\hat{R}_+$. The proof of Theorem 1 is complete.

References

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