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P-VECTORFIELDS ON TWO-MANIFOLDS

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Abstract

The concept of *P*-vectorfields on two-manifolds is introduced and some properties of *P*-vectorfields are proved. Particularly, it is shown that some *P*-vectorfields on the torus have no nontrivial recurrent orbits. Also, the absence of closed orbits for the left invariant vectorfields on two-dimensional Lie group is discussed.

§ 1. Introduction

The concept of *P*-vectorfields on the plane was introduced in [1]. In this paper we generalize the concept of *P*-vectorfields to two-dimensional manifolds.

First we set out the definition and properties of *P*-vectorfields on the plane[1, 2] needed for understanding later sections and give simpler proofs for some theorems in [1] and [2].

Let $X=X_1\frac{\partial}{\partial x_1}+X_2\frac{\partial}{\partial x_2}$ and $D=D_1\frac{\partial}{\partial x_1}+D_2\frac{\partial}{\partial x_2}$, $x=(x_1, x_2)\in R^2$, be C^1 vectorfields on the x_1x_2-plane. We define functions

$$\Delta=X_1[X, D]_2-X_2[X, D]_1,\ B=X_1D_2-X_2D_1, \tag{1.1}$$

and sets

$$W=\{x\,|\,\Delta=0,\ x\in R^2\},\ H=\{x\,|\,B=0,\ x\in R^2\}, \tag{1.2}$$

where $[X, D]=[X, D]_1\frac{\partial}{\partial x_1}+[X, D]_2\frac{\partial}{\partial x_2}$ is the Lie bracket of X and D (see [7, p. 212]).

Definition 1.1. *Let X be a C^1 vectorfield on the x_1x_2-plane. If there exists a C^1 vectorfield D such that*

(1) $\Delta\geqslant 0$, (1.3)

(2) *the set W has no two-dimensional subset,*

then X is called a P-vectorfield with aid D.

Remark 1.2. This definition of *P*-vectorfields is slightly different from that in [1]. It weakens the conditions in [1]. Hence some conclusions in [1] and [2] may be changed a little.

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Taking $\omega = X_1dx_2 - X_2dx_1$ in the following formula [7, p.292]

$$d\omega(X, D) = X(\omega(D)) - D(\omega(X)) - \omega([X, D]),$$

we get

$$\frac{\partial B}{\partial x_1} X_1 + \frac{\partial B}{\partial x_2} X_2 - B(\operatorname{div} X) = \Delta. \tag{1.4}$$

Lemma 1.3. *Let X be a P-vectorfield with aid D. If L is a closed orbit of X and $L \cap H \neq \varnothing$, then $L \subset H \cap W$.*

Proof Let L: $x = x(t)$, be a closed orbit with period T, and $L \cap H \neq \varnothing$. Without loss of generality, we suppose that $x(0) \in H$, i. e., $B(x(0)) = 0$. From (1.4) we have

$$\begin{aligned} &\int_0^T \Delta(x(t)) \exp\left\{-\int_0^t (\operatorname{div} X)(x(t))dt\right\} dt \\ &= \int_0^T d\left(B(x(t)) \exp\left\{-\int_0^t (\operatorname{div} X)(x(t))\,dt\right\}\right) \\ &= 0 \quad (B(x(T)) = B(x(0)) = 0). \end{aligned}$$

The above equality and (1.3) mean $L \subset W$. On the other hand, for any $a > 0$, $\tilde{L} = \{x(t),\ 0 \leqslant t \leqslant a\} \subset W$, so

$$\begin{aligned} 0 &= \int_0^a \Delta(x(t)) \exp\left\{-\int_0^t (\operatorname{div} X)(x(t))dt\right\} dt \\ &= B(x(a)) \exp\left\{-\int_0^a (\operatorname{div} X)(x(t))dt\right\}, \end{aligned}$$

i. e., $x(a) \in H$. Hence $L \subset H$ from the arbitrary choice of a. The proof is complete.

Theorem 1.4. *Let X be a P-vectorfield with aid D and L a closed orbit of X. Then*

(1) *L is a nonhyperbolic closed orbit if $L \subsetneqq H$ and $L \subset W$,*

(2) *L is a hyperbolic stable (unstable) closed orbit if $L \subsetneqq W$ and $L \subset \{x \mid B > 0, x \in R^2\}$ $(\{x \mid B < 0, x \in R^2\})$.*

Proof Let L: $x = x(t)$, be a closed orbit with period T. If $L \subsetneqq H \cap W$, then $L \cap H = \varnothing$ (Lemma 1.3), i.e., $B \neq 0$ on L.

From (1.4) we get

$$\int_0^T (\operatorname{div} X)(x(t))dt = -\int_0^T (\Delta(x(t))/B(x(t)))dt.$$

Hence the proof follows from the Poincaré criterion for stability.

Corollary 1.5. *Any P-vectorfield has no period cycles and its compound cycles are in $H \cap W$* [1].

Theorem 1.6. *If X is a P-vectorfield, then its homoclinic orbits connecting a hyperbolic saddle point and heteroclinic orbits connecting two hyperbolic saddle points are in W* [1].

Remark 1.7. (1) If for X there is a C^1 vectorfield D such that $\Delta \leqslant 0$, then we take$(-D)$ instead of D to satisfy the condition (1) in Definition 1.1.

(2) Instead of (1.1), we define

$$\Delta=g(X_1[X, D]_2-X_2[X, D]_1) \text{ and } B=g(X_1D_2-X_2D_1),$$

where g is a C^1 function with constant sign which does not vanish at any regular point of X. Then the above results are all right. (Replacing Δ and B in the above proofs by Δ/g and B/g respectively.) Hence such a factor g is not essential to the definition of P-vectorfield.

Other interesting conclusions and applications of P-vectorfield can be found in [1] and [2].

In Section 2 we define the P-vectorfield on two-manifolds and give its properties.

Section 3 is devoted to the P-vectorfield on the torus, which have more striking behavior.

Finally, we discuss the absence of closed orbits for the left invariant vectorfields on two-dimensional Lie groups in Section 4.

§ 2. Definition and Basic Theorems

We now introduce the concept of P-vectorfields on two-manifolds. For simplicity, we only consider the two-dimensional differentiable manifolds of class C^∞.

Let X and D be C^1 vectorfields and ω a C^1 differential 1-form on two-manifold M. We define functions

$$\Delta=\omega([X, D]), \quad B=\omega(D), \tag{2.1}$$

and sets

$$W=\{m|\Delta=0, m\in M\}, \quad H=\{m|B=0, m\in M\}. \tag{2.2}$$

Definition 2.1. *Let X be a C^1 vectorfield on two-manifold M. If there exists a C^1 vectorfield D and a C^1 differential 1-form ω on M such that*

(1) $\omega(X)=0$, *{singular points of ω}={singular points of X},*

(2) $\Delta\geqslant 0$,

(3) *the set W has no two-dimensional subsets,*

then X is called a P-vectorfield on M with aid D and differential ω.

Remark 2.2. (1) If $M=R^2=$the x_1x_2-plane, then we write $X=X_1\dfrac{\partial}{\partial x_1}+X_2\dfrac{\partial}{\partial x_2}$ and the ω satisfying the condition (1) of Definition 2.1 must be $g(X_1dx_2-X_2dx_1)$, where g is a C^1 function with constant sign which does not vanish at any regular point of X. Hence Definition 2.1 coincides with Definition 1.1 in this case (cf. Remark 1.7(2)).

(2) If M is a symplectic manifold with the non-degenerate closed differential 2-form Ω, then we can take $\omega=\Omega(X, \cdot)$ to satisfy the condition (1) of

Definition 2.1.

Let F: $M\to N$ be a diffeomorphism of class C^2 between two-manifolds M and N. Let X be a P-vectorfield on M with aid D and differential 1-form ω. Then F_*X is a C^1 vectorfield on N, where F_*: $TM\to TN$ is the tangent bundle map induced by F, and $(F^{-1})^*\omega$ is a C^1 differential 1-form on N, where $(F^{-1})^*$: $T^*M\to T^*N$ is the cotangent bundle map induced by F^{-1} [7].

Now we consider the vectorfields F_*X, F_*D and the differential 1-form $(F^{-1})^*\omega$ on N. Define functions

$$\Delta_*=((F^{-1})^*\omega)([F_*X,\ F_*D]),\ B_*=((F^{-1})^*\omega)(F_*D) \tag{2.3}$$

and sets

$$W_*=\{n|\Delta_*=0,\ n\in N\},\ H_*=\{n|B_*=0,\ n\in N\}. \tag{2.4}$$

We have

$$((F^{-1})^*\omega)(F_*X)=\omega((F^{-1})_*F_*X)=\omega(X), \tag{2.5}$$

$$B_*=\omega(D), \tag{2.6}$$

$$\Delta_*=\omega((F^{-1})_*[F_*X,\ F_*D])=\omega([X,\ D]). \tag{2.7}$$

(2.7) and (2.6) mean that

$$W_*=F(W)\quad \text{and}\ H_*=F(H). \tag{2.8}$$

From (2.5), (2.7) and (2.8) we obtain the following lemma.

Lemma 2.3. *Let F: $M\to N$ be a C^2 diffeomorphism between two-manifolds M and N. If X is a P-vectorfield on M, then F_*X is a P-vectorfield on N.*

We know that a closed curve L on M is one-sided (two-sided) if there exists a neighborhood of L on M which is homeomorphic to a Möbius band (an annular region on plane) [4].

Let L be a two-sided closed orbit of P-vectorfield X. Then there is an open neighborhood U of L which is diffeomorphic to an open annular region V on the x_1x_2-plane [4]. Hence U and V are also C^2 diffeomorphic ([5, Chapter 2, Theorem 2.7]). Let F: $U\to V$ be a C^2 diffeomorphism. Then F_*X is a P-vectorfield on $V\subset R^2$ (Lemma 2.3) and $F(L)$ is its closed orbit. Hence we come to the following conclusions from Remark 2.2 (1), Lemma 1.3, Theorem 1.4, Corollary 1.5 and (2.8).

Theorem 2.4. *Let X be a P-vectorfield on M with aid D and differential 1-form ω. If L is a two-sided closed orbit of X, then*

(1) *$L\subset H\cap W$ if $L\cap H\neq\varnothing$;*

(2) *L is a nonhyperbolic closed orbit if $L\subsetneqq H$ and $L\subset W$;*

(3) *L is a hyperbolic stable (unstable) closed orbit if $L\subsetneqq W$ and*

$$L\subset\{m|B>0,\ m\in M\}(\{m|B<0,\ m\in M\}).$$

Corollary 2.5. *Any P-vectorfield on M has no period cycles and its compound cycles are in $H\cap W$.*

Theorem 2.6. *Let X be a P–vectorfield on M with aid D and differential 1–form ω. Then the one–sided closed orbits of X intersect H and W.*

Proof If L: $m=m(t)$ is a one–sided closed orbit of P–vectorfield X and $L\cap H=\varnothing$, then $\omega(D)\neq 0$ on L. On the other hand, $\omega(X)=0$. Hence $\{X(m(t)), D(m(t))\}$ at each point $m(t)$ is a tangent 2–frame to M and it is a C^1 frame field on L. It is impossible for one–sided closed orbit L. (It is a well–known fact in topology.) So we have $L\cap H\neq\varnothing$. Similarly we can obtain $L\cap W\neq\varnothing$.

Let F: $N\to M$ be a covering map. (The definition of covering map is that in [3].) If there exists, around each point $m\in M$, a neighborhood $U_j\subset M$ whose complete inverse image $F^{-1}(U_j)$ is the union $V_{1j}\cup V_{2j}\cup\cdots$ of a set of pairwise disjoint regions V_{kj}, $k=1, 2, \cdots$, with the property that the restriction $F|_{V_{kj}}$: $V_{kj}\to U_j$, of F to each region V_{kj} is a C^2 diffeomorphism between V_{kj} and U_j, and moreover M is covered by finitely or countably many such regions, then for P–vectorfield X on M with aid D and differential 1–form ω, we can define C^1 vectorfields $(F^{-1})_*X$, $(F^{-1})_*D$ (F is a local diffeomorphism) and C^1 differential 1–form $F^*\omega$ on N. We define functions

$$\Delta^*=(F^*\omega)([(F^{-1})_*X,\ (F^{-1})_*D]),\quad B^*=(F^*\omega)((F^{-1})_*D), \tag{2.9}$$

and sets

$$W^*=\{n\,|\,\Delta^*=0,\ n\in N\},\quad H^*=\{n\,|\,B^*=0,\ n\in N\}. \tag{2.10}$$

Evidently,

$$(F^*\omega)((F^{-1})_*X)=\omega(F_*(F^{-1})_*X)=\omega(X), \tag{2.11}$$

$$B^*=\omega(D); \tag{2.12}$$

$$\Delta^*=\omega([X,\ D]). \tag{2.13}$$

From (2.13) and (2.12) we get

$$W^*=F^{-1}(W)\quad \text{(complete inverse image of } W), \tag{2.14}$$

$$H^*=F^{-1}(H)\quad \text{(complete inverse image of } H). \tag{2.15}$$

From (2.11), (2.13) and (2.14) we have the following lemma.

Lemma 2.7. *Let F: $N\to M$ be a covering map satisfying the above assumption. If X is a P–vectorfield on M, then $(F^{-1})_*X$ is a P–vectorfield on N.*

Theorem 2.8. *let F: $R^2\to M$ be a covering map satisfying the above assumption. If X is a P–vectorfield on M, then its homoclinic orbits connecting a hyperbolic saddle point and heteroclinic orbits connecting two hyperbolic saddle points are in W.*

Proof If X is a P–vectorfield on M, then $(F^{-1})_*X$ is a P–vectorfield on R^2 (Lemma 2.7). We note that any homoclinic orbit or heteroclinic orbit of X is lifted as homoclinic or heteroclinic orbits of $(F^{-1})_*X$ and the homoclinic orbits connecting a hyperbolic saddle point or the heteroclinic orbits connecting two hyperbolic saddle points of $(F^{-1})_*X$ are in W^*. Hence the conclusion of Theorem 2.8 follows from (2.14).

Corollary 2.9. *If X is a P–vectorfield on the torus T^2 (the sphere S^2, the projective plane RP^2, the Klein bottle K^2), then its homoclinic orbits connecting a hyperbolic saddle point and heteroclinic orbits connecting two hyperbolic saddle points are in W.*

Proof It is known that R^2 is a covering space of the torus T^2 and the covering map F: $R^2 \to T^2$ is a C^∞ local diffeomorphism[[6, p.181]]. So the corollary is proved for T^2 (Theorem 2.8).

An example in [3] ([3, Part II, example (e), p. 151]) shows that the torus T^2 is a 2-sheeted covering space of the Klein bottle K^2. We can choose a covering map F: $T^2 \to K^2$ which is a C^∞ local diffeomorphism (using Proposition 2.7, Proposition 0.12 of Chapter 1 in [6] and Theorem 18.3.1 in [3, Part II]). If X is a P–vectorfield on K^2, then $(F^{-1})_* X$ is a P–vectorfield on T^2 and any homoclinic orbit or heteroclinic orbit of X is lifted as homoclinic orbits or heteroclinic orbits of $(F^{-1})_* X$. Hence Corollary 2.9 is true for K^2 from (2.14) and what we have just proved.

If X is a P–vectorfield on the sphere S^2, we choose points A and B, a pair of diametrically opposite points of S^2 which are not on the homoclinic orbits and heteroclinic orbits of X. Taking A as the north pole of S^2, we consider the stereographic projection of S^2 onto the plane F: $S^2 \to R^2$ (see [3, Part I, p. 87]). It is easy to verify that F is a C^∞ diffeomorphism between $S^2 - \{A, B\}$ and $R^2 - \{(0, 0)\}$. Hence $F_* X$ is a P–vectorfield on $R^2 - \{(0, 0)\}$ and the conclusion of Corollary 2.9 for S^2 follows from Theorem 1.6 and (2.8).

The example (c) in [3] ([3, Part II, p. 150]) shows that the sphere S^2 is a 2–sheeted covering space of the projective plane RP^2. It is easy to choose a covering map F: $S^2 \to RP^2$ which is a C^∞ local diffeomorphism. Hence Corollary 2.9 is true for RP^2 from that for S^2.

§ 3. P–Vectorfields on the Torus

We are interested in vectorfields on the torus T^2 because of the existence of the nontrivial recurrent orbits. (Singular points and closed orbits are trivial recurrent.)

The torus T^2 can be obtained from the square $I = [0, a] \times [0, a]$ on the x_1x_2–plane by identifying points $(0, x_2)$ and $(x_1, 0)$ with (a, x_2) and (x_1, a) respectively. We call points $(0, x_2)$ and (a, x_2) (or $(x_1, 0)$ and (x_1, a)) a pair of identical points.

A vectorfield (or differential 1–form) on I is a vectorfield (or differential 1–form) on T^2 if it takes the same value at each pair of identiacl points. A vectorfield

X (or differential 1-form ω) on I is a C^1 vectorfield (or C^1 differential 1-form) on T^2 if it is a C^1 vectorfield (or C^1 differential 1-form) on I and X, $\partial X/\partial x_1$, $\partial X/\partial x_2$ (or ω, $\partial\omega/\partial x_1$, $\partial\omega/\partial x_2$) take the same value at each pair of identical points respectively. Hence any vectorfield X (or differential 1-form ω) on T^2 can be written in the form $X=X_1\frac{\partial}{\partial x_1}+X_2\frac{\partial}{\partial x_2}$ (or $\omega=A_1dx_1+A_2dx_2$), and the ω satisfying the condition (1) of Definition 2.1 must be $g(X_1dx_2-X_2dx_1)$, where g is a C^1 function on T^2 with constant sign which does not vanish at any regular point of X. So we always take $\omega=X_1dx_2-X_2dx_1$ in Definition 2.1 for T^2 as in Definition 1.1. (cf. Remark 2.2 (1) and Remark 1.7 (2))

Theorem 3.1. *If X is a P-vectorfield without singular points on T^2, then the nontrivial recurrent orbits of X are in W.*

Proof Suppose that X is a P-vectorfield without singular points on T^2 and its aid is $D(\omega=X_1dx_2-X_2dx_1)$. Then

$$\Delta=X_1[X,D]_2-X_2[X,D]_1\geqslant 0. \tag{3.1}$$

If X has a nontrivial recurrent orbit $L\subsetneqq W$, then we can choose $p\in L$ and a neighborhood U of p such that $\bar{U}\cap W=\varnothing$ (because W is a closed subset of T^2 and it has no other two-dimensional subset) and $\bar{U}$ does not intersect any closed orbit of X (as the number of closed orbits of X is finite near point p (Corollary 2.5).) So $\min\{\Delta|x\in\bar{U}\}>0$.

We consider the flow box F constructed in the proof of Lemma 2.4 in [6] (cf. [6, pp. 146—147]) and suppose that $F\subset U$. From there we know there exists a family of vectorfields $Z(u)=X+\varepsilon uY$, $\varepsilon>0$, $0\leqslant u\leqslant 1$, (where Y vanishes outside F), and for some $0<u_1<1$, $Z(u_1)$ has a closed orbit $L(u_1)$ through F.

Since $\min\{\Delta|x\in\bar{U}\}>0$, we can choose $\varepsilon>0$ so small that

$$\Delta(u)=Z(u)_1[Z(u),D]_2-Z(u)_2[Z(u),D]_1>0 \text{ on } \bar{U}. \tag{3.2}$$

Hence $Z(u)$, $0\leqslant u\leqslant 1$, is a family of P-vectorfields with the aid of X (Note that $Z(u)=X$ outside F) and every vectorfield in the family has no nonhyperbolic closed orbits through F ((3.2) and Theorem 2.4).

On the other hand, if u tends to zero from u_1, the closed orbit $L(u_1)$ of $Z(u_1)$ must disappear. ($\bar{U}$ does not intersect any closed orbit of X). Hence there is u_2 such that $0<u_2\leqslant u_1$ and $L(u_2)$ is a nonhyperbolic closed orbit of $Z(u_2)$ which intersects F. It contradicts the above conclusion. So the nontrivial recurrent orbits of X are in W.

Theorem 3.2. *If X is a P-vectorfield on T^2 and the singular points of X are all hyperbolic, then the nontrivial recurrent orbits of X are in W.*

Proof If X has aid D (differential 1-form $\omega=X_1dx_2-X_2dx_1$), then

$$\Delta=X_1[X,D]_2-X_2[X,D]_1\geqslant 0.$$

If X has a nontrivial recurrent orbit $L \subsetneqq W$, then we choose $p \in L$ and a neighborhood U of p such that $\overline{U} \cap W = \varnothing$. Hence $\min\{\Delta | x \in \overline{U}\} > 0$.

We construct a flow box F as that in the proof of Lemma 2.5 in [6] (cf. [6, pp. 148—150]) and suppose that $F \subset U$. From there we know that there is a family of vectorfields $Z(u) = X + \varepsilon u Y$, $\varepsilon > 0$, $0 \leqslant u \leqslant 1$, ($Y$ vanishes outside F), and for some $0 < u_0 \leqslant 1$, $Z(u_0)$ has one homoclinic or heteroclinic orbit L_0 through F.

Since $\min\{\Delta | x \in \overline{U}\} > 0$, we can choose $\varepsilon > 0$ so small that

$$\Delta(u_0) = Z(u_0)_1[Z(u_0), D]_2 - Z(u_0)_2[Z(u_0), D]_1 > 0 \text{ on } \overline{U}. \tag{3.3}$$

Hence $Z(u_0)$ is a P-vectorfield with aid D ($Z(u_0) = X$ outside F) and it has no homoclinic or heteroclinic orbit through F ((3.3) and Theorem 2.8). It contradicts the existence of L_0. Therefore the nontrivial recurrent orbits of X are in W.

The above theorems can be used to deny the existence of nontrivial recurrent orbits. We give two simple examples on T^2 obtained from $[0, 2\pi] \times [0, 2\pi] \subset R^2$.

Example 1. Consider vectorfield

$$X = (\sin(x_1 + x_2) + b(x_2)\cos(x_1 + x_2))\frac{\partial}{\partial x_1} + \sin(x_1 + x_2)\frac{\partial}{\partial x_2},$$

where $b(x_2) > 0$ is a C^1 function with period 2π. We take $D = -\frac{\partial}{\partial x_1}$. Then

$$\Delta = b(x_2) > 0, \; B = \sin(x_1 + x_2).$$

Hence X is a P-vectorfield without singular points on T^2 and it has no nontrivial recurrent orbits (Theorem 3.1).

In fact, $H = \{(x_1, x_2) | \sin(x_1 + x_2) = 0, \; (x_1, x_2) \in T^2\}$ contains two closed curves which divide T^2 into two regions, and X has exactly one closed orbit in each region (cf. [2, Theorem 2]).

Example 2. Consider vectorfield

$$X = -(\cos x_1 + \sin x_1)(\cos x_1 \cos x_2 - \sin x_1 \sin x_2)\frac{\partial}{\partial x_1}$$
$$+ (\sin x_1 \cos x_2 + \cos x_1 \sin x_2)\frac{\partial}{\partial x_2}.$$

Let

$$D = (\cos x_1 + \sin x_1)\frac{\partial}{\partial x_1}.$$

Then

$$\Delta = (\cos x_1 + \sin x_1)^2 X_2^2 + X_1^2 \geqslant 0,$$

and X is a P-vectorfield on T^2. X has four singular points:

$$(3\pi/4, \pi/4), \; (3\pi/4, 5\pi/4), \; (7\pi/4, \pi/4), \; (7\pi/4, 5\pi/4)$$

which are all hyperbolic. The set W contains two closed curves consisting of orbits of X. Therefore X has no nontrivial recurrent orbits (Theorem 3.2).

§ 4. An Application to Lie Group

We now consider the left invariant vectorfields on a two-dimensional Lie group G, which are C^∞ vectorfields ([7, Proposition 2, Chapter 10]).

Let $\widetilde{X}_1$ and $\widetilde{X}_2$ be linearly independent left invariant vectorfields. Then

$$[\widetilde{X}_i, \widetilde{X}_j] = \sum_{k=1}^{2} C_{ij}^k \widetilde{X}_k, \quad 1 \leqslant i, j \leqslant 2, \tag{4.1}$$

where C_{ij}^k are the constants of structure G ([7, p. 537]).

$\{X_1 X_2\}$ is a basis of left invariant vectorfields on G, its dual basis is $\{\omega^1, \omega^2\}$ consisting of left invariant 1-forms. So we have

$$\omega^i(\widetilde{X}_j) = \delta_j^i, \quad 1 \leqslant i, j \leqslant 2. \tag{4.2}$$

Every left invariant vectorfield $\widetilde{X}$ is a linear combination of $\widetilde{X}_1$ and $\widetilde{X}_2$:

$$\widetilde{X} = C_1\widetilde{X}_1 + C_2\widetilde{X}_2, \ C_1 \text{ and } C_2 \text{ are constants.} \tag{4.3}$$

Theorem 4.1. *If constants C_1 and C_2 satisfy $C_2C_{12}^1 - C_1C_{12}^2 \neq 0$, then the left invariant vector field $\widetilde{X} = C_1\widetilde{X}_1 + C_2\widetilde{X}_2$ has no closed orbits.*

Proof If $C_2C_{12}^1 - C_1C_{12}^2 > 0$, by taking

$$D = -C_2\widetilde{X}_1 + C_1\widetilde{X}_2, \quad \omega = C_2\omega^1 - C_1\omega^2, \tag{4.4}$$

we have ((4.2))

$$\omega(\widetilde{X}) = 0,$$

$$\omega([\widetilde{X}, D]) = (C_1^2 + C_2^2)(C_2C_{12}^1 - C_1C_{12}^2) > 0. \tag{4.5}$$

Hence $\widetilde{X}$ is a P-vectorfield on G. If $C_2C_{12}^1 - C_1C_{12}^2 < 0$, we can come to the same conclusion by taking $D = C_2\widetilde{X}_1 - C_1\widetilde{X}_2$ in (4.4).

If $\widetilde{X}$ has a closed orbit $L \subset G$, then $aL = \{ab \mid b \in L\}$, $a \in G$, is also a closed orbit of $\widetilde{X}$ (because $\widetilde{X}$ is a left invariant vectorfield), i. e., L is a period cycle. It contradicts Corollary 2.5. The proof is complete.

If G is a two-dimensional compact Lie group, then G is a torus T^2 (G is orientable ([7, Corollary 3, p. 507]) and any nontrivial left invariant vectorfield has no singular points).

Corollary 4.2. *If Lie group G is a torus T^2, then $C_{12}^1 = C_{12}^2 = 0$.*

Proof If $C_{12}^1 \neq 0$, then we take $C_2 = 1$, $C_1 = 0$ in (4.3). From Theorem 4.1 we know that $\widetilde{X}$ has no closed orbits. So $\widetilde{X}$ has a nontrivial recurrent orbit (it is a well-known fact in the theory of dynamical systems), which contradicts Theorem 4.1. Therefore we have $C_{12}^1 = 0$. Similarly we can prove $C_{12}^2 = 0$.

References

[1] Chen Yiyuan, P-vectorfields on plane, *Ann. of Diff. Eqs.*, **4**: 4 (1988), 405—409.

[2] Chen Yiyuan, The number of limit cycles of the plane autonomous systems, *Chin. Ann. of Math.*, **10B**: 2 (1989), 277—284.

[3] Dubrovin, B. A., Fomenko, A. T. & Novikov, S. P., Modern geometry-methods and applications, Part I. II, Springer-Verlag, New York, Berlin, 1984.

[4] Godbillon, C., Dynamical systems on surfaces, Springer-Verlag, New York, Berlin, 1983.

[5] Hirsch, M., Differential topology, Graduate texts in mathematics, Springer-Verlag, New York, 1976.

[6] Palis, J. & Melo, W , Geometric theory of dynamical systems, Springer-Verlag, New York, 1982.

[7] Spivak, M., Differential geometry, Vol. 1, Publish and Perish, Inc., Berkeley, 1979