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THE COMPENSATED COMPACTNESS METHOD APPLIED TO QUASILINEAR PARABOLIC EQUATIONS OF HIGHER ORDER WITH DOUBLY STRONG DEGENERATION

YIN JINGXUE (尹景学)*

Abstract

This paper applies the compensated compactnessmethod to the study of a class of quasilinear parbolic equations of higher order with doubly strong degeneration. Some existence and uniqueness results are proved under certain conditions.

§ 1. Introdution

The recently developed compensated compactness method has found its applications to many fields of partial differential equa-tions such as nonlinear hyperbolic systems, degenerate quasilinear parabolic equations (see Tartar [1], Zhao Junning [2]).

In this paper, we apply the compensated compactness method to the study of the first boundary value problem for degenerate quasilinear parabolic equations of the form

$$\frac{\partial u}{\partial t}+(-1)^m D^m A(D^m B(u))=0, \tag{1.0}$$

where $D=\frac{\partial}{\partial x}$,

$$A(s)=\int_0^s a(\sigma)d\sigma,\ B(s)=\int_0^s b(\sigma)d\sigma,$$

with $a(s), b(s)$ being nonnegative and appropriately smooth functions. The case $A(s) = |s|^{M-1}\text{sgn } s$, $B(s)=|s|^{N-1}\text{sgn } s$, $(M, N>1)$, of the equation (1.0) was investigated by Bernis[3], in which the existence of so called "energy solutions" was proved by means of the theory of monotone operators deriving from convex functionals. When $m=1$, equation (1.0) becomes the nonlinear diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}A\left(\frac{\partial}{\partial x}B(u)\right), \tag{1.0$'$}$$

which was studied in several articles (see [4—7]).

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* Department of Mathematics, Jilin University, Changchun, Jilin, China.

This work is an extension to Bernis'. The existence results are new in some respects even for the second order case $m=1$. Since the method used in what follows is suitable to any positive integer m, we need only to discuss the fourth order case $m=2$ for simplicity, that is,

$$\frac{\partial u}{\partial t}+D^2A(D^2B(u))=0,\ (t,x)\in Q_T\equiv(0,T)\times(0,1), \tag{1.1}$$

$$B(u(t,0))=B(u(t,1))=DB(u(t,0))=DB(u(t,1))=0, \tag{1.2}$$

$$u(0,x)=u_0(x). \tag{1.3}$$

We assume

(HA) There exist two nonnegative and appropriately smooth functions $a_1(s)$, $a_2(s)$ such that $a(s)=a_1(s)a_2(s)$, where $a_1(s)$, $a_2(s)$ satisfy the following conditions: there exists a constant $M>0$ such that for $|s|\geqslant M$,

(i) $$|A_1(s)|\geqslant\alpha_{11}|s|^{p_1+1},\ a_1(s)\leqslant\alpha_{12}|s|^{p_1},$$

(ii) $$\alpha_{21}|s|^{p_2}\leqslant a_2(s)\leqslant\alpha_{22}|s|^{p_2},$$

where α_{11}, α_{12}, α_{21}, $\alpha_{22}>0$, $0\leqslant p_1\leqslant 1$, $p_2\geqslant 0$ and

$$A_1(s)=\int_0^s a_1(\sigma)d\sigma.$$

(HB1) The closure of the set S of all points at which the generalized inverse function $B^{-1}(s)$ of $B(s)$ is multi-valued is at most countable.

(HB2) $$\lim_{s\to+\infty}B(s)=+\infty,\ \lim_{s\to-\infty}B(s)=-\infty.$$

All the above assumptions permit $a(s)$ and $b(s)$ to have strong degeneration. Some examples and related propositions will be given in Section 6.

The method used in [3] depends heavily on the convexity of the functions $A(s)$ and $B(s)$ and so is not suitable for establishing the existence of generalized solutions of problem (1.1), (1.2), (1.3), under weak assumptions (HA), (HB1) and (HB2). The main difficulties are caused not only by the double nonlinearity of equation (1.1) but also by the strong degeneration of $a(s)$ and $b(s)$.

Here we abopt the method of parabolic regularization. Differing from the case of second order, the solvability of regularized problems is not known and hence should be discussed in advance. We apply Lerey-Schauder fixed point principle to the proof of the existence of classical solutions of the regularized problems. The energy method is used to establish the needed Schauder type priori estimates, which are relatively less seen in the study of quasilinear parabolic equations of higher order.

The generalized solutions of problem (1.1)—(1.3) can be obtained as the limits of some subsequence of the family of solutions $\{u_\varepsilon\}$ of the regularized problems. The crucial step is to pass the limiting process by means of the relatively less estimates on the approximate solutions. To do this, we use the compensated compactness method. Precisely speaking, a delicate analysis of the family $\{u_\varepsilon\}$ and the regularized problems themselves, together with the application of some basic facts in the theory of compensated compactness, are surely enough to complete the needed limiting processes concerning the two factors of nonlinearities.

We also discuss the uniqueness of generalized solutions of problem (1.1)—(1.3). For technical reason, we only discuss the case $B(s)=s$ and the case $A(s)=s$. The proof of the uniqueness in the case $A(s)=s$ has been done in the previous paper [8]. We note that the uniquess has not been discussed in Bernis[3].

§ 2. Some Lemmas about the Compensated Compactness Method

One of the main difficulties in the study of degenerate parabolic equations lies in the following fact: if the nonlinear factors are the same, then in the degenerate case one can only establish less estimates than in the uniform case. So, the strong compactness results can not be used to complete the limiting process concerning the given nonlinear functions. But, degenerate quasilinear parabolic equations are always associated with some functions which have some property of monotonicity. By means of this special property, we may make use of the basic theory of compensated compactness to prove some weak compactness results which are enough not only for the study of the problem(1.1), (1.2), (1.3) but also for other degenerate problems.

First we need the following lemma in the theory of compensated compactness, whose proof can be found in [9].

Lemma 2.1. *Let Ω be an open set in $\mathbf{R}^n$ and $\{u_k\}\subset L^\infty(\Omega)$ be such that $|u_k(x)|\leqslant M$, a. e. $x\in\Omega$. Then there exist a subsequence $\{u_{k_j}\}$of $\{u_k\}$ and a family of probobility measures $\{u_x\}_{x\in\Omega}$ on $\mathbf{R}$ depending measurably on x with supp $\mu_x\subset[-M, M]$ such that for any continuous function F: $\mathbf{R}\to\mathbf{R}$ there holds*

$$F(u_{k_j}(x))\overset{*}{\rightharpoonup}\int_{\mathbf{R}}F(\lambda)\mu_x(d\lambda)\ \text{in}\ L^\infty(\Omega).$$

Using Lemma 2.1, we prove the following lemma on weak compactness.

Lemma 2.2. *Let Ω be a bounded domain in $\mathbf{R}^n$ and $u_k\overset{*}{\rightharpoonup}u$ in $L^\infty(\Omega)$. Let $A(s)$ and $B(s)$ be continuous in $\mathbf{R}$ and $A(s)$ be non decreasing. If for any $\alpha\in A(\mathbf{R})$, the set $B(A^{-1}(\alpha))$ contains only one single point and*

$$A(u_k(x))\to w(x),\ B(u_k(x))\to v(x),\ \text{a. e. in}\ \Omega,$$

then there holds

$$A(u(x))=w(x),\ B(u(x))=v(x),\quad \text{a. e. in}\ \Omega.$$

Proof From Lemma 2.1, there exists a family of probability measure $\{\mu_x\}_{x\in\Omega}$

with supp $\mu_x \subset [-M, M]$, $M \geqslant \sup |u_k(x)|$, such that

$$\varphi(w(x)) = \lim_{j\to\infty} \varphi(A(u_{k_j}(x))) = \int_{\mathbf{R}} \varphi(A(\lambda))\mu_x(d\lambda), \tag{2.1}$$

for some subsequence $\{u_{k_j}\}$ and any continuous function $\varphi(s)$ at any point x in $\Omega \backslash E_\varphi$, with E_φ depending on φ and mes $E_\varphi = 0$. Denote

$$K = \sup_{|s| < M} |A(s)|.$$

Choose a sequence $\{\varphi_k\}$ from $C([-K, K])$ such that $\{\varphi_k\}$ is dense in $C([-K, K])$ and set

$$E = \bigcup_{k=1}^{\infty} E_{\varphi_k}.$$

It is easy to see that mes $E = 0$ and (2.1) holds for any continuous function $\varphi(s)$ at any point $x \in \Omega \backslash E$.

Again using Lemma 2.1, we have

$$u_{k_j}(x) \overset{*}{\rightharpoonup} \int_{\mathbf{R}} \lambda \mu_x(d\lambda) = u(x), \text{ in } L^\infty(\Omega). \tag{2.2}$$

Without loss of generality, we may assume that

$$|u_{k_j}(x)| \leqslant M, \int_{\mathbf{R}} \lambda\mu_x(d\lambda) = u(x), \quad \forall x \in \Omega \backslash E.$$

For any fixed $x \in \Omega \backslash E$, we conclude that the set

$$F_x \equiv \{s \in \mathbf{R};\ A(s) = w(x)\}$$

is nonempty. In fact, if $F_x = \phi$ for some x, then from the monotonicity of the function $A(s)$, we have $w(x) \overline{\in} [A(-M), A(M)]$. Again from the monotonicity,

$$A(u_{k_j}(x)) \in [A(-M), A(M)],$$

which implies that

$$w(x) = \lim_{j\to\infty} A(u_{k_j}(x)) \in [A(-M), A(M)].$$

A contradicition. So, the set F_x is nonempty and hence is a closed interval or a single point due to the continuity and the monotonicity of $A(s)$.

Now we prove the followinginclusion

$$\operatorname{supp} \mu_x \subset F_x, \quad \forall x \in \Omega \backslash E. \tag{2.3}$$

Suppose the contrary. Then there exists a point $x_0 \in \Omega \backslash E$ such that

$$\operatorname{supp} \mu \equiv \operatorname{supp} \mu_{x_0} \not\subset F_{x_0} \equiv F,$$

which is equivalent to

$$\mu([-M, M] \backslash F) > 0.$$

Setting $\varphi(s) = -|s - w_0|$, $w_0 = w(x_0)$, we get from (2.1) and the definition of F,

$$(w(x_0)) = 0 = \int_{\mathbf{R}} \varphi(A(\lambda))\mu(d\lambda) = -\int_{-M}^{M} |A(\lambda) - w_0| \mu(d\lambda)$$

$$= -\int_{[-M, M]\backslash F} |A(\lambda) - w_0| \mu(d\lambda) < 0.$$

A contradiction. Hence (2.3) holds.

By (2.2) and (2.3) we see that $u(x)=\int_{\mathbf{R}}\lambda\mu_x\ (d\lambda)\ \in F_x$, which implies that $A(u(x))=w(x)$.

Set
$$E_x=\{s\in\mathbf{R};\ B(s)=v(x)\}.$$
We conclude that $F_x\subset E_x$ a. e. $x\in\Omega$. Once this is done, we immediately come to the conclusion $B(u(x))=v(x)$ from $u(x)\in F_x$.

Let $x_0\in\Omega\backslash E$ and set
$$s_k=|A(u_k(x_0))-w(x_0)|.$$
Then
$$[a_k,\ b_k]\equiv A^{-1}([w(x_0)-s_k,\ w(x_0)+s_k])\to A^{-1}(w(x_0)),$$
and hence from the continuity of $B(s)$,
$$[\alpha_k,\ \beta_k]\equiv B([a_k,\ b_k])\to B(A^{-1})(w(x_0))).$$
Since $u_k(x_0)\in[a_k,\ b_k]$, we have $B(u_k(x_0))\in[\alpha_k,\beta_k]$ and hence $v(x_0)\in B(A^{-1}(w(x_0)))$, which implies that $\{v(x_0)\}=B(A^{-1}(w(x_0)))$ by the assumption that the set $B(A^{-1}(\alpha))$ is a single point. This shows that $A^{-}(w(x_0))\subset B^{-1}(v(x_0))$, that is, $F_{x_0}\subset E_{x_0}$. We have thus proved Lemma 2.2.

§ 3. Regularized Problem

In this section, we prove the existence of classical solutions of problem (1.1), (1.2), (1.3), under the conditions

(R1) $$a(s)=(a_1(s)+\alpha)(a_2(s)+\alpha),\quad (\alpha>0),$$

(R2) $$b(s)\geqslant\alpha,$$

where $a_1(s)$, $a_2(s)$, $b(s)$ are sufficiently smooth, $a_1(s)$, $a_2(s)$ satisfy (HA)—(i), (ii) respectiuely.

First we change equation (1.1) into the form
$$\frac{\partial u}{\partial t}+a(D^2B(u))b(u)D^4u=\sum_{i=1}^{3}b_i(u,\ Du,\ D^2u,\ D^3u)D^iu, \tag{3.1}$$
where
$$\sum_{i=1}^{3}b_i(u,\ Du,\ D^2u,\ D^3u)D^iu=-a'(D^2B(u))(D^3B(u))^2-$$
$$-a(D^2B(u))[4b'(u)DuD^3u+3b'(u))(D^2u)^2+6b''(u)(Du)^2D^2u$$
$$+b'''(u)(Du)^4].$$
Denote
$$X=\{u\in C^{(3+\alpha)/4,3+\alpha}(\bar{Q}_T);\ u(t,\ 0)=u(t,\ 1)=Du(t,\ 0)=$$
$$=Du(t,\ 1)=0,\ u(0,\ x)=u_0(x)\},\quad 0<\alpha<1,$$
and define the operator
$$T:\ X\to X,\ u\mapsto w,$$
where w is determined by the following linear problem

$$\frac{\partial w}{\partial t}+a(D^2B(u))b(u)D^4w=\sum_{i=1}^{3}b_i(u,\ Du,\ D^2u,\ D^3u)D^iw,$$

$$w(t,\ 0)=w(t,\ 1)=Dw(t,\ 0)=Dw(t,\ 1)=0,$$

$$w(0,\ x)=u_0(x).$$

The operator T is well-defined by the classical linear theory (see Theorm 3, § 4 of [10]), provided that $a(s)$, $b(s)$ are sufficiently smooth, u_0 is sufficiently smooth and satisfies some compatible conditions. The main purpose is to show that the operator T has a fixed point in X. To do this, we need some priori estimates on the classical solutions of problem (1.1), (1.2), (1.3).

Set
$$\mathscr{A}(s)=\int_0^s A(\sigma)d\sigma.$$

Multiply (1.1) by $\frac{\partial}{\partial t}B(u)$ and integrate the resulting relation over $Q_t\equiv(0,\ t)\times(0,\ 1)$. Integrating by parts, we have

$$\begin{aligned}0&=\iint_{Q_t} b(u)\left(\frac{\partial u}{\partial t}\right)^2 ds\,dx+\iint_{Q_t} A(D^2B(u))\frac{\partial}{\partial t}D^2B(u)ds\,dx\\&=\iint_{Q_t} b(u)\left(\frac{\partial u}{\partial t}\right)^2 ds\,dx+\iint_{Q_t}\frac{\partial}{\partial t}\mathscr{A}(D^2B(u))ds\,dx\\&=\iint_{Q_t} b(u)\left(\frac{\partial u}{\partial t}\right)^2 ds\,dx+\int_0^1\mathscr{A}(D^2B(u(t,\ x)))dx\\&\quad-\int_0^1\mathscr{A}(D^2B(u_0(x)))dx.\end{aligned}\tag{3.2}$$

From assumption (R2), it follows that

$$\iint_{Q_t}\left(\frac{\partial u}{\partial t}\right)^2 ds\,dx\leqslant\frac{1}{\alpha}\int_0^1\mathscr{A}(D^2B(u_0(x)))dx, \tag{3.3}$$

From assumption (HA) and Proposition 6.1, we may conclude that there exist constants $M_2>M$, $\mu_2>0$ such that

$$\mathscr{A}(s)\geqslant\mu_2|s|^{p+2},\ \text{whenever}\ |s|\geqslant M_2.$$

By virtue of this, we get from (3.2)

$$\begin{aligned}&\sup_{0<t<T}\int_0^1|D^2B(u(t,\ x))|^{p+2}dx\\&\leqslant\sup_{0<t<T}\int_{E^t}|D^2B(u(t,\ x))|^{p+2}dx+\sup_{0<t<T}\int_{\mathscr{C}E^t}|D^2B(u(t,\ x))|^{p+2}dx\\&\leqslant M_2^{p+2}+\frac{1}{\mu_2}\sup_{0<t<T}\int_0^1\mathscr{A}(D^2B(u(t,\ x)))dx\\&\leqslant M_2^{p+2}+\frac{1}{\mu_2}\int_0^1\mathscr{A}(D^2B(u_0(x)))dx,\end{aligned}\tag{3.4}$$

where $\mathscr{C}E^t=(0,\ 1)\setminus E^t$ and

$$E^t=\{x;\ |D^2B(u(t,\ x))|\leqslant M_2\}.$$

Using (3.4) and the boundary value condition (1.2), we have

$$\sup_{Q_T}|B(u(t,x))|\leqslant C,\ \sup_{Q_T}|DB(u(t,x))|\leqslant C$$

and hence

$$\sup_{Q_T}|u(t,x)|\leqslant C,\ \sup_{Q_T}|Du(t,x)|\leqslant C. \tag{3.5}$$

Next, we set $v=\frac{\partial u}{\partial t}$. Differentiate equation (1.1) with respect to t,

$$\frac{\partial v}{\partial t}+D^2[a(D^2B(u))D^2(b(u)v)]=0. \tag{3.6}$$

Multiplying (3.6) by v and integrating the resulting relation over Q_t, we have by integrating by parts

$$\begin{aligned}0&=\frac{1}{2}\int_0^1 v^2(t,x)dx-\frac{1}{2}\int_0^1 v^2(0,x)dx+\iint_{Q_t} a(D^2B(u))D^2(b(u)v)D^2v ds\,dx\\&=\frac{1}{2}\int_0^1 v^2(t,x)dx-\frac{1}{2}\int_0^1 v^2(0,x)dx+\iint_{Q_t} a(D^2B(u))b(u)(D^2v)^2 ds\,dx+\\&\quad+2\iint_{Q_t} a(D^2B(u))b'(u)DuDvD^2v\,ds\,dx+\\&\quad+\iint_{Q_t} a(D^2B(u))[b'(u)D^2u+b''(u)(Du)^2]vD^2v\,ds\,dx.\end{aligned} \tag{3.7}$$

We conclude that

$$\sup_{Q_T}|D^2u|^{p+2}\leqslant C\int_0^1 v^2(0,x)dx+C. \tag{3.8}$$

In fact, by setting

$$\sup_{Q_T}|D^2u|=|D^2u(t_0,x_0)|,\quad (t_0,x_0)\in\bar{Q}_T,$$

and noticing that

$$|D^2u|\leqslant\frac{1}{\alpha}(|D^2B(u)|+|b'(u)||Du|^2),$$

we get from (3.4), (3.5) and Proposition 6.1,

$$\begin{aligned}&|D^2u(t_0,x_0)|^{p+1}\leqslant C\sup_{0\leqslant x\leqslant 1}|D^2B(u(t_0,x))|^{p+1}+C\\&\leqslant\sup_{0\leqslant x\leqslant 1}|A(D^2B(u(t_0,x)))|+C\\&\leqslant C\Big[\int_0^1|DA(D^2B(u(t_0,x)))|^{\frac{p+2}{p+1}}dx\Big]^{\frac{p+1}{p+2}}+C\Big[\int_0^1|A(D^2B(u(t_0,x)))|^{\frac{p+2}{p+1}}dx\Big]^{\frac{p+1}{p+2}}+C\\&\leqslant\varepsilon\Big[\int_0^1|D^2A(D^2B(u(t_0,x)))|^{\frac{p+2}{p+1}}dx\Big]^{\frac{p+1}{p+2}}+C_\varepsilon\Big[\int_0^1|A(D^2B(u(t_0,x))|^{\frac{p+2}{p+1}}dx\Big]^{\frac{p+1}{p+2}}+C\\&\leqslant\varepsilon\Big[\int_0^1|D^2A(D^2B(u(t_0,x)))|^2dx\Big]^{\frac{1}{2}}+C_\varepsilon\Big[\int_0^1|D^2B(u(t_0,x))|^{p+2}dx\Big]^{\frac{p+1}{p+2}}+C\\&\leqslant\varepsilon\Big[\int_0^1|D^2A(D^2B(u(t_0,x)))|^2dx\Big]^{\frac{1}{2}}+C_\varepsilon\end{aligned}$$

and hence

$$|D^2u(t_0, x_0)|^{2p+2} \leqslant \varepsilon \int_0^1 |D^2A(D^2B(u(t_0, x)))|^2 dx + C_\varepsilon. \tag{3.9}$$

To obtain (3.8), we need some estimates on the last three terms in (3.7). Set

$$Q_0^M = \{(t, x) \in Q_{t_0};\ |D^2B(u(t, x))| \geqslant M\}.$$

From assumptions (R1), (R2) and (HA)—(i), (ii), we have

$$\iint_{Q_{t_0}} a(D^2B(u))b(u)(D^2v)^2 ds\, dx$$

$$\geqslant \alpha^3 \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \alpha^2\alpha_{21} \iint_{Q_0^M} |D^2B(u)|^{p_2}(D^2v)^2 ds\, dx. \tag{3.10}$$

By (3.5) and assumption (HA)—(i), (ii),

$$|2\iint_{Q_{t_0}} a(D^2B(u))b'(u)DuDvD^2v\, ds\, dx|$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} a_2(D^2B(u))(D^2v)^2 ds\, dx + C_\varepsilon \iint_{Q_{t_0}} a_1^2(D^2B(u))a_2(D^2B(u))(Dv)^2 ds\, dx$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \iint_{Q_0^M} |D^2B(u)|^{p_2}(D^2v)^2 ds\, dx +$$

$$+ C_\varepsilon \sup_{Q_{t_0}} a_1^2(D^2B(u))a_2(D^2B(u)) \iint_{Q_{t_0}} (Dv)^2 ds\, dx.$$

From assumption (HA)—(i), (ii), the estimate (3.3) and the inequality

$$\iint_{Q_{t_0}} (Dv)^2 ds\, dx = -\iint_{Q_{t_0}} vD^2v ds\, dx \leqslant \left(\iint_{Q_{t_0}} v^2 ds\, dx\right)^{1/2} \left(\iint_{Q_{t_0}} (D^2v)^2 ds\, dx\right)^{1/2},$$

it follows that

$$|2\iint_{Q_{t_0}} a(D^2B(u))b'(u)DuDvD^2v\, ds\, dx|$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \iint_{Q_0^M} |D^2B(u)|^{p_2}(D^2v)^2 ds\, dx + C_\varepsilon \sup_{Q_{t_0}} |D^2B(u)|^{4p_1+2p_2}.$$

And so from (3.9) and the fact $4p_1 + 2p_2 \leqslant 2p+2$,

$$|2\iint_{Q_{t_0}} a(D^2B(u))b'(u)DuDvD^2v\, ds\, dx|$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \iint_{Q_{t_0}} |D^2B(u)|^{p_2}(D^2v)^2 ds\, dx + \varepsilon \int_0^1 |D^2A(D^2B(u(t_0, x)))|^2 dx + C_\varepsilon. \tag{3.11}$$

Reasoning as for (3.11), we also have

$$\left|\iint_{Q_{t_0}} a(D^2B(u))b''(u)(Du)^2 vD^2v\, ds\, dx\right|$$

$$\leqslant C \sup_{Q_{t_0}} |D^2B(u)|^p \left(\iint_{Q_{t_0}} v^2 ds dx\right)^{1/2} \left(\iint_{Q_{t_0}} (D^2v)^2 ds\, dx\right)^{1/2}$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + C_\varepsilon \sup_{Q_{t_0}} |D^2B(u)|^{2p}$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + C_\varepsilon \sup_{Q_{t_0}} |D^2B(u)|^{2p+2} + C_\varepsilon$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \int_0^1 |D^2A(D^2B(u(t_0, x)))|^2 dx + C_\varepsilon, \tag{3.12}$$

$$\left| \iint_{Q_{t_0}} a(D^2B(u)) b'(u) D^2u D^2v \cdot v\, ds\, dx \right|$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} a_2(D^2B(u)) (D^2v)^2 ds\, dx + C_\varepsilon \iint_{Q_{t_0}} a_1^2(D^2B(u)) a_2(D^2B(u)) (D^2u)^2 v^2\, ds\, dx$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \iint_{Q_0^M} |D^2B(u)|^{p_2} (D^2v)^2 ds\, dx +$$

$$+ C_\varepsilon \sup_{Q_{t_0}} a_1^2(D^2B(u)) a_2(D^2B(u)) (D^2u)^2 \iint_{Q_{t_0}} v^2\, ds\, dx$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \iint_{Q_0^M} |D^2B(u)|^{p_2} (D^2v)^2 ds\, dx + C_\varepsilon \sup_{Q_{t_0}} |D^2B(u)|^{2p_1+p_2+2}$$

$$\leqslant \varepsilon \iint_{Q_{t_0}} (D^2v)^2 ds\, dx + \varepsilon \iint_{Q_0^M} |D^2B(u)|^{p_2} (D^2v)^2 ds\, dx + \varepsilon \int_0^1 |D^2A(D^2B(u(t_0, x)))|^2 dx$$
$$+ C_\varepsilon. \tag{3.13}$$

Substituting (3.10)—(3.13) into (3.7), we have

$$\int_0^1 v^2(t_0, x) dx \leqslant \int_0^1 v^2(0, x) dx + C,$$

which together with (3.9) implies (3.8).

Notice that

$$\int_0^1 v^2(0, x) dx = \int_0^1 |D^2A(D^2B(u_0(x)))|^2 dx.$$

From (3.3), (3.5), (3.7), (3.8) and assumption (HA)—(i), (ii), we have

$$\int_0^1 v^2(t, x) dx + 2\alpha^3 \iint_{Q_t} (D^2v)^2 ds\, dx$$

$$\leqslant C + C \iint_{Q_t} |Dv|\, |D^2v|\, ds\, dx + C \iint_{Q_t} |v|\, |D^2v|\, ds\, dx$$

$$\leqslant \alpha^3 \iint_{Q_t} (D^2v)^2 ds\, dx + C.$$

Thus

$$\int_0^1 v^2(t, x) dx \leqslant C, \tag{3.14}$$

$$\iint_{Q_t} (D^2v)^2 ds dx \leqslant C. \tag{3.15}$$

By (3.8), assumption (HA)—(i), (ii) and Proposition 6.1.

$$\int_0^1 |A(D^2B(u(t,\ x)))|^2dx \leqslant C\sup_{Q_T}|D^2B(u)|^{2p+2} \leqslant C.$$

So it follows from (3.14) that

$$\sup_{Q_T}|DA(D^2B(u))| \leqslant C, \tag{3.16}$$

which together with assumption (R1) implies that

$$\sup_{Q_T}|D^3B(u)| \leqslant C. \tag{3.17}$$

Set $g(t,\ x)=D^3u(t,\ x)$. Then g satisfies

$$\frac{\partial g}{\partial t}=D^3v. \tag{3.18}$$

By the estimates (3.14), (3.17) and assumption (R1), we have

$$\int_0^1 |D^4B(u(t,\ x))|^2dx \leqslant C.$$

Thus

$$|D^3B(u(t,\ x_1))-D^3B(u(t,\ x_2))| \leqslant C|x_1-x_2|^{1/2},\quad x_1,\ x_2\in(0,\ 1).,$$

and hence

$$|g(t,\ x_1)-g(t,\ x_2)| \leqslant C|x_1-x_2|^{1/2},\quad x_1,\ x_2\in(0,\ 1). \tag{3.19}$$

Now we fix $x,\ t_1,\ t_2,\ 0\leqslant x\leqslant 1/2,\ 0<t_1<t_2<T,\ (\Delta t)^\alpha\leqslant 1/4,\ \Delta t=t_2-t_1$, where α will be specified later. Integrating (3.18) over $(t_1,\ t_2)\times(y,\ y+(\Delta t)^\alpha)$, we have

$$\begin{aligned}
&\int_y^{y+(\Delta t)^\alpha}[g(t_2,\ z)-g(t_1,\ z)]dz\\
&=(\Delta t)^\alpha\int_0^1[g(t_2,\ y+\theta(\Delta t)^\alpha)-g(t_1,\ y+\theta(\Delta t)^\alpha]d\theta\\
&=\int_{t_1}^{t_2}[D^2v(s,\ y+(\Delta t)^\alpha)-D^2v(s,\ y)]ds.
\end{aligned}$$

Integrating this equality with respect to y over $(x,\ x+(\Delta t)^\alpha)$, we get

$$\begin{aligned}
&(\Delta t)^\alpha\int_0^1\int_x^{x+(\Delta t)^\alpha} g(t_2,\ y+\theta(\Delta t)^\alpha-g(t_1,\ y+\theta(\Delta t)^\alpha)]d\theta\,dy\\
&=\int_{t_1}^{t_2}\int_x^{x+(\Delta t)^\alpha} D^2v(s,\ y+(\Delta t)^\alpha)-D^2v(s,\ y)]ds\,dy.
\end{aligned} \tag{3.20}$$

Using the mean value theorem, there exists a point $x^*=y^*+\theta^*(\Delta t)^\alpha$ such that

$$\begin{aligned}
&(\Delta t)^\alpha[g(t_2,\ x^*)-g(t_1,\ x^*)]\\
&=\int_0^1\int_x^{x+(\Delta t)^\alpha}[g(t_2,\ y+\theta(\Delta t)^\alpha)-g(t_1,\ y+\theta(\Delta t)^\alpha)]d\theta\,dy.
\end{aligned}$$

Taking this and (3.15) into account, we have from (3.20)

$$g|(t_2,\ x^*)-g(t_1,\ x^*)| \leqslant C(\Delta t)^{(1-3\alpha)/2},$$

which together with (3.19) implies that

$$\begin{aligned}
&|g(t_1,\ x)-g(t_2,\ x)|\\
&\leqslant|g(t_1,\ x)-g(t_1,\ x^*)|+|g(t_1,\ x^*)-g(t_2,\ x^*)|+|g(t_2,\ x^*)-g(t_2,\ x)|\\
&\leqslant C(\Delta t)^{\alpha/2}+C(\Delta t)^{(1-3\alpha)/2}.
\end{aligned}$$

The choice $\alpha=\frac{1}{4}$ gives

$$|g(t_1, x)-g(t_2, x)|\leqslant C|t_1-t_2|^{1/8}. \tag{3.21}$$

Rewrite (1.1) as the form

$$\frac{\partial u}{\partial t}+a(t, x)D^4u=f(t, x),$$

where

$$a(t, x)=a(D^2B(u(t, x)))b(u(t, x)t),$$

$$f(t, x)=\sum_{i=1}^{3}b_i(u(t, x), Du(t, x), D^2u(t, x), D^3u(t, x))D^iu(t, x).$$

The estimates (3.5), (3.19) and (3.21) give

$$\|a\|_{C^{1+\beta/4,1+\beta}(\overline{Q}_T)}\leqslant C, \quad \|f\|_{C^{\beta/4,\beta}(\overline{Q}_T)}\leqslant C,$$

with $\beta\in(0, 1)$, $C>0$ depending only on the known quantities. Then by the classical linear theory (see Theorem 1, § 4 of [10]), we see that

$$\|u\|_{C^{1+\beta/4,4+\beta}(\overline{Q}_T)}\leqslant C. \tag{3.22}$$

In particular, for α given in the definition of X, we have

$$\|a\|_{C^{(1+\alpha)/4,1+\alpha}(\overline{Q}_T)}\leqslant C, \quad \|f\|_{C^{\alpha/4,\alpha}(\overline{Q}_T)}\leqslant C$$

and hence

$$\|u\|_{C^{1+\alpha/4,4+\alpha}(\overline{Q}_T)}\leqslant C. \tag{3.23}$$

For any fixed $\sigma\in(0, 1]$, if some function u in X satisfies $u=\sigma Tu$, then from the definition of the operator T, u satisfies (1.1), (1.2) and $u(0, x)=\sigma u_0(x)$. Thus u satisfies the estimate (3.23) with C depending only on the known quantities. Applying Leray-Schauder's fixed point theorem to the operator T, we see that T has a fixed point u, which is the desired classical solution of problem (1.1), (1.2), (1.3). We have thus proved the following

Theorem 3.1. *Let $a(s)$, $b(s)$ be sufficiently smooth functions satisfying $(R1)$ and $(R2)$ respectively. Let $u_0\in C_0^\infty(I)$. Then problem* (1.1), (1.2), (1.3) *admits a classical solution which is sufficiently smooth.*

Remark. The conclusion in Theorem 3.1 is enough for the purpose of constructing a suitable sequence of approximate solutions of problem (1.1), (1.2), (1.3) in degenerate case. However, by an approximation, we may prove that to ensure the solvability of problem (1.1), (1.2), (1.3) in uniform case in the space $C^{1+\alpha/4,4+\alpha}(\overline{Q}_T)$, it suffices to assume $a(s)\in C^{1+\alpha}(\mathbf{R})$, $b(s)\in C^{3+\alpha}(\mathbf{R})$, $u_0\in C^{4+\alpha}(\overline{I})$, $D^i u_0(0)=D^iu_0(1)=0$, $(i=0, 1, 2, 3, 4)$.

§ 4. The Existence of Generalized Solutions

We are in a position now to approach the question of the existence of generalized solutions of problem (1.1), (1.2), (1.3) under the structural conditions (HA), (HB1) and (HB2). The treatment here will be accomplished by the method of parabolic regularization, where the difficult step for passage to the limit will be completed by means of the lemmas about the compensated compactness method stated in section 2.

First we state the following

Definition 4.1. *A function $u\in L^\infty(Q_T)$ is said to be a generalized solution of problem* (1.1), (1.2), (1.3), *if the following conditions are fulfilled:*

1.° $B(u)\in C^\alpha(\bar{Q}_T)\cap L^\infty(0, T; W_0^{2,p+2}(I))$.

2.° *For any $\varphi\in C^\infty(\bar{Q}_T)$ with $\varphi(t, 0)=\varphi(t, 1)=D\varphi(t, 0)=D\varphi(t, 1)=\varphi(T, x)=0$, there holds the following integral equality*

$$\iint_{Q_T} u\frac{\partial\varphi}{\partial t}\,dt\,dx-\iint_{Q_T} A(D^2B(u))D^2\varphi\,dt\,dx+\int_0^1 u_0(x)\varphi(0, x)dx=0.$$

The basic result in this section is

Theorem 4.2. *Let $u_0\in C^2(\bar{I})$, $u_0(0)=u_0(1)=u_0'(0)=u_0'(1)=0$ and the assumptions* (HA), (HB1) *and* (HB2) *be fulfilled. Then problem* (1.1), (1.2), (1.3) *admits a generalized solution in the sense of Definition* 4.1.

Proof Consider the regularized problems

$$\frac{\partial u_\varepsilon}{\partial t}+D^2A_\varepsilon(D^2B_\varepsilon(u_\varepsilon))=0, \quad \text{in } Q_T=(0, T)\times(0, 1), \tag{4.1}$$

$$u_\varepsilon(t, 0)=u_\varepsilon(t, 1)=Du_\varepsilon(t, 0)=Du_\varepsilon(t, 1)=0, \tag{4.2}$$

$$u_\varepsilon(0, x)=u_0^\varepsilon(x), \tag{4.3}$$

where
$$A_\varepsilon(s)=\int_0^s(a_1(\sigma)+\varepsilon)(a_2(\sigma)+\varepsilon)d\sigma,$$

$$B_\varepsilon(s)=\int_0^s(b(\sigma)+\varepsilon)d\sigma$$

and $u_0^\varepsilon\in C_0^\infty(I)$ satisfying $|D^2u_0^\varepsilon(x)|\le C$ and

$$\|u_0^\varepsilon-u_0\|_{H^1}\to 0, \quad (\varepsilon\to 0).$$

By Theorem 3.1, the regularized problems admit classical solutions u_ε which are sufficiently smooth. We need some estimates on u_ε.

First we multiply equation (4.1) by $\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)$ and integrate the resulting relation over Q_t

$$\iint_{Q_t}(b(u_\varepsilon)+\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 ds\,dx+\iint_{Q_t} D^2A_\varepsilon(D^2B_\varepsilon(u_\varepsilon))\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)ds\,dx=0.$$

Integration by parts gives

$$\iint_{Q_t}(b(u_\varepsilon)+\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 ds\,dx+\iint_{Q_t}\frac{\partial}{\partial t}\mathscr{A}_\varepsilon(D^2B_\varepsilon(u_\varepsilon))ds\,dx=0,$$

where
$$\mathscr{A}_\varepsilon(s)=\int_0^s A_\varepsilon(\sigma)d\sigma.$$

It follows that

$$\sup_{0<t\leq T}\int_0^1 \mathscr{A}_\varepsilon(D^2B_\varepsilon(u_\varepsilon(t, x)))dx\leq\int_0^1 \mathscr{A}_\varepsilon(D^2B_\varepsilon(u_0^\varepsilon))dx, \tag{4.4}$$

$$\iint_{Q_t}(b(u_\varepsilon)+\varepsilon)\left(\frac{\partial u_\varepsilon}{\partial t}\right)^2 ds\,dx\leq\int_0^1 \mathscr{A}_\varepsilon(D^2B_\varepsilon(u_0^\varepsilon))dx. \tag{4.5}$$

Recalling that

$$\mathscr{A}(s)\equiv\int_0^s A(\sigma)d\sigma\geq\mu_2|s|^{p+2}, \text{ whenever } |s|\geq M_2,$$

we get from (4.4)

$$\int_0^1 |D^2B_\varepsilon(u_\varepsilon(t, x))|^{p+2}dx\leq C. \tag{4.6}$$

Thus

$$\sup_{Q_T}|B_\varepsilon(u_\varepsilon(t, x))|\leq C, \quad \sup_{Q_T}|DB_\varepsilon(u_\varepsilon(t, x))|\leq C \tag{4.7}$$

and hence, from assumption (HB2),

$$\sup_{Q_T}|u_\varepsilon(t, x)|\leq C, \tag{4.8}$$

which together with (4.5) imply that

$$\iint_{Q_T}\left(\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)\right)^2 dt\,dx\leq C, \tag{4.9}$$

$$\iint_{Q_T}(b(u_\varepsilon)+\varepsilon)\,[D^2A_\varepsilon(D^2B_\varepsilon(u_\varepsilon))]^2dtdx\leq C. \tag{4.10}$$

From (4.7), we obtain

$$|B_\varepsilon(u_\varepsilon(t, x_1))-B_\varepsilon(u_\varepsilon(t, x_2))|\leq C|x_1-x_2|. \tag{4.11}$$

Moreover, we may prove

$$|B_\varepsilon(u_\varepsilon(t_1, x))-B_\varepsilon(u_\varepsilon(t_2, x))|\leq C|t_1-t_2|^{1/3}, \tag{4.12}$$

similar to the proof of (3.21).

By the estimates (4.8), (4.6), (4.9), (4.11) and (4.12), we can extract a subsequence from $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, such that

$$u_\varepsilon \overset{*}{\rightharpoonup} u, \text{ in } L^\infty(Q_T),$$

$$D^2B_\varepsilon(u_\varepsilon)\overset{*}{\rightharpoonup} D^2w, \text{ in } L^\infty(0, T; W_0^{2,p+2}(I)),$$

$$\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)\rightharpoonup\frac{\partial}{\partial t}w, \text{ in } L^2(Q_T),$$

$$B_\varepsilon(u_\varepsilon)\to w, \text{uniformly in } Q_T.$$

The limiting function $w\in C^\alpha(\overline{Q}_T)\cap L^\infty(0, T; W_0^{2,p+2}(I))$ for some $\alpha\in(0, 1)$ and from Lemma 2.2, $w=B(u)$.

To show that u is a generalized solution of problem (1.1), (1.2), (1.3), it remains to prove that u satisfies the integral equality in Definition 4.1. This will be done by taking the limit as $\varepsilon\to 0$ in the following equality

$$\iint_{Q_T} u_\varepsilon \frac{\partial \varphi}{\partial t} dt\,dx - \iint_{Q_T} A_\varepsilon(D^2B_\varepsilon(u_\varepsilon))D^2\varphi dt\,dx + \int_0^1 u_0^\varepsilon(x)\varphi(0, x)dx = 0,$$

where φ is an arbitrary function in $C^\infty(\overline{Q}_T)$ with $\varphi(t, 0) = \varphi(t, 1) = D\varphi(t, 0) = D\varphi(t, 1) = \varphi(T, x) = 0$. Since $u_\varepsilon \overset{*}{\rightharpoonup} u$ in $L^\infty(Q_T)$, $u_0^\varepsilon \to u_0$ uniformly in $(0, 1)$, it suffices to show

$$\lim_{\varepsilon\to 0} \iint_{Q_T} A_\varepsilon(D^2B_\varepsilon(u_\varepsilon))D^2\varphi\, dt\, dx = \iint_{Q_T} A(D^2B(u))D^2\varphi\, dt\, dx. \tag{4.13}$$

The crucial steps are to show the following

$$\lim_{\varepsilon\to 0} \iint_{Q_T} \left[u_\varepsilon \frac{\partial}{\partial t} B_\varepsilon(u_\varepsilon) - u\frac{\partial}{\partial t} B(u)\right] dt\, dx = 0, \tag{4.14}$$

$$\lim_{\varepsilon\to 0} \iint_{Q_T} a_\varepsilon^* [D^2B_\varepsilon(u_\varepsilon) - D^2B(u)]^2 dt\, dx = 0, \tag{4.15}$$

where

$$a_\varepsilon^* = \int_0^1 [a_1(\lambda D^2B_\varepsilon(u_\varepsilon) + (1-\lambda)D^2B(u)) + \varepsilon]\,[a_2(\lambda D^2B_\varepsilon(u_\varepsilon) + (1-\lambda)D^2B(u)) + \varepsilon]\, d\lambda.$$

First we prove (4.14). Set

$$F = \{(t, x); B(u(t, x)) \in \overline{S}\},$$
$$E = \{(t, x); B(u(t, x)) \in \mathscr{C}\overline{S}\}.$$

Since $B(u) \in C^\alpha(\overline{Q}_T)$, we see that F is a closed subset of $\overline{Q}_T$ and E a relatively open subset of $\overline{Q}_T$. Denote

$$\mathscr{C}\overline{S} = \bigcup_k (\alpha_k, \beta_k),$$

$$E_k = \{(t, x); B(u(t, x)) \in (\alpha_k, \beta_k)\}.$$

Clearly $\bigcup_k E_k = E$ and hence $\sum_k \text{mes } E_k = \text{mes } E$. For any fixed $\delta > 0$, we choose N, such that $\text{mes} \bigcup_{k>N} E_k < \delta$. Then we choose $\eta > 0$ such that

$$\text{mes } E^\eta \equiv \text{mes}\{(t, x); B(u(t, x)) \in \bigcup_{k=1}^N (\alpha_k, \alpha_k+\eta) \cup (\beta_k - \eta, \beta_k)\} < \delta.$$

Denoting $Q_N^\eta = \bigcup_{k>N} E_k \cup E^\eta$, we have from the estimates (4.8), (4.9)

$$\left|\iint_{Q_N^\eta} \left(u_\varepsilon \frac{\partial}{\partial t} B_\varepsilon(u_\varepsilon) - u\frac{\partial}{\partial t} B(u)\right) dt\, dx\right| \le C\sqrt{\delta}. \tag{4.16}$$

Denote $\alpha_k' = \alpha_k + \eta$, $\beta_k' = \beta_k - \eta$ and set

$$E_k^\eta = \{(t, x); B(u(t, x)) \in [\alpha_k', \beta_k']\}.$$

Since $B_\varepsilon(u_\varepsilon) \to B(u)$ uniformly in Q_T, we may conclude that there exists $\varepsilon_0 > 0$ such that

$$B(u_\varepsilon),\ B_\varepsilon(u_\varepsilon) \in [\alpha_k' - \eta/2, \beta_k' + \eta/2], \text{ whenever } 0 < \varepsilon < \varepsilon_0.$$

By definition, $B(s)$ is strictly increasing on any of the intervals $[\alpha_k' < \eta/2, \beta_k' + \eta/2]$, $(k = 1, 2, \cdots, N)$. This suggests $u_\varepsilon \to u$ uniformly in E_k^η, $(k = 1, 2, \cdots, N)$. It follows from (4.8), (4.9) that

$$\left|\sum_{k=1}^{N}\iint_{E_k^2}(u_\varepsilon-u)\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)dt\,dx\right|$$

$$\leqslant\sup_{E_k^2}|u_\varepsilon-u|\left[\iint_{Q_T}\left(\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)\right)^2dt\,dx\right]^{1/2}(\text{mes }Q_T)^{1/2}\to 0\ (\varepsilon\to 0).$$

By virtue of this and the fact that

$$\sum_{k=1}^{N}\iint_{E_k^2}u\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)dt\,dx\to\sum_{k=1}^{N}\iint_{E_k^2}u\frac{\partial}{\partial t}B(u)dt\,dx\ (\varepsilon\to 0),$$

we see that

$$\sum_{k=1}^{N}\iint_{E_k^2}u_\varepsilon\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)dt\,dx\to\sum_{k=1}^{N}\iint_{E_k^2}u\frac{\partial}{\partial t}B(u)dt\,dx\ (\varepsilon\to 0). \tag{4.17}$$

Set

$$\mathscr{B}^*(s)=\int_0^s\sigma b(\sigma)d\sigma,\ \mathscr{B}_\varepsilon^*(s)=\int_0^s\sigma(b(\sigma)+\varepsilon)d\sigma.$$

As before, from the estimates (4.7), (4.8), (4.10) and equation (4.1), we can prove that there exists a subsequence of $\{u_\varepsilon\}$, denoted also by $\{u_\varepsilon\}$, such that

$$\mathscr{B}_\varepsilon^*(u_\varepsilon(t,x))\to v(t,x),\ \text{uniformly in } Q_T,$$

$$\frac{\partial}{\partial t}\mathscr{B}_\varepsilon^*(u_\varepsilon)\to\frac{\partial}{\partial t}v,\ \text{ in } L^2(Q_T).$$

Notincig that $\mathscr{B}^*(s)$ is nonincreasing in $(-\infty,0)$ and nondecreasing in $(0,+\infty)$, we immediately obtain $v=\mathscr{B}^*(u)$ by using Lemma 2.2.

Set

$$\bar S=\bigcup_k\{\lambda_k\},\ F_k=\{(t,x);B(u(t,x))=\lambda_k\}.$$

Then from the definition of $\mathscr{B}^*(s)$, $\mathscr{B}^*(u)=\mu_k$ in F_k for some constants μ_k, $k=1,2,\cdots$. Thus

$$\iint_F u\frac{\partial}{\partial t}B(u)dt\,dx=\iint_F\frac{\partial}{\partial t}\mathscr{B}^*(u)dt\,dx=0$$

and hence

$$\lim_{\varepsilon\to 0}\iint_F u_\varepsilon\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)dtdx=\lim_{\varepsilon\to 0}\iint_{Q_T}\chi_F(t,x)\frac{\partial}{\partial t}\mathscr{B}_\varepsilon^*(u_\varepsilon)dt\,dx$$

$$=\iint_{Q_T}\chi_F(t,x)\frac{\partial}{dt}\mathscr{B}^*(u)dt\,dx=\iint_F u\frac{\partial}{\partial t}B(u)dt\,dx. \tag{4.18}$$

Combining (4.16), (4.17) with (4.18), we have

$$\varlimsup_{\varepsilon\to 0}\left|\iint_{Q_T}\left(u_\varepsilon\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon)-u\frac{\partial}{\partial t}B(u)\right)dt\,dx\right|\leqslant C\sqrt{\delta}$$

and so (4.14) follows by the arbitrariness of δ.

Using (4.14), we may prove (4.15) immediately from

$$\lim_{\varepsilon\to 0}\iint_{Q_T} a_\varepsilon^* [D^2B_\varepsilon(u_\varepsilon) - D^2B(u)]^2 dt\,dx$$
$$= \lim_{\varepsilon\to 0}\iint_{Q_T} [A_\varepsilon(D^2B_\varepsilon(u_\varepsilon)) - A_\varepsilon(D^2B(u))](D^2B_\varepsilon(u_\varepsilon) - D^2B(u)]\,dt\,dx$$
$$= -\lim_{\varepsilon\to 0}\iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t}(B_\varepsilon(u_\varepsilon) - B(u))dt\,dx$$
$$= -\lim_{\varepsilon\to 0}\int_0^1 u_\varepsilon(T, x)(B_\varepsilon(u_\varepsilon(T, x)) - B(u(T, x)))dx$$
$$+\lim_{\varepsilon\to 0}\int_0^1 u_\varepsilon^0(B_\varepsilon(u_0^\varepsilon) - B(u(0, x)))dx$$
$$+\lim_{\varepsilon\to 0}\iint_{Q_T} u_\varepsilon\Big(\frac{\partial}{\partial t}B_\varepsilon(u_\varepsilon) - \frac{\partial}{\partial t}B(u)\Big)dt\,dx$$
$$= \lim_{\varepsilon\to 0}\iint_{Q_T}(u - u_\varepsilon)\frac{\partial}{\partial t}B(u)\,dt\,dx = 0.$$

Finally, we show (4.13) by (4.15). By the estimate (4.6) and assumption (HA), we have

$$\iint_{Q_T} a_\varepsilon^*(t, x)\,dt\,dx \leqslant C.$$

Therefore

$$\Big|\iint_{Q_T} A_\varepsilon(D^2B_\varepsilon(u_\varepsilon))D^2\varphi\,dt\,dx - \iint_{Q_T} A_\varepsilon(D^2B(u))D^2\varphi\,dt\,dx\Big|$$
$$\leqslant \Big\{\iint_{Q_T} a_\varepsilon^*[D^2B_\varepsilon(u_\varepsilon) - D^2B(u)]^2 dt\,dx\Big\}^{1/2}\Big\{\iint_{Q_T} a_\varepsilon^*(D^2\varphi)^2 dt\,dx\Big\}^{1/2} \to 0 \quad (\varepsilon\to 0).$$

By virtue of this and the fact that

$$\iint_{Q_T} A_\varepsilon(D^2B(u))D^2\varphi\,dt\,dx \to \iint_{Q_T} A(D^2B(u))D^2\varphi\,dt\,dx \quad (\varepsilon\to 0)$$

we obtain (4.13) and hence complete the proof of the theorem.

Corollary 4. 3. *If in addition to the assumptions in Theorem 4 2, the function $B(s)$ is strictly increasing, then the solution obtained in Theorem 4.2 is continuous.*

If $$|B(s_1) - B(s_2)| \geqslant \alpha_m |s_1 - s_2|^m, \quad \alpha_0 > 0, \ m > 0,$$
then thesolution is Hölder continuous.

Remark. By approximation, we can prove that to ensure the conclusion in Theorem 4.2 the smoothness assumptions for $a(s)$, $b(s)$, $u_0(x)$ are needed only to be

$$a(s)\in C(\mathbf{R}),\ b(s)\in C^1(\mathbf{R}),\ u_0\in C^2(\bar I),\ u_0(0) = u_0(1) = u_0'(0) = u_0'(1) = 0.$$

If $B(s)$ is strictly increaiing, then we may assume

$$A(s)\in C(\mathbf{R}),\ w_0 \equiv B(u_0)\in C^2(\bar I),\ w_0(0) = w_0(1) = w_0'(0) = w_0'(1) = 0,$$

instead of the original assumptions on $A(s)$ and u_0. In this case, the initial value condition for regularized problems can be taken as

$$u_\varepsilon(0, x) = \Psi_\varepsilon(w_0^\varepsilon),$$

where $\Psi_\varepsilon(s)$ is the inverse function of $B_\varepsilon(s) = \int_0^s b_\varepsilon(\sigma) + \varepsilon) d\sigma$, $w_0^\varepsilon \in C_0^\infty(I)$, $b_\varepsilon(s) \in C^\infty(\mathbf{R})$, $|D^2 w_0^\varepsilon(x)| \leqslant C$,

$$w_0^\varepsilon(x) \to w_0(x), \text{ uniformly in } (0, 1),$$

$$b_\varepsilon(s) \to b(s), \quad \text{uniformly in any segment.}$$

§5. Uniqueness Theorem

We prove here the following uniqueness result for solutions of problem (1.1), (1.2), (1.3) in the case $B(s) = s$. For another case $A(s) = s$, see [4].

Theorem 5.1. *Suppose that $A(s)$ satisfies assumption (HA), $B(s) = s$, $u_0 \in H_0^2(I) \cup C^4(\bar{I})$. Then the generalized solutions with the property* $\sup|D^2 u| \leqslant C$ *of problem* (1.1), (1.2), (1.3) *is unique.*

Proof Since $B(s) = s$, the regularized equation (4.1) can be taken as

$$\frac{\partial u_\varepsilon}{\partial t} + D^2 A_\varepsilon(D^2 u_\varepsilon) = 0,$$

from which and (4.2), (4.3) we may prove the following estemate

$$\sup_{0<t<T} \int_0^1 [D^2 A_\varepsilon(D^2 u_\varepsilon(t, x))]^2 dx \leqslant C.$$

By virtue of this and the estimate (4.6), it is easily seen that

$$\sup_{Q_T} |D^2 u_\varepsilon(t, x)| \leqslant C.$$

Then problem (1.1), (1.2), (1.3) has a generalized solution with the property $\sup |D^2 u| \leqslant C$.

Now we turn to the question of the uniqueness. Let u_1, u_2 be the generalized solutions of problem (1.1), (1.2), (1.3),

$$\sup|D^2 u_1| \leqslant C, \ \sup|D^2 u_2| \leqslant C.$$

Then by definition, for any $\varphi \in C^\infty(\bar{Q}_T)$ with $\varphi(t, 0) = \varphi(t, 1) = D\varphi(t, 0) = D\varphi(t, 1) = \varphi(T, x) = 0$, we have

$$\iint_{Q_T} (u_1 - u_2) \frac{\partial \varphi}{\partial t} \, dt \, dx = \iint_{Q_T} \hat{a}(D^2 u_1 - D^2 u_2) D^2 \varphi \, dt \, dx, \tag{5.1}$$

where

$$\hat{a} = \int_0^1 a(\lambda D^2 u_1 + (1 - \lambda) D^2 u_2) d\lambda.$$

Choose a sequence $\{\hat{a}_\varepsilon\} \subset C_0^\infty(Q_T)$ such that

$$0 \leqslant \hat{a}_\varepsilon(t, x) \leqslant M, \ \iint_{Q_T} |\hat{a}_\varepsilon - \hat{a}|^2 dt dx < \varepsilon^2, \quad \varepsilon > 0. \tag{5.2}$$

Consider the following problem

$$\frac{\partial\varphi_\varepsilon}{\partial t}=D^2[(\hat{a}_\varepsilon+\varepsilon)D^2\varphi_\varepsilon]+f, \quad \text{in } Q_T,$$
$$\varphi_\varepsilon(t,\ 0)=\varphi_\varepsilon(t,\ 1)=D\varphi_\varepsilon(t,\ 0)=D\varphi_\varepsilon(t,\ 1)=0,$$
$$\varphi_\varepsilon(T,\ x)=0,$$

where f is an arbitrary function in $C_0^\infty(Q_T)$. By classical linear theory the above problem has classical solutions φ_ε, which satisfy the following estimate

$$\iint_{Q_T}(\hat{a}_\varepsilon+\varepsilon)(D^2\varphi_\varepsilon)^2 dt\,dx\leqslant\frac{1}{2}e^T\iint_{Q_T}f^2\,dt\,dx. \tag{5.3}$$

In fact, multiplying the equation, satisfied by φ_ε, by φ_ε and integrating the resulting relation over $(t,\ T)\times(0,\ 1)$, we have

$$\int_t^T\int_0^1\frac{1}{2}\frac{\partial}{\partial t}\varphi_\varepsilon^2 dt\,dx=\int_t^T\int_0^1 D^2[(\hat{a}_\varepsilon+\varepsilon)D^2\varphi_\varepsilon]\varphi_\varepsilon\,dt\,dx+\int_t^T\int_0^1 f\varphi_\varepsilon f dt\,dx.$$

Integrating by parts, we have

$$-\frac{1}{2}\int_0^1\varphi_\varepsilon^2(t,\ x)dx=\int_t^T\int_0^1(\hat{a}_\varepsilon+\varepsilon)(D^2{}_\varepsilon\varphi)^2 ds\,dx+\int_t^T\int_0^1 f\varphi_\varepsilon ds\,dx$$

and hence

$$\frac{1}{2}\int_0^1\varphi_\varepsilon^2(t,\ x)dx+\int_t^T\int_0^1(\hat{a}_\varepsilon+\varepsilon)(D^2\varphi_\varepsilon)^2 ds\,dx$$
$$\leqslant\frac{1}{2}\iint_{Q_T}f^2 ds\,dx+\frac{1}{2}\int_t^T\int_0^1\varphi_\varepsilon^2(s,\ x)ds\,dx,$$

from which the estimate (5.3) follows.

Replacing φ by φ_ε in equality (5.1), we have

$$\iint_{Q_T}(u_1-u_2)D^2[(\hat{a}_\varepsilon+\varepsilon)D^2\varphi_\varepsilon]\,dt\,dx+\iint_{Q_T}(u_1-u_2)f\,dt\,dx$$
$$=\iint_{Q_T}\hat{a}(D^2u_1-D^2u_2)D^2\varphi_\varepsilon\,dt\,dx.$$

Thus by (5.2) and (5.3), we obtain

$$\left|\iint_{Q_T}(u_1-u_2)f\,dt\,dx\right|$$
$$\leqslant\left|\iint_{Q_T}(D^2u_1-D^2u_2)(\hat{a}_\varepsilon-\hat{a})D^2\varphi_\varepsilon\,dt\,dx\right|+\left|\iint_{Q_T}\varepsilon(D^2u_1-D^2u_2)D^2\varphi_\varepsilon\,dt\,dx\right|$$
$$\leqslant C\left\{\iint_{Q_T}(\hat{a}_\varepsilon-\hat{a})^2 dt\,dx\right\}^{1/2}\left\{\iint_{Q_T}(D^2\varphi_\varepsilon)^2 dt\,dx\right\}^{1/2}$$
$$+C\varepsilon\left\{\iint_{Q_T}(D^2\varphi_\varepsilon)^2 dt\,dx\right\}^{1/2}$$
$$\leqslant C\sqrt{\varepsilon}\left\{\iint_{Q_T}\varepsilon(D^2\varphi_\varepsilon)^2 dt\,dx\right\}^{1/2}$$

$$\leqslant C\sqrt{\varepsilon}\to 0 \quad (\varepsilon\to 0).$$

Since f is arbitrary, we immediately obtain $u_1(t, x)\equiv u_2(t, x)$ and complete the proof of the theorem.

§ 6. Some Examples and Propositions

A simple example that satisfies assumptions (HA), (HB1) and (HB2) is the function $|s|^p s$ with $p>0$, which has one point of degeneracy. However many interesting examples can be illustrated which have more than one point of degeneracy, even infinite number of degenerate intervals.

Example 1. Let $\alpha(s)$ be a C^∞ function with the property that $\alpha(s)>0$ in $(-1, 1)$, supp $\alpha(s)=[-1, 1]$ and $\int_{-1}^{1}\alpha(s)ds=1$. Let E be the union of all sets of the form $[4k-1, 4k+1]$ with k any integer. Set

$$f(s)=\int_{\mathbf{R}}\alpha(4s-4\sigma)\chi_E(\sigma)d\sigma.$$

If $a_1(s)=f(s)$, $a_2(s)\equiv 1$, then $a(s)=a_1(s)a_2(s)$ satisfies assumption (HA) with $p_1=p_2=0$, $\alpha_{11}=\frac{1}{32}$, $\alpha_{12}=\frac{1}{4}$, $\alpha_{21}=\alpha_{22}=1$.

Example 2. Let

$$g(s)=\alpha_0(|s|-\beta_0)_+^p\equiv\begin{cases}\alpha_0(|s|-\beta_0)^p, & |s|>\beta_0,\\ 0, & |s|\leqslant\beta_0,\end{cases}\quad (\alpha_0, \beta_0, P>0).$$

Then $g(s)$ satisfies assumption (HA).

Moreover, if we take $b(s)=f(s)$ or $g(s)$, then $B(s)$ satisfies assumptions (HB1) and (HB2).

Now we give an assumption which is stronger than (HA) but easy to apply.

(HA)′. There exists a constant $M_1>0$ such that for $|s|\geqslant M_1$

$$\alpha_{31}|s|^{p+1}\leqslant|A(s)|\leqslant\alpha_{32}|s|^{p+1},$$

where α_{32}, $\alpha_{32}>0$, $p=p_1+p_2$.

The following proposition shows the relationship between assumptions (HA) and (HA)′.

Proposition 6.1. (HA) *implies* (HA)′.

Proof For simplicity, we discuss for $s>0$. Choose $\alpha_0>0$ such that

$$A_1(M)\leqslant\frac{\alpha_0\alpha_{12}}{p_1+1}M^{p_1+1}.$$

Then for $s\geqslant M$,

$$A_1(s)=\int_0^s a_1(\sigma)d\sigma=A_1(M)+\int_M^s a_1(\sigma)d\sigma$$

$$\leqslant A_1(M)+\int_0^s\alpha_{12}\sigma^{p_1}d\sigma$$

$$\leqslant \frac{\alpha_0\alpha_{12}}{p_1+1} s^{p_1+1} + \frac{\alpha_{12}}{p_1+1} s^{p_1+1}$$
$$= \frac{(\alpha_0+1)\alpha_{12}}{p_1+1} s^{p_1+1}.$$

For $\lambda>1$, $|s|\geqslant\lambda M$,

$$A(s)=\int_0^s a(\sigma)d\sigma\geqslant\int_{\frac{s}{\lambda}}^s a(\sigma)d\sigma=\int_{\frac{s}{\lambda}}^s a_1(\sigma)a_2(\sigma)d\sigma$$
$$\geqslant\alpha_{21}\left|\frac{s}{\lambda}\right|^{p_2}\int_{\frac{s}{\lambda}}^s a_1(\sigma)d\sigma$$
$$=\alpha_{21}\frac{s^{p_2}}{\lambda^{p_2}}\left[A_1(s)-A_1\left(\frac{s}{\lambda}\right)\right]$$
$$\geqslant\frac{\alpha_{21}}{\lambda^{p_2}}s^{p_2}\left[\alpha_{11}s^{p_1+1}-\frac{(\alpha_0+1)\alpha_{12}}{p_1+1}\left(\frac{s}{\lambda}\right)^{p_1+1}\right]$$
$$=\frac{\alpha_{21}}{\lambda^{p_2}}\left[\alpha_{11}-\frac{(\alpha_0+1)\alpha_{21}}{p_1+1}\cdot\frac{1}{\lambda^{p_1+1}}\right]s^{p+1}.$$

The first inequality of the conclusion follows by setting

$$\lambda=1+\left[\frac{\alpha_{21}(\alpha_0+1)}{\alpha_{11}(p_1+1)}\right]^{\frac{1}{p_1+1}},\quad \alpha_{31}=\frac{\alpha_{11}\alpha_{21}}{2\lambda^{p_2}},\quad M_1=\lambda M.$$

We can prove the second inequality by the same method and hence complete the proof of the proposition.

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