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INVERSE THEOREMS IN L^p FOR SOME MULTIDIMENSIONAL POSITIVE LINEAR OPERATORS**

ZHOU DINGXUAN (周定轩)*

Abstract

Let $\{L_n\}_{n\in N}$ be positive linear operators in $L_p(I)$, $I=[0, 1]$ or $[0, \infty)$. This paper considers their variants in $L_p(I\times I)$

$$L_{n,m}(F; x, y)=L_n(L_m(F(u, v); y); x)=L_m(L_n(F(u, v); x); y), n, m\in N.$$

The characterization problem for these operators is solved which gives the inverse theorems in L_p for multidimensional Bernstein type operators.

§ 1. Introduction

Many papers have been devoted to the problem of extending Bernstein polynomials or Szász–Mirakjan operators to L_p space. Saturation and characterization are the most important in the research of these operators[1-14].

In this paper we want to solve the characterization problem for these operators in the multidimensional case.

We carry over our results only in two dimension. The higher dimensional case can be treated. similarly.

In the following discussion we always set $1\leqslant p<\infty$, $0<\alpha<1$, $I=[0, 1]$ or $[0, \infty)$.

For our positive linear operators $\{L_n\}$, our variant in two dimension is the operator

$$L_{nm}(F; x, y)=L_n(L_m(F(u, v); y); x)=L_m(L_n(F(u, u); x); y), \quad F\in L_p(I\times I)$$

with two parameters $m, n\in N$.

Let
$$\varphi(x)=\sqrt{x(1-x)} \text{ or } \sqrt{x},$$
$$D=\{g\,|\,g\in L_p(I), g'\in A.C_{\text{loc.}}, \varphi^2g''\in L_p(I)\}.$$

For $g\in D$, let $S(g)=(g, \varphi^2g'')$,
$$\|S(g)\|_p=\|g\|_p+\|\varphi^2g''\|_p.$$

For $f\in L_p(I)$,

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* Department of Mathematics, Zhejiang University, Advanced Institute of Mathematics, Hangzhou, Zhejiang, China.

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$$K(f, t)=\inf_{g\in D}\{\|f-g\|_p+t\|S(g)\|_p\},\ t>0$$

is the so-called K-functional.

For $0\leqslant h\leqslant 1$, let

$$h^*=\frac{h^2}{1+h^2} \text{ or } h^2,$$

$$h^{**}=\frac{1}{1+h^2} \text{ or } \infty$$

responding to $I=[0, 1]$ or $[0, \infty)$.

For $f\in L_p(I)$ and $0\leqslant t\leqslant 1$, let

$$v(f, t)=\sup_{0<h<t}\|\Delta^2_{h\varphi}f\|_{L_p(h^*, h^{**})},$$

where $\Delta^2_s f(x)=f(x-s)-2f(x)+f(x+s)$.

For $F\in L_p(I\times I)$, let

$$F_x(y)=F(x, y),\ y\in I,$$

$$F^y(x)=F(x, y),\ x\in I$$

be the sectional functions.

Let us mention that the analogues of our results can also be obtained for $C([0, 1]\times[0, 1])$.

§2. Lemmas

Lemma 1[9]. *There exists a constant K such that*

$$\|f'\|_p\leqslant K\|S(f)\|_p$$

holds for any $f\in D$.

Lemma 2[9]. *There is a constant K such that*

$$\frac{1}{K}v(f, t)\leqslant K(f, t^2)\leqslant K\int_0^t\frac{v(f, u)}{u}du$$

holds for all $f\in L_p(I)$ and $0\leqslant t\leqslant t_0$.

Lemma 3. *For $F\in L_p(I\times I)$, the following statements are equivalent:*

(I) $$\|K(F_x, h)\|_{L_p(I)}=O(h^\alpha),$$
$$\|K(F^y, h)\|_p=O(h^\alpha),$$

(II) $$\|v(F_x, h)\|_p=O(h^{2\alpha}),$$
$$\|v(F^y, h)\|_p=O(h^{2\alpha}).$$

It must be interesting to characterize the above statements by

$$\|\Delta^2_{h\varphi}F^y\|_{L_p([h^*, h^{**}]\times I)} \text{ and } \|\Delta^2_{h\varphi}F_x\|_{L_p(I\times[h^*, h^{**}])}.$$

Remark. The retstriction of $[h^*, h^{**}]$ is natural since $x\pm h\varphi(x)\in I$ iff $x\in[h^*, h^{**}]$.

Proof (I)$\Rightarrow$(II). By Lemma 2, it is trivial.

(II)$\Rightarrow$(I). Suppose $\|v(F_x, t)\|_p\leqslant Mt^{2\alpha}$ for any $t_0\geqslant t\geqslant 0$. By the generalized

Minkowski inequality, we have for $0\leqslant h\leqslant t_0$

$$\|K(F_x,h^2)\|_p\leqslant\left\{\int_I\left(K\int_0^h\frac{v(F_x,u)}{u}du\right)^p dx\right\}^{1/p}$$
$$\leqslant K\int_0^h\left(\int_I\left(\frac{v(F_x,u)}{u}\right)^p dx\right)^{1/p}du$$
$$\leqslant K\int_0^h\frac{Mu^{2\alpha}d}{u}du=\frac{KM}{2\alpha}h^{2\alpha},$$

and
$$\|K(F^y,h)\|_p=O(h^\alpha)$$
can be easily obtained.

§ 3. Main Result

Definition. *A real sequence $\{\lambda_n\}$ is called intermediate, if $\lambda_n\searrow 0$ and $\lambda_1\leqslant 1$ and for all $b>0$, there exists $a>0$ and ψ: $N\cup\{0\}\to N$, $\psi(0)=1$, $\psi(n)\nearrow\infty$ with*

$$a\leqslant\frac{\lambda_{\psi(n+1)}}{\lambda_{\psi(n)}}\leqslant b.$$

We set up our theorem more generally.

Theorem. *For positive linear operators $\{L_n\}$ in L_p, suppose*

(1) $L_n(L_m(F;y);x)=L_m(L_n(F;x);y)$ *for* $F\in L_p(I\times I)$;

(2) $L_n 1=1$;

$$L_n f\xrightarrow{L_p} f \text{ for any } f\in L_p(I);$$

(3) $\sup\limits_n\|L_n\|\leqslant M$;

(4) $\|S(L_nf)\|_p\leqslant M\lambda_n^{-1}\|f\|_p$ *for* $f\in L_p(I)$;

(5) $\|S(L_nf)\|_p\leqslant M\|S(f)\|_p$ *for* $f\in D$;

(6) $\|L_nf-f\|_p\leqslant M\lambda_n\|S(f)\|_p$ *for* $f\in D$.

Here λ_n is an intermediate sequence and M is a constant independant of f, n and λ_n. Then for $F\in L_p(I\times I)$,

$$\|L_{nm}(F)-F\|_p=O(\lambda_n^\alpha+\lambda_m^\alpha)\ (*)$$

is equivalent to

$$\Big\|\sup_{0\leqslant h\leqslant t}\|\Delta^2_{h\varphi}F^y\|_{L_p[h^*,h^{**}]}\Big\|_p=O(t^{2\alpha})\text{ and }\Big\|\sup_{0\leqslant h\leqslant t}\|\Delta^2_{h\varphi}F_x\|_{L_p[h^*,h^{**}]}\Big\|_p=O(t^{2\alpha}),\ (**)$$

which is also equivalent to the statements in Lemma 3.

Proof

$$L_{nm}(F;x,y)-F(x,y)$$
$$=L_m(L_n(F(u,v)-F(x,v);x);y)+L_m(F_x;y)-F_x(y)$$
$$=I_1+I_2.$$

Now suppose (**) holds. By Lemma 3, we can assume

$$\|K(F_x,h)\|_p\leqslant M_1h^\alpha\text{ for }h>0.$$

For any fixed $x\in I$ and any $g\in D$

$$\|L_m(F_x)-F_x\|_p\leqslant(M+1)\|F_x-g\|_p+M\lambda_m\|S(g)\|_p.$$

Hence $$\|L_m(F_x)-F_x\|_p\leqslant(M+1)K(F_x,\lambda_m).$$

Thus $$\|I_2\|_{L_p(I\times I)}\leqslant(M+1)\|K(F_x,\lambda_m)\|_p\leqslant(M+1)M_1\lambda_m^\alpha.$$

By $\|L_m\|\leqslant M$, we have

$$\|I_1\|_{L_p(I\times I)}\leqslant\Big(\int_I M^p\int_I|L_n(F(u,y)-F(x,y);x)|^p dy\,dx\Big)^{1/p}$$
$$\leqslant M\Big(\int_I\|L_n(F^y)-F^y\|_p^p\,dy\Big)^{1/p}$$
$$\leqslant M(M+1)\|K(F^y,\lambda_n)\|_p\leqslant M(M+1)M_1\lambda_n^\alpha.$$

Here the fact

$$\|L_n(F^y)-F^y\|_p\leqslant(M+1)K(F^y,\lambda_n)$$

can be proved in the same way as for F_x. Therefore we have proved (*).

Now suppose (*) holds. We only need to prove (I) of Lemma 3.

Step 1: Since $|F|^p\in L_1(I\times I)$, we have a set $E\subset I$ with $mE=0$ such that for any $x\in I-E$,

$$|F_x|^p\in L_1(I).$$

Therefore $$\|L_m(F_x)-F_x\|_{L_p(I)}\to 0\quad(m\to\infty).$$

Thus $$\lim_{m\to\infty}\|I_2\|_p^p=\int_{I-E}\lim_{m\to\infty}\|L_m(F_x)-F_x\|_{L_p(I)}^p\,dx=0.$$

Here we should note that for $x\in I-E$

$$\int_I|L_m(F_x)(y)-F_x(y)|^p dy\leqslant 2^p(\|F_x\|_p^p+M^p\|F_x\|_p^p)$$

while the last term is in $L_1(I)$.

In the same way

$$\int_I\int_I|L_n(F^v)(x)-F^v(x)|^p\,dx\,dv\leqslant 2^p(M^p+1)\|F\|_{L_p(I\times I)}^p.$$

We know that for almost every $x\in I$

$$L_n(F^v)(x)-F^v(x)\in L_p(I)\text{ with respect to }v.$$

Hence $$L_m(L_n(F^v)(x)-F^v(x))(y)\xrightarrow{L_p}L_n(F^y)(x)-F^y(x)\quad(m\to\infty).$$

Thus we have

$$\lim_{m\to\infty}\|I_1\|_{L_p(I\times I)}^p=\int_I\lim_{m\to\infty}\int_I|L_m(L_n(F^v)(x)-F^v(x))(y)|^p dy\,dx$$
$$=\int_I\int_I|L_n(F^y)(x)-F^v(x)|^p dy\,dx$$

since

$$\int_I|L_m(L_n(F^v)(x)-F^v(x))(y)|^p dy\leqslant M^p\int_I|L_n(F^y)(x)-F^y(x)|^p dy$$

which is fixed in $L_1(I)$.

Now suppose

$$\|L_{nm}(F)-F\|_p\leqslant M_1(\lambda_n^\alpha+\lambda_m^\alpha)$$

for all $m, n\in N$. Let $m\to\infty$. By the above discussion we have

$$\left(\int_I\int_I |L_n(F^y)(x)-F^y(x)|^p dy\, dx\right)^{1/p}\leqslant M_1\lambda_n^\alpha.$$

Step 2: For any $1\geqslant h>0$, any fixed $y\in I$ and $g\in D$, we have

$$\begin{aligned}K(F^y, h)&\leqslant\|F^y-L_n(F^y)\|_p+h\|S(L_n(F^y-g))\|_p+h\|S(L_ng)\|_p\\&\leqslant\|F^y-L_n(F^y)\|_p+hM\lambda_n^{-1}\|F^y-g\|_p+hM\|S(g)\|_p,\end{aligned}$$

which implies

$$K(F^y, h)\leqslant\|F^y-L_n(F^y)\|_p+hM\lambda_n^{-1}K(F^y, \lambda_n).$$

Denoting

$$K^*(h)=\left(\int_0^1\left(K(F^y, h)\right)^p dy\right)^{1/p}=\|K(F^y, h)\|_p,$$

Which is obviously monotone non-decreasing function, we have

$$K^*(h)\leqslant M_1\lambda_n^\alpha+Mh\lambda_n^{-1}K^*(\lambda_n).$$

Step 3: Let $b=\min\{(2M)^{1/(\alpha-1)}, 1/2\}$.

we can choose $a>0$ and ψ astin the definition of intermediate sequence such that

$$a\leqslant\frac{\lambda_{\psi(n+1)}}{\lambda_{\psi(n)}}\leqslant b.$$

For $m\in N\cup\{0\}$, let $t_m=\lambda_{\psi(m)}$ and $n=\psi(m)$.

$$\begin{aligned}K^*(t_{m+1})&\leqslant M_1t_m^\alpha+M\frac{t_{m+1}}{t_m}K^*(t_m)\\&\leqslant t_{m+1}^\alpha\max\left\{2M_1\left(\frac{t_m}{t_{m+1}}\right)^\alpha, 2M\left(\frac{t_{m+1}}{t_m}\right)^{1-\alpha}t_m^{-\alpha}K^*(t_m)\right\}\\&\leqslant t_{m+1}^\alpha\max\{2M_1a^{-\alpha}, t_m^{-\alpha}K^*(t_m)\}.\end{aligned}$$

Thus we have by induction

$$K^*(t_m)\leqslant(2M_1a^{-\alpha}+\lambda_1^{-\alpha}K^*(\lambda_1))t_m^\alpha.$$

From $b\leqslant1/2$, we know that $t_{m+1}\leqslant t_m/2$.

For any $\lambda_1\geqslant h>0$, we can choose m such that

$$t_{m+1}<h\leqslant t_m,$$

$$\begin{aligned}K^*(h)\leqslant K^*(t_m)&\leqslant(2M_1a^{-\alpha}+\lambda_1^{-\alpha}K^*(\lambda_1))\left(\frac{t_m}{t_{m+1}}\right)^\alpha t_{m+1}^\alpha\\&\leqslant(2M_1a^{-\alpha}+\lambda_1^{-\alpha}K^*(\lambda_1))a^\alpha h^\alpha.\end{aligned}$$

Here the constant only depends on α and λ, We have

$$K^*(h)=\|K(F^y, h)\|_p=O(h^\alpha).$$

$$\|K(F_x, h)\|_p=O(h^\alpha)$$

can be obtained similarly.

The proof is complete.

Remark. The theorem is also valid for non-positive linear operators. Noting that $\lambda_n=1/n$ is intermediate, we have

Corollary. *For* $F\in L_p(I\times I)$,

$$\|L_{nm}F - F\|_{L_p(I\times I)} = O\Big(\frac{1}{n^{\alpha}} - \frac{1}{m^{\alpha}}\Big)$$

if and only if

$$\|\sup_{0<h\le t}\|\Delta^2_{h\varphi}F^y\|_{L_p[h^*,h^{**}]}\|_p = O(t^{2\alpha}) \text{ and } \|\sup_{0\le h\le t}\|\Delta^2_{h\varphi}F_x\|_{L_p[h^*,h^{**}]}\|_p = O(t^{2\alpha}).$$

Here $\{L_n\}$ can be one of the following classes of operators:

Kantorovich operators [2, 4, 9];

Szász–Mirakjan–Kantorovich operators [6];

Meyer–König and Zeller Type operators [8];

Baskakov–Kantorovich operators [9];

Bernstein–Durrmeyer operators [12];

Szász–Mirakjan type operators [14].

Thus we see that L_p-approximation by positive linear operators in two dimension is parallel to the results in one dimension.

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