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THE CONNES SPECTRUM OF FIRITE DIMENSIONAL HOPF ALGEBRA

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Abstract

Let H be a finite-dimensional Hopf algebra and A an H-module algebra. Criteria for the smash product $A\#H$ to be prime are obtained. In particular, a correct statement of S. Montgomery's question is given.

The Arveson-Connes theory of the spectrum of actions of an abelian group was very successful and played a vital role in the structure analysis of a factor of type III [6]. When A is a ring graded by a finite group G, in [4], S. Montgomery and D. Passman introduced and studied the Connes subgroup of G, and obtained criteria for $A\#G^*$ to be prime or simple. If G is a finite abelian subgroup of automorphisms group of the k-algebra A and the field k contains $|G|^{-1}$ and all $|G|^{th}$ roots of unity, then the skew group ring AG has a natural smash product structure $A\#k[\hat{G}]^*$, where $\hat{G}$ is the dual group of G. The Connes subgroup of $\hat{G}$ determines when AG is prime. S. Montgomery conjectured that when A is a prime k-algebra with a finite group of automorphisms and $|G|^{-1}\in k$, AG is prime if and only if $\mathrm{Irr}(G)=\Gamma$, where $\mathrm{Irr}(G)=\{$all irreducible reperesentations of $G\}$ and $\Gamma=\{\rho\in\mathrm{Irr}(G)/B_\rho\neq 0$, for all G-hereditary subalgebras $B\}$.

In [2]. the author obtained a criterion for the skew group ring AG to be prime for certain finite non-abelian groups G. This result gives a negative answer for the sufficiency of Montgomery's question. The positiue answer for the necessity was obtained by D. Passman [5]. The Corollary 5 will give a correct statement of the question.

Let H be a Hopf algebra over a field k, with comultiplication Δ: $H\to H\otimes H$, $\Delta(h)=\sum_{(h)} h_1\otimes h_2$, counit: ε: $H\to k$. Let A be a k-algebra with identity1_A. For readers unfamiliar with Hopf algebra, the basicreferece is Sweedler's book [9]. We say that A is an H-module algebra if

i) A is an H-module, the action of H on A is denotedby $h\cdot a$,

ii) $h\cdot(ab)=\sum_{(h)}(h_1\cdot a)(h_2\cdot b)$, where $\Delta(h)=\sum_{(h)} h_1\otimes h_2$:

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iii) $h\cdot 1_A=\varepsilon(h)1_A$.

Given a Hopf algebra H and an H-module algebra A, we can form the smash product $A\#H$ which is a k-algebra defined as follows:

i) As a vector space, $A\#H=A\otimes H$, $a\otimes h$ will be written $a\#h$;

ii) Multiplication is defined by

$$(a\#h)(b\#g)=\sum_{(h)}a(h_1\cdot b)\#h_2g,\quad \Delta(h)=\sum_{(h)}h_1\otimes h_2.$$

It is easy to verify that A and H can be algebraically embedded in $A\#H$ via $a\to a\#1$ and $h\to 1\#h$ respectively. So we can identify A with $A\#1$ and H with $1\#H$.

Given an H-module algebra A, A is a left $A\#H$-module via:

$$(\sum_n a_h\#h)\cdot b=\sum_h a_h(h\cdot b)\ \text{ for any } b\in A.$$

The H-invariants subalgebra A^H of A is defined as follows:

$$A^H=\{a\in A/h\cdot a=\varepsilon(h)a,\ \text{for all } h\in H\}.$$

A subset S of A is called H-stable if $h\cdot S\subseteq S$ for all $h\in H$. A subalgebra (without 1_A) B of A is called a hereditary subalgebra if $B=RL$ where R is a non-zero H-stable right ideal of A and L is a non-zero H-stable left ideal of A.

We denote $\mathrm{Irr}(H)=\{$all irreducible reperesentations of $H\}$. Since every finite-dimensional Hopf algebra is Frobenius [8, p.78], any irreducible H-module is isomorphic to a minimal left ideal of H. For any hereditary subalgebra B, B can be decomposed as a direct sum, $B=\bigoplus_{\tau\in Irr(H)}B_\tau$, where B_τ are the homogeneous components of B corresponding irreducible reperesntations τ. We define the Connes spectrum $\Gamma_A(H)$ as follows:

$\Gamma_A(H)=\{\rho\in\mathrm{Irr}(H)/B_\rho\neq 0$, for all hereditary subalgebras $B\}$.

If $H=k[G]^*$ for finite group G, then kp_x, $x\in G$, are all minimal left ideals of $k[G]^*$ and $\mathrm{Irr}\ (k[G]^*)=\{\rho_x/x\in G\}$, where $\rho_x(p_y)1_k=\delta_{x,y}=p_y(x)$. Thus, we identify $\mathrm{Irr}(k[G]^*)=G$, and then Γ is the Connes subgroup of G when A is G-graded semiprime.

Now we are ready to prove the following

Theorem 1. *Let H be a finite-dimensional Hopf algebra over a field k, and let A be an H-module algebra. Then the smash product $A\#H$ is prime if and only if*

i) A is a faithful left $A\#H$-module and

ii) $f\cdot B\neq 0$ for all hereditary subalgebras B,

where f is a non-zero integral for H (i. e. $hf=\varepsilon(h)f$ for all $h\in H$).

Proof $(\Rightarrow)$ If $0\neq\sum_h a_h\#h,\ \in A\#H$, then $(\Sigma a_h\#h)(A\#f)\neq 0$ since $A\#H$ is prime and $A\#f$ is a non-zero leftideal. But

$$(\Sigma a_h\#h)(A\#f)=\{\sum_h a_h(h\cdot r)\#f/r\in A\}.$$

Thus, there exists $r\in A$ with

$$(\Sigma a_h\# h)\rightharpoonup r=\Sigma a_h(h\cdot r)\neq 0.$$

A is a faithful left $A\# H$-module.

Let $B=RL$ be a hereditary subalgebra. Then $L\# f$ is a non-zero left ideal of $A\# H$, and

$$0\neq (fR)(L\# f)=f\cdot(RL)\# f=f\cdot B\# f.$$

It follows that $f\cdot B\neq 0$ for any hereditary subalgebra B.

($\Leftarrow$) Let I be a non-zero ideal of $A\# H$, and let $0\neq\Sigma a_h\# h\in I$. By i) there exists $r\in A$ with $\Sigma a_h(h\cdot r)\neq 0$, and then

$$0\neq(\Sigma a_h(h\cdot f))\# f=(\Sigma a_h\# h)(r\# f)\in I.$$

Denote
$$L=\{r\in A/r\# f\in I\}.$$

We can see that L is a non-zero H-stable left ideal of A. By ii), $f\cdot L\neq 0$. Let $0\neq a\in f\cdot L$. Then $fa=a\# f\in I$ since $f\cdot L\subseteq A^H\cap L$.

Let J be a non-zero ideol of $A\# H$. Then there exists $0\neq b\in A^H$ with $Ab\# f\subseteq J$. Therefore.

$$IJ\supseteq(fa)(Ab\# f)=(f\cdot(aAb))\# f.$$

Since aAb is a hereditary subalgebra of A, $f\cdot(aAb)\neq 0$. Thus $IJ\neq 0$. This shows that $A\# H$ is prime.

Remark. From the proof of the theorem, we can see that the condition ii) in Theorem 1 can be replaced by the condition ii′) $h\cdot B\neq 0$ for all $0\neq h\in H$ and all hereditary subalgebras B.

When H is a semisimple Hopf algebra, we have the following result.

Corollary 2. *Let H be a semisimple Hopf algebra over a field k, and let A be an H-module algebra. Then the smash product $A\# H$ is prime if and only if*

i) A is a faithful left $A\# H$-module and

ii″) $\Gamma_A(H)=\mathrm{Irr}(H)$.

Proof Since H is a Frobenius algebra, every irreducible H-module M is isomorphic to a minimal left ideal Hh of H. If $h\cdot B\neq 0$ for a hereditary subalgebra B, then there exists $b\in B$ with $h\cdot b\neq 0$, and thus we have H-module isomorphisms $M\cong Hh\cong(Hh)\cdot b\subseteq B$. It follows that $\Gamma_A(H)=\mathrm{Irr}(H)$ by Theorem 1 and the above Remark. We have proved the necessity.

Conversely, let B be any hereditary subaglebra. For $s\in\mathrm{Irr}(H)$, $B_s\neq 0$ by the condition ii″). Since H is semisimple, by the result of Sweedler [7, p.329] we see that H contains an integral f with $s(f)=1$. For any $a\in A^H$, $f\cdot a=s(f)a=a$. Thus, $B_s=f\cdot B$. By Theorem 1, $A\# H$ is prime.

When $H=k[G]^*$ for a finite group G and A is G-graded k-algebra, $A=\bigoplus_{x\in G}A(x)$, $A(x)A(y)\subseteq A(xy)$ for all $x,y\in G$. then $p_x\cdot B\neq 0$ for all $x\in G$ and all hereditary subalgebras B implies that A is a faithful $A\#k[G]^*$-module. Indeed, for any $0\neq a\in A(x)$, aA is a hdreditary subalgebra of A. Then $aA(y)=p_{xy}\cdot(aA)\neq 0$. By [4, Lemma 2.5(2)], A is a faithful left $A\#k[G]^*$-module.

Also, we can see that ii′) and ii″) are equivalent, since, for any $x\in G$, $p_x\cdot B=B\cap A(x)=B_p$ for all hereditary subalgebras B.

Therefore, we have the following result.

Corollary 3 ([4, Corollary 2.8]). *Let A be a G-graded k-algebra for a finite group G. Then $A\#k[G]^*$ is prime if and only if $\Gamma_A(k[G]^*))=\mathrm{Irr}(k[G]^*)$.*

Since a faithful grading is a nondegererate grading, any non-zero one side grading ideal intersects $A(1)$ non-trivially if i) holds. Thus, if the condition i) holds, then the condition ii) is equal to that $A(1)$ is prime. Hence we have

Corollary 4 ([3, Theorem 2.10 (3)⇔(4)]). *$A\#k[G]^*$ is prime if and only if $A(1)$ is prime and the grading is faithful.*

If $H=k[G]$ for finite group G and $|G|^{-1}\in k$, then H is semi-simple. By Corollary 2, we obtain a correct statement of *S*. Montgomery's conjecture.

Corollary 5. *Let A be a k-algebra with a finite group G of automorphisms and $|G|^{-1}\in k$. Then the skew group ring AG is prime if and only if A is a faithful AG-module and $\Gamma_A(r[G])=\mathrm{Irr}(k[G])$.*

Next, we consider connection between A^H and $A\#H$. If H is a finite-nsional Hopf algebra, then the following properties go down from $A\#H$ to A^H: dime being prime, primitive, simple Artinian see ([1, Proposition 2.4 (1), Theorem 3.3(2), Theorem 2.9(1) 1]. Furthermore, if H is semisimple, the simpleness goes down (see [1, Theorem 2.7]). However, these properties do not go up in general, as the following example shows (see [2] for details).

Example 6. Let $A=M_2(k)$, k being the complex number field, and let w be a cube root, n and h the inner automorphisms induced by $\begin{pmatrix} w & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$ respectively. Then $G=\langle n, h\rangle=S_3$. Only A is the hereditary subalgebra of A. $\Gamma_A(k[G])=\mathrm{Irr}(k[G])$. However,

$$\left(\begin{pmatrix} 0 & 1\\ 0 & 0\end{pmatrix}\#(1+wn+w^2n^2)\right)\rightharpoonup A=0;$$

thus, A is not a faithful AG-module. It follows that AG is not prime, by Theorem 1. But. $A^H=k$.

The general going up problem appears intractable. However, something can be said. We close this papar by the following

Proposition 7. *Let H be a semisimple Hopf algebra over a field k (with an integral f such that $\varepsilon(f)=1$), and let A be an H-module algebra. Then enery non-*

zero ideal of $A\# H$ intersects $A^H\# f$ non-trivially if and only if A is a faithful left $A\# H$-module and $f\cdot L\neq 0$ for all non-zero H-stable left ideals L of A. In particular, if $A\# H$ is prime, then every non-zero ideal of $A\# H$ intersects $A^H\# f$ non-trivilly.

Proof ($\Rightarrow$) If $I=\operatorname{ann}_{A\# H}(A)\neq 0$, then $I\cap(A^H\# f)\neq 0$ by the assumption. Let $0\neq a\# f\in I$. Then

$$a=\varepsilon(f)a=(a\# f)\rightharpoonup 1_A\in(a\# f)\rightharpoonup A=0,$$

a contradiction. Thus A is a faithful $A\# H$-module.

Let L be a non-zero H-stable left ideal of A. Then $(L\# f)A$ is a non-zero ideal of $A\# H$, and thus $(L\# f)A\cap A^H\# f\neq 0$. Since $f^2=f$, $f(A^H\# f)f=A^H\# f$. It follows that $f(L\# f)Af\neq 0$. But

$$f(L\# f)Af=((f\cdot L)\# f)Af=(f\cdot L)A^H\# f.$$

Thus, $f\cdot L\neq 0$ for all non-zero H-stable left ideal L of A.

($\Leftarrow$) Let I be any non-zero ideal of $A\# H$. By the assumption of faithfulness of $A\# H$-module A, $I\supseteq L\# f$ for some non-zero H-stable left ideal of A (it was shown in the proof of Theorem 1). By the assumption of $f\cdot L\neq 0$, $0\neq f\cdot L\# f\subseteq I\cap(A^H\# f)$. This shows that every non-zero deal of $A\# H$ intersects $A^H\# f$ non-trivially.

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