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WEAK TYPE (1, 1) BOUNDEDNESS OF RIESZ TRANSFORM ON POSITIVELY CURVED MANIFOLDS

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Abstract

For complete Riemannian manifold M, it is proved that $\nabla(-\Delta)^{-1/2}$ is bounded from $L^1(M)$ to weak-$L^1(M)$ if $\mathrm{Ric}(M)\geqslant 0$.

§ 1. Introduction

Let M be a complete Riemannian manifold, Δ its Laplacian, ∇ gradient operator, $\{H_t\}_{t>0}$ heat-diffusion semigroup, $\{h_t\}_{t>0}$ heat kernel, $\Delta\triangleq\Delta+\left(\frac{\partial}{\partial t}\right)^2$, $\nabla\triangleq\left(\nabla, \frac{\partial}{\partial t}\right)$. Let $V_x(r)$ denote the volume of geodesic ball $B_x(r)$ with center x and radius r. Riesz transform $\nabla(-\Delta)^{-1/2}$ was introduced by R. S. Strichartz first, and earlier on, E. M. Stein had introduced Riesz transform on Lie groups in a different way[9,7]. Strichartz proved[9] its L^2-boundedness, Bakry proved [1] its L^p-boundedness $(2<p<\infty)$ under some additional conditions, Lohoue considered[4] the case when M is negatively curved. In this paper, we get

Theorem. $\nabla(-\Delta)^{-1/2}$ *is boanded from* $L^1(M)$ *to weak*-$L^1(M)$ *if* $\mathrm{Ric}(M)\geqslant 0$.

§ 2. Proof

At first, we give the following kernel representation theorem of $\nabla(-\Delta)^{-1/2}$.

Proposition 1. *For* $f\in\mathscr{D}_{1/2}$ *(the domain of* $(-\Delta)^{-1/2}$*)*, $\nabla(-\Delta)^{-1/2}(f)=L^2-\lim\limits_{\varepsilon\to 0} R_\varepsilon(f)$ *where*

$$\begin{cases} R_\varepsilon(f)(x)=\int_M K_\varepsilon(x, y)f(y)dy, \\ K_\varepsilon(x, y)=\Gamma^{-1}(1/2)\int_\varepsilon^{\varepsilon^{-1}} s^{1/2}\nabla_x h_s(x, y)ds. \end{cases} \tag{1}$$

Proof By spectral decomposition of $-\Delta$, we have $-\Delta=\int_0^{+\infty} t dE_t$. Thus

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$$(-\Delta)^{-1/2}(f)=\int_0^{+\infty} t^{-1/2}\,dE_t(f)$$

$$=L^2-\lim_{\varepsilon\to 0}\int_0^{+\infty}\left(\Gamma^{-1}(1/2)\int_\varepsilon^{\varepsilon^{-1}} s^{-1/2}e^{-st}\,ds\right)dE_t(f)$$

(by Lebesgue dominant convergence theorem)

$$=L^2-\lim_{\varepsilon\to 0}\int_\varepsilon^{\varepsilon^{-1}} \Gamma^{-1}(1/2)s^{-1/2}e^{-s\Delta}(f)\,ds.$$

By [9, Corollary 2.6], it is not hard to show

$$\nabla(-\Delta)^{-1/2}(f)=L^2-\lim_{\varepsilon\to 0}\nabla\int_\varepsilon^{\varepsilon^{-1}} \Gamma^{-1}(1/2)s^{-1/2}e^{-s\Delta}(f)\,ds$$

$$=L^2-\lim_{\varepsilon\to 0}\int_M f(y)\left(\int_\varepsilon^{\varepsilon^{-1}}\Gamma^{-1}(1/2)s^{-1/2}\nabla_x h_s(x,y)\,ds\right)dy.$$

Proposition 1 is proved.

Proposition 2. *If* $\operatorname{Ric}(M)\geqslant 0$, *then*

$$|\nabla_y K_\varepsilon(x,y)|\leqslant C_n V_x^{-1}(d(x,y))d^{-1}(x,y)\quad(\forall 0<\varepsilon<1),\tag{2}$$

where $d(x,y)$ is the geodesic distance between x and y.

To prove this proposition, we need some lemmas.

Lemma 3. *If* $\operatorname{Ric}(M)\geqslant -k\,(k\geqslant 0)$, *then*

$$|\nabla_x h_{t+s}(x,y)|\leqslant\int_M|\nabla_z h_t(z,y)|\,h_s(x,z)\,dz.\tag{3}$$

Proof For fixed $y\in M$, let $v(x,t)=\nabla_x h_t(x,y)$ and $u_s(x,t)=H_t(v(\cdot,s))(x)=\int_M h_t(x,z)v(z,s)\,dz$. Then, it is enough to prove $u_s\geqslant v(\cdot,\cdot+s)$. Now, let

$$F_s(x,t)=\max\{0,\ v(x,t+s)-u_s(x,t)\}$$

and $f_s(x)=\int_0^T\exp(-kt)F_s(x,t)\,dt$ where $T(>0)$ is arbitrary. Then

$F_s\geqslant 0$ (by definition of F_s)

$\lim_{t\to 0}F_s(x,t)=0$ (by [9, Theorem 3.5])

$\left(\Delta_x-\dfrac{\partial}{\partial t}\right)F_s(x,t)\geqslant -kF_s(x,t)$ (by [2, Lemma6]).

Thus

$$\Delta f_s(x)=\int_0^T\exp(-kt)\Delta F_s(x,t)\,dt$$

$$\geqslant\int_0^T\exp(-kt)\frac{\partial}{\partial t}F_s(x,t)\,dt-\int_0^T\exp(-kt)kF_s(x,t)\,dt$$

$$=\int_0^T\frac{\partial}{\partial t}(\exp(-kt)F_s(x,t))\,dt=\exp(-kT)F_s(x,T)\geqslant 0$$

and

$$\int_M f_s^2(x)\,dx\leqslant T\int_M\int_0^T\exp(-2kt)F_s^2(x,t)\,dx\,dt$$

$$\leqslant T\int_M\int_0^T v^2(x,t+s)\,dx\,dt+T\int_M\int_0^T u_s^2(x,t)\,dx\,dt$$

$$\leqslant 2T^2\sup_{0\leqslant t\leqslant T}\|v(\cdot,t+s)\|_2^2$$

because $\{H_t\}_{t>0}$ is a contract semigroup and

$$\int_M |\nabla_x h_t(x, y)| dy \leqslant C_{x, s} \quad (\forall s<t<s^{-1}).$$

So, f_s is a nonnegative L^2-subharmonic function on M, and it must be constant. Thus $F_s=0$, i. e., $u_s \geqslant v(.,.+s)$.

Lemma 4. *If* $\mathrm{Ric}(M) \geqslant -k (k\geqslant 0)$, *then*

a) $\int_M h_t^2(x, y) dy \leqslant C_n V_x^{-1}(t^{1/2}/2) \exp(C_n kt)$,

b) $\int_{M-B_x(\rho)} h_t^2(x, y) dy \leqslant C_n V_x^{-1}(t^{1/2}/2) \exp(C_n kt - \rho^2/3t)$,

c) $\int_M |\nabla_z h_t(x, z)|^2 dz \leqslant C_n V_x^{-1}(t^{1/2}) \exp(C_n kt)$,

d) $\int_{M-B_x(\rho)} |\nabla_z h_t(x, z)|^2 dz \leqslant C_n (t^{-1}+\rho^{-2}) V_x^{-1}(t^{1/2}/2) \cdot \exp(C_n kt - \rho^2/3t)$.

Proof a) and b) can be obtained from the proof of [6, Theorem 4.6]. c) can be obtained from a) and

$$\int_M |\nabla_z h_t(x, z)|^2 dz = \int_M h_t(x, z) \Delta_z h_t(z, x) dz,$$

$$\left|\int_M h_t(x, z) \Delta_z^k h_t(x, z) dz\right| \leqslant C_k t^{-k} \int_M h_{t/2}^2(x, z) dz$$

(by [3, Lemma 7]).

Similarly, d) can be obtained.

Now, we have

Proposition 5. *If* $\mathrm{Ric}(M) \geqslant -k (k\geqslant 0)$, *then*

$$|\nabla_x h_t(x, y)| \leqslant C_n t^{-1/2} V_x^{-1/2}(t^{1/2}/3) V_y^{-1/2}(t^{1/2}/3) \cdot \exp(C_n kt - d^2(x, y)/24t). \tag{4}$$

Proof When $t \geqslant d^2(x, y)/8$, by Lemma 3 and Lemma 4

$$\begin{aligned} |\nabla_x h_t(x, y)| &\leqslant \left(\int_M h_{t/2}(x, z) dz\right)^{1/2} \left(\int_M |\nabla_z h_{t/2}(z, y)|^2 dz\right)^{1/2} \\ &\leqslant C_n V_x^{-1/2}(t^{1/2}/3) \exp(C_n kt) t^{-1/2} V_y^{-1/2}(t^{1/2}/3) \exp(C_n kt) \\ &\leqslant C_n t^{-1/2} V_x^{-1/2}(t^{1/2}/3) V_y^{-1/2}(t^{1/2}/3) \exp(C_n kt - d^2(x, y)/24t). \end{aligned} \tag{5}$$

When $t \leqslant d^2(x, y)/8$, let $\tilde{d} = d(x, y)/2$. Then

$$\begin{aligned} &\int_M h_{t/2}(x, z) |\nabla_z h_{t/2}(z, y)| dz \\ &= \left(\int_{B_y(\tilde{d})} + \int_{M-B_y(\tilde{d})}\right) h_{t/2}(x, z) |\nabla_z h_{t/2}(z, y)| dz \\ &\triangleq I + II. \end{aligned}$$

For II, we have

$$\begin{aligned} II &\leqslant \left(\int_M h_{t/2}^2(x, z) dz\right)^{1/2} \left(\int_{M-B_y(\tilde{d})} |\nabla_z h_{t/2}(z, y)|^2 dz\right)^{1/2} \\ &\leqslant C_n V_x^{-1/2}(t^{1/2}/3) V_y^{-1/2}(t^{1/2}/3) \exp(C_n kt - d^2(x, y)/24t) t^{-1/2}. \end{aligned} \tag{6}$$

For I, because $B_y(\tilde{d})\subset M-B_x(d-\tilde{d})$, we have

$$I\leqslant\left(\int_{B_y(\tilde{d})} h^2_{t/2}(x,z)dz\right)^{1/2}\left(\int_{B_y(\tilde{d})} |\nabla_z h_{t/2}(z,y)|^2dz\right)^{1/2}$$

$$\leqslant\left(\int_{M-B_x(d-\tilde{d})} h^2_{t/2}(x,z)dz\right)^{1/2}\left(\int_M |\nabla_z h_{t/2}(z,y)|^2dz\right)^{1/2}$$

$$\leqslant C_n t^{-1/2}V_x^{-1/2}(t^{1/2}/3)V_y^{-1/2}(t^{1/2}/3)\exp(C_n kt+d^2(x,y)/24t). \quad (7)$$

From (5), (6) and (7), we can get (4) easily.

As a corollary of Proposition 5, we have

Lemma 6. *If* $\mathrm{Ric}(M)\geqslant 0$, *then*

$$|\nabla_x\nabla_y h_t(x,y)|\leqslant C_n t^{-1}V_x^{-1}(t^{1/2})\exp(-d^2(x,y)/100t). \quad (8)$$

Proof At first, we have (see[p. 12; 6])

$$V_x(r)/V_x(R)\geqslant(r/R)^n(0<r<R<\infty). \quad (9)$$

Now, let

$$A_1=\int_M |\nabla_x h_t(x,z)|^p dz,$$

$$A_2=\int_{M-B_x(\rho)} |\nabla_x h_t(x,z)|^p dz.$$

By Proposition 5 and (9)

$$A_1\leqslant C_{n,p}\int_M \exp(-pd^2(x,y)/25t)dyV_x^{-p}(t^{1/2})t^{-p/2}$$

$$=C_{n,p}t^{-p/2}V_x^{-p}(t^{1/2})\left(\sum_1^\infty\int_{2^{k-1}t<d^2\leqslant 2^kt}+\int_{d^2\leqslant t}\cdot\right)$$

$$\leqslant C_{n,p}t^{-p/2}V_x^{-p+1}(t^{1/2})\left(1+\sum_1^\infty\exp(-2^{k-1}p/25)\cdot 2^{kn}\right)$$

$$\leqslant C_{n,p}t^{-p/2}V_x^{-p+1}(t^{1/2}) \quad (10)$$

$$A_2\leqslant C_{n,p}t^{-p/2}V_x^{-p}(t^{1/2})\int_{M-B_x(\rho)}\exp(pd^2(x,z)/25t)dz$$

$$\leqslant C_{n,p}t^{-p/2}V_x^{-1}(t^{1/2})\exp(-p\rho^2/25t). \quad (11)$$

Now

$$|\nabla_x\nabla_y h_t(x,y)|\leqslant\int_M |\nabla_x h_{t/2}(x,z)|\cdot|\nabla_y h_{t/2}(z,y)|dz$$

$$\triangleq\int_{B_y(\rho)}\cdot+\int_{M-B_y(\rho)}\cdot$$

and $(d\triangleq d(x,y)>\rho>0)$

$$\int_{B_y(\rho)}\cdot\leqslant\left(\int_{M-B_x(d-\rho)}|\nabla_x h_{t/2}(x,z)|^2dz\right)^{1/2}\left(\int_M|\nabla_y h_{t/2}(y,z)|^2dz\right)^{1/2}$$

$$\leqslant C_n t^{-1}V_x^{-1}(t^{1/2})\exp(-(d-\rho)^2/25t)\ (\text{by (9), (10) and (11)})$$

$$\int_{M-B_y(\rho)}\cdot\leqslant\left(\int_{M-B_y(\rho)}|\nabla_y h_{t/2}(y,z)|^2dz\right)^{1/2}\left(\int_M|\nabla_x h_{t/2}(x,z)|^2dz\right)^{1/2}$$

$$\leqslant C_n t^{-1}V_x^{-1}(t^{1/2})\exp(-(d-\rho)^2/25t)\ (\text{by (9), (10), (11)}).$$

Taking $\rho=d/2$, we get

$$|\nabla_x\nabla_y h_t(x,y)|\leqslant C_n t^{-1}V_x^{-1}(t^{1/2})\exp(-d^2(x,y)/100t).$$

Lemma 6 is proved.

Lemma 7. *If* $\mathrm{Ric}(M)\geqslant 0$, *then for* $a>0$, *there holds*

$$\int_0^\infty s^{-a}V_x^{-1}(s^{1/2})\exp(-t^2/cs)ds/s\leqslant C_{n,a}V_x^{-1}(t)t^{-2a}.$$

Proof By (9), $V_x(s^{1/2})V_t^{-1}(t)\geqslant(s^{1/2}/t)^n$ for $0<s<t^2$. Thus

$$\int_0^{t^2}\cdot ds/s\leqslant t^nV_x^{-1}(t)\int_0^{t^2}s^{-a-n/2}\exp(-t^2/cs)ds/s$$
$$\leqslant C_{n,a}t^{-2a}V_x^{-1}(t)$$
$$\int_{t^2}^\infty\cdot ds/s\leqslant V_x^{-1}(t)\int_{t^2}^\infty s^{-a}\exp(-t^2/cs)ds/s$$
$$\leqslant C_{n,a}V_x^{-1}(t)t^{-2a}.$$

Combining above two inequalities, we get (12). When M is compact, proof is similar.

Proof of Proposition 2 By (1) and Lemmas 6—7

$$|\nabla_y K_s(x,y)|\leqslant C_n\int_0^\infty s^{-1/}V_x^{-1}(s^{1/2})\exp(-d^2(x,y)/100s)ds/s$$
$$\leqslant C_nV_x^{-1}(d(x,y))d^{-1}(x,y).$$

Finally, by standard Calderon-Zygmund method[3,8], we can get the $L^1(M)\to$ weak-$L^1(M)$ boundedness of Riesz transform.

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