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THE INVARIANT FORMS ON THE GRADED MODULES OF THE GRADED CARTAN TYPE LIE ALGEBRAS

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Abstract

This paper determines the graded modules of graded Cartan type Lie algebras which possess nondegenerate invariant form.

§ 1. Introduction

Let L be a Lie algebra over a field F of characteristic $p>0$, and V an L-module. A symmetric (or skew symmetric) bilinear form λ: $V\times V\to F$ is called invariant on V, if $\lambda(x\cdot v,\ w)=-\lambda(v,\ x\cdot w)\forall x\in L,\ \forall v,\ w\in V$. The subspace rad (λ): $=\{v\in V|(\lambda(v,\ w)=0,\ \forall w\in V\}$ is called the radical of λ. The form λ is nondegenerate if rad $(\lambda)=0$.

In 1986, R. Farnsteiner determined in his paper [4] those graded Lie algebras of Cartan type which possess a nondegenerate symmetric invariant (i. e., associative) form on the adjoint module. Our results generalize Farnsteiner's results and determine the graded modules of graded Cartan type Lie algebras which possess a nondegenerate invariant symmetric bilinear form.

In § 2, we establish several general properties of the induced modules and coinduced modules of graded Lie algebras. § 3 provides the basic tools for the study of the invariant forms so that we can reduce the problem of existence of the nondegenerate invariant forms on an induced module of a graded Lie algebra $L=\bigoplus_{i=-r}^{s} L_{[i]}$ to that of the nondegenerate invariant form on its base space (regarded as $L_{[0]}$-module). The last section is devoted to the existence of the invariant forms on the graded modules of graded Cartan type Lie algebras.

All Lie algebras and modules treated in the present article are assumed to be finite dimensional.

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§ 2. The Induced Modules And Coinduced Modules

Throughout this section we shall be concerned with a graded Lie algebra $L=\bigoplus_{i=-r}^{s}L_{[i]}$ over a field F of positive characteristic p>0. Let $L_i=\bigoplus_{j\geq i}L_{[j]}$ and $L^-=\bigoplus_{i<0}L_{[i]}$. Let $U(L)$ and $U(L_0)$ be the universal enveloping algebras of L and L_0, respectively. Let $\{y_1, \cdots, y_n\}$ be the basis of L^-. Then there exist $m_1, \cdots, m_k\in N$ such that

$$(\text{ad } y_1)^{p^{m_i}}=0, 1\leqslant i\leqslant k.$$

Thus the elements $z_i:=y_i^{p^{m_i}}$ belong to the center of $U(L)$ and the subalgebra $\theta(L, L_0)$ of $U(L)$, which is generated by $U(L_0)$ and $\{z_1, \cdots, z_k)$, is isomorphic to $F[z_1, \cdots, z_k]\otimes_F U(L_0)$, the first factor being a polynomial ring in k indeterminates. By *P-B-W* theorem, we can show that $U(L)$ is a free loft $\theta(L, L_0)$-module with basis $\{y_1^{a_1}\cdots y_k^{a_k}\ 0\leqslant a_i\leqslant p^{m_i}-1,\ i=1, \cdots, k\}$. Let V be an arbitrary $\theta(L, L_0)$-module. Consider the coinduced module $\text{Hom}_{\theta(L,L_0)}(U(L), V)$ with $U(L)$-action given via $(u\cdot f)(k):=f(xu)$. Set $\pi=(\pi_1, \cdots, \pi_k)=(p^{m_1}-1, \cdots, p^{m_k}-1)$. For $v\in V$ and $0\leqslant\alpha=(a_0, \cdots, a_k)\leqslant\pi$, let $X_k^{(\alpha)}$ be the element of $\text{Hom}_{\theta(L, L_0)}(U(L), V)$ which sends $y_1^{\beta_1}\cdots y_k^{\beta_k}$ $(\beta=(\beta_1\cdots\beta_k)\leqslant\pi)$ onto $\delta(\alpha, \beta)v$. Note that if V is an L_0-module, then the action of $U(L_0)$ on V can be extended to $\theta(L, L_0)$ by letting the polynomial algebra $F[z_1, \cdots, z_k]$ operate via its canonical supplementation. Henceforth all L_0-modules will be considered as $\theta(L, L_0)$-modules in this fashion. Let ζ denote the natural representation of L_0 in L/L_0. Then there exists unique homomorphism σ: $U(L_0)\to F$ of F-algebras such that $\sigma(x)=\text{tr}(\zeta(x))$. We introduce a twisted action on V by setting $x\cdot v:=xv+\sigma(x)v$. The new L_0-module will be denoted by V_σ. By [6, Theorem 1.4], there is a natural isomophism of $U(L)$-modules between the induced module and the coinuced module.

$$\Phi: U(L)\otimes_{\theta(L,L_0)}, V_\sigma\xrightarrow{\cong}\text{Hom}_{\theta(L,L_0)}(U(L), V), \tag{2.1}$$

where $\Phi\ (u\otimes v)=u\cdot X_v^{(\pi)}$, $\forall u\in U(L)$, $\forall u\in V$. If V is an $L_{(0)}$-module, then we can extend the operations on V to L_0 by letting L_1 act trivially and regard it as an L_0-module.

Remark 2.1. Note that $P=\text{Hom}_{\theta(L,L_0)}(U(L), V)$ possesses a positively graded L-module structure by setting

$$P_i:=\{f\in P|f(U(L)_j)=0, \forall j\neq -i\},$$

where $U(L)=\bigoplus_{i\geqslant 0}U(L)_i$ is graded by using P-B-W theorem, such that P_0 is isomorphic to V as an L_0-module and P is transitively graded (cf. [5, Proposition 4.2]).

Now we shall give the property of adjoint isomorphism.

Proposition 2.2. *Let V be an L_0-module. Then*

$$(U(L)\otimes_{\theta(L,L_0)}(V)^*\cong \mathrm{Hom}_{\theta(L,L_0)}(U(L), V^n).$$

Proof This follows by adjointness between Hom and $\otimes$ (cf. [8, Theorem 2.11]).

Remark 2.3 For $x_1, \cdots, x_n \in L$, set

$$(x_1\cdots x_n)^T: =(-1)^n x_n x_{n-1}\cdots x_1.$$

Then the principal anti-automorphism of $U(L)$ is defined by $u\mapsto u^T$. For $\psi\in (U(L)\otimes_{\theta(L,L_0)}, V)^*$ and $u\in U(L)$, let $\tilde{\psi}(u)$ be the linear function on V defined by $v\mapsto \psi(u^T\otimes v)$. Then Ψ: $\psi\mapsto\tilde{\psi}$ is the adjoint isomorphism of $(U(L)\otimes_{\theta(L,L_0)}V)^*$ onto $\mathrm{Hom}_{\theta(L,L_0)}(U(L), V^*)$. Conversely, let $f\in \mathrm{Hom}_{\theta(L,L_0)}(U(L), V^*)$. Define $\psi(u\otimes v)$: $=f(u^T)(v)$. Then $\Psi(\psi)=f$.

Proposition 2.4. *Let V be an L_0-module. Then $(U(L)\otimes_{\theta(L,L_0)}V_\sigma)^*\cong U(L)\otimes_{\theta(L,L_0)}V_\sigma$ if and only iff $(V_\sigma)\cong V$.*

Proof If $(U(L)\otimes_{\theta(L,L_0)}V)^n\cong U(L)\otimes_{\theta(L,L_0)}V_\sigma$, by Proposition 2.2 and (2.1), we have

$$\mathrm{Hom}_{\theta(L,L_0)}(U(L), (V_\sigma)^*)\cong \mathrm{Hom}_{\theta(L,L_0)}(U(L), V). \tag{2.2}$$

Then by Remark 2.1, we have

$$(V_\sigma)^*\cong V.$$

Conversely, the sufficiency is clear.

§ 3. Invariant Forms

Let L be a Lie algebra over F and V an L-module. A bilinear form λ: $V\times V\to F$ is called symmetric (resp. skew symmetric) if $\lambda(v,w)=\lambda(w,v)$ (resp. $-\lambda(w,v)$), $\forall v, w\in V$.

Proposition 3.1. *Let L be a Lie algebra over F and V an L-module. Then there exists a nondegenerate symmetric (or skew symmetric) invariant form λ on V if and only if there exists an L-module isomorphism ψ: $V\to V^*$ such that $\psi(v)(w)=\psi(w)(v)$ (or $-\psi(w)(v)$), $\forall v, w\in V$.*

Proof Let λ be a nondegenerate symmetric (or skew symmetric) invariant form λ on V. Define ψ: $V\to V^*$ by $\psi(v)(w)$: $=\lambda(v, w)$. Clearly, ψ is a linear mapping such that $\ker\psi=\mathrm{rad}(\lambda)=0$ and $\psi(v)(w)=\pm\psi(w)(v)$, $\forall v, w\in V$. Hence ψ is injective. Since $\dim V=\dim V^*$, ψ is bijective. For $x\in L$ and $v\in V$, we have

$$\psi(xv)(w)=\lambda(xv, w)=-\lambda(v, xw)$$
$$=-\psi(v)(xw)=(x\cdot\psi(v))(w),\ \forall\in V.$$

Thus ψ is the desired isomorphism of L-modules. Conversely, let ψ: $V\to V^*$ be an L-module isomorphism such that $\psi(v)(w)=\psi(w)(v)$ (or $-\psi(w)(v)$), $\forall v, w\in V$.

Put $\lambda(v, w):=\psi(v)(w)$. Thus λ is a symmetric (or skew symmetric) bilinear form. Furthermore, $\lambda(xv, w)=\psi(xv)(w)=(v\cdot\psi(v))(w)=-\psi(v)(xw)=-\lambda(v, xw)$, $\forall x\in L$. $\forall v, w\in V$. Hence λ is invariant. As $\operatorname{rad}(\lambda)=\ker\psi=0$, λ is nondegenerate.

Proposition 3.2. *Let L be a Lie algebra over F and V an irreducible L-module. If $V\cong V^*$ (regarded as L-modules), then there exists a nondegenerate invariant form on V which is either symmetric or skew symmetric.*

Proof By the proof of Proposition 3.1, there exists a nondegenerate invariant form β on V. Let

$$\lambda(v, w)=\beta(v, w)+\beta(w, v),\ \forall v, w\in V.$$

Then λ is a symmetric invariant form. Since V is irreducible, λ is either nondegenerate or 0. It implies that either β or λ is the desired form.

Theorem 3.3. *Let $L=\bigoplus_{i=-r}^{s} L_{[i]}$ be a graded Lie algebra over F and V an $L_{[0]}$-module. Then the following statements are equivalent.*

(1) *There exists a nondegenerate invariant symmetric (or skew symmetric) form on the induced module $U(L)\otimes_{\theta(L,L_0)}V_\sigma$.*

(2) *There exists an $L_{[0]}$-module isomorphism $\zeta: V\to(V_\sigma)^*$ such that*

$$\zeta(v)(w)=\zeta(w)(v)(\text{or}-\zeta(w)(v)),\ \forall v, w\in V.$$

Proof (1) $\Rightarrow$ (2). Suppose that there exists a nondegenerate invariant symmetric (or skew symmetric) form on $U(L)\otimes_{\theta(L,L_0)}V_\sigma$. By Propositions 2.4 and 3.1, there exists an L-module isomorphism ψ: $U(L)\otimes_{\theta(L,L_0)}V_\sigma\to(U(L)\otimes_{\theta(L_0L_0)}V_\sigma)^*$ such that

$$\psi(u_1\otimes v_1)(u_2\otimes v_2)=\psi(u_2\otimes v_2)(u_1\otimes v_1)(\text{or}-\psi(u_2\otimes v_2)(u_1\otimes v_1)),\qquad(3.1)$$

$\forall u_1, u_2\in U(L)$, $\forall u_1, v_2\in V$ and there exists an $L_{[0]}$-module isomorphism ζ: $V\to(V_\sigma)^*$. Let $u_1=y_1^{\alpha_1}\cdots y_k^{\alpha_k}\in U(L^-)$ and $u_2=y_1^{\beta_1}\cdots y_k^{\beta_k}\in U(L^-)$, where $\alpha=(\alpha_1,\cdots,\alpha_k)\leqslant\pi$ and $\beta=(\beta_1,\cdots,\beta_k)\leqslant\pi$. Then by the proof of Proposition 2.4 and Remark 2.3, we have

$$\psi(u_1\otimes v_1)(u_2\otimes v_2)=X^{(\pi)}_{\zeta(v_1)}(u_2^Tu_1^T)(v_2)=\delta(\pi, \alpha+\beta)\zeta(v_1)(v_2),\qquad(3.2)$$

$\forall v_1, v_2\in V$. By (3.1) and (3.2), we have

$$\zeta(v_1)(v_2)=\zeta(v_2)(v_1)\ (\text{or}-\zeta(v_2)(v_1)).$$

(2)$\Rightarrow$(1). This also follows from Proposition 2.4, Proposition 3.1 and Remark 2.3.

§ 4. The Invariant Forms on the Graded Cartan Type Lie Algebras

We adopt the notation of [2] (or [9, 10, 11]). If L is a graded Lie algebra of Cartan type of type $W(n, \boldsymbol{m})(=W)$, $S(n,\boldsymbol{m})(=S)$, $H(n, \boldsymbol{m})(=H)$, then $\{D_1, \cdots,$

$D_n\}$ is the canonical basis of $L^-(=L_{[-1]})$ and we can define $z_i:=D_i^{p^{m_i}}$, $i=1, \cdots n$. By [5, Theorem 4.4], if V_0 is an $L_{[0]}$-module, then the coinduced module $\mathrm{Hom}_{\theta(L,L_0)}(U(L), V_0)$ is isomorphic to the mixed product $\tilde{V}_0$ (i. e., $\mathscr{U}(n, \boldsymbol{m})\rtimes V_0$), where $\mathscr{U}(n, \boldsymbol{m})=\langle x^{(\alpha)}\mid 0\leqslant\alpha\leqslant\pi\rangle$ is the divided power algebra, $\boldsymbol{m}=(m_1, \cdots, m_n)$ and $\pi=(p^{m_1}-1, \cdots, p^{m_n}-1)$. By [11, Proposition 1.2], if V_0 is $L_{[0]}$-irreducible, then $\tilde{V}_0$ is L-irreducible unless V_0 is trivial or a highest weight module with a fundamental weight as its highest weight. If L is any one of W, S and H, then $L_{[0]}$ is isomorphic to $gl(n)$, $sl(n)$ and $sp(n)$, respectively. Let λ_i ($i=1, \cdots, n$ for $L_{[0]}=gl(n)$; $i=1, \cdots, n-1$ for $L_{[0]}=sl(n)$; $i=1, \cdots, n/2$ for $L_{[0]}=sp(n)$) be the fundamental weights of $L_{[0]}$. Every weight of $sl(n)$ or $sp(n)$ is a linear combination of fundamental weights. We denote the $L_{[0]}$-irreducible module with the highest weight λ by $V(\lambda)$.

Theorem 4.1. *Let $L=W(n, \boldsymbol{m})$ and $V(\lambda)$ be the $L_{[0]}(=gl(n))$-module with the highest weight $\lambda=\sum_{i=1}^{n} a_i\lambda_i$. Then there exists a nondegenerate invariant form on $U(L)\otimes_{\theta(L,L_0)}V(\lambda)_\sigma(\cong\tilde{V}(\lambda))$ if and only if $2\sum_{i=1}^{n} ia_i\equiv n \bmod(p)$ and $a_i=a_{n-i}$, $i=1, \cdots, n-1$.*

Proof If there exists a nondegenerate invariant symmetric (or skew symmetric) form on $\tilde{V}(\lambda)$, then by Theorem 3.3 we have

$$V(\lambda)\cong(V(\lambda)_\sigma)^*. \tag{4.1}$$

Let $I=E_{11}+E_{22}+\cdots+E_{nn}\in gl(n)=W_{[0]}$. Then the action of I on $V(\lambda)$ and $(V(\lambda)_\sigma)^*$ are the scalar multiplication by $\sum_{i=1}^{n} ia_i$ and $-\sum_{i=1}^{n} ia_i+n$, respectively. It implies that $2\sum_{i=1}^{n} ia_i\equiv n \bmod(p)$. Write $\lambda'=\sum_{i=1}^{n-1} a_i\lambda_i$. As $sl(n)$-modules, by (4.1), we have

$$V(\lambda')\cong(V(\lambda'))^*$$

and $\lambda'=-w_0\lambda'$, where w_0 is the unique element in the Weyl group of $sl(n)$ sending the collection of positiuve roots to the collection of negative roots. Since $-w_0\lambda'=\sum_{i=1}^{n-1} a_{n-i}\lambda_i$, we have

$$a_i=a_{n-i},\ i=1, \cdots, n-1,$$

so the necessity follows. Vice versa.

Corollary 4.2. *There exists a nondegenerate invariant symmetric form on the adjoint module $W(n, \boldsymbol{m})$ if and only if $n=1$ and $p=3$ or $n=2$ and $p=2$.*

Proof. By [9, Lemma 2.2], the adjoint module $W(n, \boldsymbol{m})$ is isomorphic to the mixed product $\tilde{V}^*(\lambda_1)$ $(\cong U(W)\otimes_{\theta(W,W_0)}V^*(\lambda_1)_\sigma)$, where $V^*(\lambda_1)$ is the dual module of $V(\lambda_1)$. We can prove easily that $V^*(\lambda_1)\cong V^*(\lambda_1)_\sigma)^*(\cong V(\lambda_1)_{-\sigma})$ if and only if $n=1$ and $p=3$ or $n=2$ and $p=2$. Thus if $n=1$ and $p=3$ or $n=2$ and $p=2$, then there exists a nondegenerate invariant form β on $\tilde{V}^*(\lambda_1)$. Using Proposition 3.2, we can also prove easily that β is symmetric. Conversely, the necessary condition is clear.

Corollary 4.3. *Let $L=S(n, \boldsymbol{m})$ and $V(\lambda)$ be the $L_{[0]}(=sl(n))$-module with the highest weight $\lambda=\sum_{i=1}^{n-1}a_i\lambda_i$. Then there exists a nondegenerate invariant form on $U(L)\otimes_{\theta(L,L_0)}V(\lambda)_\sigma(\cong\tilde{V}(\lambda))$ if and only if $a_i=a_{n-i}$, $i=1, \cdots, n-1$.*

Proof The proof is similar to Theorem 4.1.

As in [10], if V_0 is $L_{[0]}(\cong sl(n))$-irrdeucible then we denote the unique minimum submodule $U(S)(1\otimes V_0)$ of $\tilde{V}_0$ by $(\tilde{V}_0)_{\min}$. By [10, Theorem 2.2], any irreducible graded S-module with base space V_0 is isomorphic to $(\tilde{V}_0)_{\min}$. Note that $(\tilde{V}(\lambda))_{\min}\subsetneqq V(\lambda)$ if and only if $\lambda=\lambda_0, \cdots, \lambda_{n-1}$ and $(\tilde{V}(\lambda_0))_{\min}=F$ for $p>2$. For covenience, denote $(\tilde{V}(\lambda_i))_{\min}$ by $\tilde{M}(\lambda_i)$.

Theorem 4.4. *Let $L=S(n, \boldsymbol{m})$ and $p>2$. Then there exists a nondegenerate invariant symmetric form on $\tilde{M}(\lambda_i)$ if and only if $i=0$ or n is odd and $i=\frac{n+1}{2}$.*

Proof Obviously, the trivial module $\tilde{M}(\lambda_0)=F$ admits a nondegenerate invariant symmetric form. Now assume that $i=1, \cdots, n-1$. By Proposition 2.2, $(\tilde{V}(\lambda_i))^*\cong\mathrm{Hom}_{\theta(S,S_0)}(V(S), (V(\lambda_i)_\sigma)^*)\cong\tilde{V}(\lambda_{n-i})$. Hence $(\tilde{M}(\lambda_i))^*$ is isomorphic to the irreducible quotient module of $\tilde{V}(\lambda_{n-i})$. By [11, Theorem 2.2], the irreducible quotient module of $\tilde{V}(\lambda_{n-i})$ is $\tilde{M}(\lambda_{n-i+1})$. It implies that $\tilde{M}(\lambda_i)\cong(\tilde{M}(\lambda_i))^*$ if and only if $i=n-i+1$, that is n is odd and $i=(n+1)/2$. Then there exists a nondegenerate invariant symmetric form on $\tilde{M}(\lambda_i)$.

It is well-known that $V(\lambda_k)$ is isomorphic to $\Lambda^k V(\lambda_k)$, where $V(\lambda_1)$ is the module of the natural representation of $S_{[0]}$. Let $\Omega^k(n, \boldsymbol{m})$ be the $S(n, \boldsymbol{m})$-modules of differential forms of k-th degree with coefficients in $\mathscr{U}(n, \boldsymbol{m})$ and $\{e_1, \cdots, e_n\}$ the natural basis of $V(\lambda_1)$. Then the linear map $x^{(\alpha)}\otimes(e_{j_1}\wedge\cdots\wedge e_{j_k})\mapsto x^{(\alpha)}dx_{j_1}\wedge\cdots\wedge dx_{j_k}$ is a module isomorphism of $\tilde{V}(\lambda_k)$ onto $\Omega^k(n, \boldsymbol{m})$. Let $I=\{1, \cdots, n\}$, $J\subset I$, and the elements of J be $j_1<j_2\cdots<j_k$. Write

$$e_J=e_{j_1}\wedge\cdots\wedge e_{j_k}. \tag{4.2}$$

If $|T|+|J|=n$ for $T\subset I$, let

$$e_T\wedge e_J=\tau(T, J)\ e_1\wedge\cdots\wedge e_n.$$

Then

$$\tau(T, J)=\begin{cases}\pm1, & \text{if } T\cup J=I,\\ 0, & \text{otherwise.}\end{cases}$$

Define the bilinear form $\beta: V(\lambda_{n-i})\times V(\lambda_i)\to F$ by

$$(e_T, e_J)\mapsto\tau(T, J).$$

It is easy to show that

$$\beta(x\, e_T, e_J)=-\beta(e_T, xe_J), \text{ for } x\in sl(n),$$

that is, β is invariant. Hence there exists an $S_{[0]}(\cong sl(n))$-module isomorphism ζ

of $V(\lambda_{n-i})$ onto $(V(\lambda_i))^*$. If we modify the proof of Theorem 3.3, then it is not difficult to prove that there exists an $S(n, \boldsymbol{m})$-module isomorphism ψ: $U(S)\otimes_{\theta(S,S_0)} V(\lambda_{n-i})_\sigma \to (U(S)\otimes_{\theta(S,S_0)} V(\lambda_i)_\sigma)^*$ such that

$$\psi(D_1^{\alpha_1}\cdots D_n^{\alpha_n}\otimes e_r)(D_1^{\alpha'_1}\cdots D_n^{\alpha'_n}\otimes e_J)$$
$$=\Pi\delta(\pi_i, \alpha_i+\alpha'_i)\beta(e_r, e_J). \tag{4.3}$$

Denote $D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ by D^α. The linear map $D^\alpha\otimes e_J\mapsto x^{(\pi-\alpha)}\otimes e_J$ is a module isomorphism of $U(S)\otimes_{\theta(S,S_0)}V(\lambda_i)$ onto $\widetilde{V}(\lambda_i)$. Write $\omega_J=dx_{j_1}\wedge\cdots\wedge dx_{j_k}$ which corresponds to e_J in (4.2). By (4.3),

$$\widetilde{\beta}: \widetilde{V}(\lambda_{n-i})\times\widetilde{V}(\lambda_i)\to F,$$
$$\widetilde{\beta}(x^{(\alpha)}\otimes\omega_r, x^{(\alpha')}\otimes\omega_J)=\delta(\pi, \alpha+\alpha')\beta(\omega_r, \omega_J), \tag{4.4}$$

is a nondegenerate invariant form. Define as usual the exterior differential operation d_k: $\widetilde{V}(\lambda_k)\to\widetilde{V}(\lambda_{k+1})$ by

$$d_k(f\omega_J)=(df)\wedge\omega_J,\ f\in\mathscr{U}(n, \boldsymbol{m}),\ |J|=k,$$

where $df=\sum_{i=1}^n(D_if)dx_i$. By [11, (2.4), (2.6), (2.7) and Theorem 2.2], we have

$$d_kd_{k-1}=0,$$
$$\widetilde{M}(\lambda_k)=d_{k-1}\widetilde{V}(\lambda_{k-1}),\ \dim\widetilde{M}(\lambda_k)=(p^{|m|}-1)C_{k-1}^{n-1},$$
$$\widetilde{M}(\lambda_k)=\widetilde{V}(\lambda_{k-1})/\ker d_{k-1}.$$

It is easy to show that

$$\widetilde{\beta}(d_{n-i}(x^{(\alpha)}\otimes\omega_r), x^{(\alpha')}\otimes\omega_J)$$
$$=(-1)^{|T|}\widetilde{\beta}(x^{(\alpha)}\otimes\omega_T, d_{i-1}(x^{(\alpha')}\otimes\omega_J))$$

for $|\widetilde{\beta}|=n-i$ and $|J|=i-1$. Hence

$$\widetilde{\beta}(f\otimes\omega_r, d_{i-1}(g\otimes\omega_J))=(-1)^{|r|}\widetilde{\beta}(d_{n-i}(f\otimes\omega_r), g\otimes\omega_r)=0, \tag{4.5}$$

for $f\otimes\omega_r\in\ker d_{n-i}$ and $g\otimes\omega_J\in\widetilde{V}(\lambda_{i-1})$. Hence we have

$$\widetilde{\beta}\ |_{\ker d_{n-i}\times\widetilde{m}(\lambda_i)}=0. \tag{4.6}$$

By virtue of (4.6), we obtain a nondegenerate invariant pairing on $(\widetilde{V}(\lambda_{n-i})/\ker d_{n-i})\times\widetilde{M}(\lambda_i)$ (i. e., $\widetilde{M}(\lambda_{n-i+1})\times\widetilde{M}(\lambda_i)$). In particular, if we take $i=(n+1)/2$ (n is odd), then we obtain a nondegenerate invariant form on $\widetilde{M}(\lambda_i)$. Furthermore, by (4.4) and (4.5), we can check that it is symmetric, as desired.

Corollary 4.5. *Suppose that $p>2$. There exists a nondegenerate invariant symmetric form on the adjoint module $S(n, \boldsymbol{m})$ if and only if $n=3$.*

Proof Since $S(n, \boldsymbol{m})$ is isomorphic to the minimum submodule of $W(n, \boldsymbol{m})\cong\widetilde{V}^*(\lambda_1)$ (regarded as an $S(n, \boldsymbol{m})$-module), we have $S(n, \boldsymbol{m})\cong\widetilde{M}(\lambda_{n-1})$. Our assertion is an immediate consequence of Theorem 4.4.

Now we consider the case $L=H(n, \boldsymbol{m})$ where $p>2$, $n=2r$ and $L_{[0]}=sp(n)$. If V_0 is $L_{[0]}$-irreducible, then we denote the unique minimum submodule of $\widetilde{V}_0$ by $(\widetilde{V}_0)_{\min}$. Since [11, Theorem 2.3] is also valid for $p>2$, $(V(\lambda_i))_{\min}\subsetneq\widetilde{V}(\lambda_i)$ if and only if $\lambda=\lambda_0, \cdots, \lambda_r$. Denote $(\widetilde{V}(\lambda_i))_{\min}$ by $M(\lambda_i)$. Then $\widetilde{V}(\lambda_0)\cong\mathscr{U}(n, \boldsymbol{m})$ and $\widetilde{M}(\lambda_1)$

$\cong \mathscr{U}'/F\cdot 1$, where $\mathscr{U}' := \bigoplus_{\alpha\neq\pi} \langle x^{(\alpha)}\rangle$.

Theorem 4.6. *Let $L=H(n, \boldsymbol{m})$ and $p>2$. Then there exists a nondegenerate invariant form on $\tilde{V}(\lambda)$ or $\tilde{M}(\lambda_i)$ for $i=0, \cdots, r$.*

Proof Since $V(\lambda)\cong(V(\lambda)_\sigma)^*$ and the top (or base) composition factor of $\tilde{V}(\lambda_i)$ is $\tilde{M}(\lambda_i)$, the theorem follows directly from Proposition 3.2 and the proof of Theorem 3.3.

Corollary 4.7. *Suppose that $p>2$. There exists a nondegenerate invariant symmetric form on the adjoint module $H(n, \boldsymbol{m})$.*

Proof Since the base space $H_{[-1]}$ of the adjoint module $H(n, \boldsymbol{m})$ is $V(\lambda_1)$, we have $H(n, \boldsymbol{m})\cong M(\lambda_1)\cong\mathscr{U}'/F\cdot 1$. Since $V(\lambda_0)=F$, by Theorem 3.3, there exists a nondegenerate invariant symmetric form β on $\tilde{V}(\lambda_0)(\cong\mathscr{U}(n, \boldsymbol{m}))$ such that

$$\beta(x^{(\alpha)}, x^{(\beta)})=\delta(\pi, \alpha+\beta).$$

It induces a nondegenerate invariant symmetric form on $\mathscr{U}'/F\cdot 1(\cong H(n, \boldsymbol{m}))$.

We finally consider the case of $K(n, \boldsymbol{m})$ $(=K)$ $(p>2)$. We put $n:=2r+1$ and adopt the notation of [12, § 4.5]. We define

$$\sigma(j)=\begin{cases}1, & j\leqslant r,\\ -1, & j\geqslant r+1,\end{cases} \qquad j'=\begin{cases}j+r, & j\leqslant r,\\ j-r, & j\geqslant r+1.\end{cases}$$

Let D_k: $\mathscr{U}(n, \boldsymbol{m})\to W(n, \boldsymbol{m})$ be the canonical linear mapping defined by

$$D_k(x^{(\alpha)})=\sum_{j=1}^{2r}\left(\sigma(j')x^{(\alpha-\varepsilon_{j'})}-x^{(\alpha-\varepsilon_n)}x^{(\varepsilon_j)}\right)D_j+\left(2-\sum_{j=1}^{2r}\alpha_j\right)x^{(\alpha)}D_n.$$

Then we can identify $f\in\mathscr{U}(n, \boldsymbol{m})$ with $D_k(f)$ such that

$$K(n, \boldsymbol{m})=\begin{cases}\mathscr{U}(n, \boldsymbol{m}), & n+3\not\equiv 0 \bmod(p),\\ \langle x^{(\alpha)}\mid\alpha\langle\pi\rangle, & n+3\equiv 0 \bmod(p).\end{cases}$$

Define $|\alpha|:=\sum_{i=1}^{n}\alpha_i$, $\|\alpha\|:=|\alpha|+\alpha_n-2$ and $\mathscr{U}(n, \boldsymbol{m})_{[i]}$: $\langle x^{(\alpha)}\mid\|\alpha\|=i\rangle$. Then $\mathscr{U}(n, \boldsymbol{m}) = \bigoplus_{i=-2}^{s}\mathscr{U}(n, \boldsymbol{m})_{[i]}(s=\|\pi\|)$ is graded. Since $(\mathrm{ad}\ D_k(x^{(\varepsilon_i)}))^{p^{m_{i'}}}=0$, $i=1, \cdots, 2r$, and $(\mathrm{ad}\ D_k(1))^{p^{m_n}}=0$, the elements

$$z_i:=D_k(x^{(\varepsilon_i)}))^{p^{m_{i'}}}, \ i=1, \cdots, 2r,$$
$$z_n:=D_k(1))^{p^{m_n}}$$

belong to the center of $U(K)$ and the subalgebra $\theta(K, K_0)$ of $U(K)$, which is generated by $U(K_0)$ and $\{z_1, \cdots, z_n\}$. Put $V_i:=K_{[i-2]}$. Then $V=\bigoplus_{i\geqslant 0}V_i$ is a positively and transitively graded K-module satisfying $z_j.\ V=0$, $1\leqslant j\leqslant n$. Let $V(\lambda, c)$ be a $K_{[0]}$-module such that it is the $\langle x^{(\varepsilon_i+\varepsilon_j)}\ 1\leqslant i, j\leqslant 2r\rangle$-module (i. e., sp$(2r)$-module) with the highest weight λ and the action of $x^{(\varepsilon_n)}$ on $V(\lambda, c)$ is the scalar multiplication by c. Since $V_0=K_{[-2]}\cong V(0, -2)$, by [5, Proposition 4.3], we can easily show that if $n+3\not\equiv 0 \bmod(p)$, then

$$K(n, \boldsymbol{m})\cong U(K)\otimes_{\theta(K, K_0)}V(0, -2)$$

and if $n+3\equiv 0 \bmod (p)$, then there is an exact sequence

$$0\to K(n, \mathbf{m})\to U(K)\otimes_{\theta(K,K_0)} V(0, -2)_\sigma \to F\to 0.$$

Suppose that $n+3\not\equiv 0 \bmod (p)$. Since $\sigma(x^{(\varepsilon_n)})=-n-1$ and $\dim V(0, -2)=1$, by Theorem 3.3, we can easily show that the adjoint module $K(n, \mathbf{m})$ possesses a nondegenerate invariant symmetric form if and only if $n+5\equiv 0 \bmod (p)$. Finally, we assume $n+3\equiv 0 \bmod (p)$. Then it is easy to show that $K(n, \mathbf{m})^*$ is isomorphic to a proper submodul of

$$U(K\otimes_{\theta(K,K_0)}, V(\lambda_1, -1)_\sigma.$$

Since the base spaces of $K(n, \mathbf{m})$ and $K(n, \mathbf{m})^*$ are $V(0, -2)$ and $V(\lambda_1, -1)$, respectively, which are not isomorphic, $K(n, \mathbf{m})$ is not isomorphic to $K(n, \mathbf{m})^*$. Hence $K(n, \mathbf{m})$ does not admit a nondegenerate invariant form. Thus we have

Theorem 4.8. *Suppose that* $p>2$. $K(n, \mathbf{m})$ *possesses a nondegenerate invariant symmetric form if and only if* $n+5\equiv 0 \bmod (p)$.

Remark 4.9. Corollary 4.2, Corollary 4.5, Corollary 4.7 and Theorem 4.8 also determine those graded Cartan type Lie algebras which possess a nondegenerate associative form. But our approach is less technical than it pursued by Farnsteiner [4].

References

[1] Chiu Sen, Central extensions and $H^1(L, L^*)$ of the graded Lie algebras of Cartan type (preprint).

[2] Chiu, S. & Yu. Shen, G., Cohomology of graded Lie algebras of Cartan type of characteristic p, *Abh. Math. Sem. Univ. Hamburg*, **57** (1987), 139—156.

[3] Dixmier, J., Enveloping alpebras, North-Hollane, Amsterdam, 1977.

[4] Farnsteiner, R., The associative forms of the graded Cartan type Lie algeblas, *Trans. Amer. Math. Soc.*, **295** (1986), 417—427.

[5] Farnsteiner, R., Extension functors of modular Lie algebras and Erobenius algebras (preprint).

[6] Farnsteiner, R. & Strade, H., Shapiro's lemma and its consequences in the cohomology theory of modular Lie algebras (prerint).

[7] Humphreys, J., Introduction to Lie algebras and representation theory, Springer, New York, 1972.

[8] Rotman, J. An introduction to homological algebra, Academic Press, New York, 1979.

[9] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (I)——mixed product of modules, *Scientia Sinica* (*Ser.* A), **29**: 6 (1986), 570—581.

[10] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (II)——positive and negative graded modules, *Scientia Sinica* (*Ser. A*), **29**: 10 (1986), 1009—1019.

[11] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (III)——irreducible modules, *Chin. Ann. of Math.*, **9B**:4 (1988), 404—417.

[12] Strade, H. & Farnsteiner, R., Modular Lie algebras and their representations, Dekker 116, New York, 1988.