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RESAMPLING METHOD UNDER DEPENDENT MODELS

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Abstract

As well known, the jackknife and the bootstrap methods fail for the mean of the dependent observations. Recently, the moving blocks jackknife and bootstrap have been proposed in the case of the dependent observations. For the mean of the strictly stationary and m-dependent observations, it has been proved that the proposed distribution and variance estimators are weakly consistent. This paper proves that the distribution and variance estimators are strongly consistent for the mean (and the regular functions of mean) of the strictly stationary and m-dependent or φ-mixing observations.

§ 1. Introduction

As two main methods in resampling theory, Quenouille-Tukey's jackknife and Efron's bootstrap have significant applications in modern statistics. We can use these methods to estimate the distribution of statistics and obtain the confidence interval of parameters, the consistent estimator of the asymptotic variance, etc. Unfortunately, even for simple statistics, such as sample mean, the ordinary resampling methods may fail when the observations are dependent. In 1981, Singh illustrated this problem with the example under m-dependent model. The practical data such as from meteorology, hydrology, etc. are usually dependent each other. How to use the resampling methods under dependent models is a very interesting topic with wide applications.

Many statisticians explored various methods to solve this problem. In general they noted the following fact: the resamples are drawn independently from the empirical distribution. So it can not simulate the dependent models. Usually there exist two methods to solve this difficulty. One is to make the resampling procedure resemble the sampling procedure. So we must know the original random sampling model and it is very restrictive. Sometime it is even impossible. The another

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method is to split the whole observations into several groups. The elements in each group must be succesive and keep their orders as in the original samples. So each group can be viewed as a random miniature of the original model. The resampling groups are drawn independently from these groups. Basing on this idea, Künsch and others proposed the movingblocks method, which has attracted considerable attention recently.

Let $X_1, \cdots, X_n$ be strictly stationary random variables with the common distribution F. Let B_j be the block consisting of b consecutive observations starting from X_j. Thus we have $(n-b+1)$ blocks, i. e., $B_1, B_2, \cdots, B_{n-b+1}$. For the moving block jackknife (MBJ), each block is deleted once to generate 'pseudo-values; for the moving block bootstrap (MBB), k blocks are drawn independently and with replacement from $B_1, B_2, \cdots, B_{n-b+1}$. All elements in k blocks are then regarded as the bootstrap samples.

Are the moving blocks methods reasonable in statistical theory? In a technical report in 1988 Liu and Singh made a preliminary exploration. For the mean of the strictly stationary and m-dependent observations, they proved the weak consistency of the MBJ and MBB variance estimators and then obtained the weak consistency of the MBB distribution. Whether it is possible to get better results under this model and whether the analogous results can be obtained under general dependent models are worthwhile to study.

This paper tries to solve the above questions. We study two weakly dependent models. In Section 2, the strong consistency of the MBJ and MBB variance estimators is proved for the mean of the strictly stationary and m-dependent observations. We also prove the strong consistency of MBB distribution. In section 3, analogous results are obtained when the observations are strictly stationary and weak φ-mixing. We also study the strong and weak consistency of MBB variance estimator under various conditions.

Throughout the paper, $\Phi(t)$ stands for the standard normal distribution; C denotes a constant which may depend on F but not on n and b.

§ 2. Strictly Stationary and m-Dependent Case

Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of random variables. $\{X_i\}_{i=1}^{\infty}$ are called strictly stationary if for any k, t and $(i_1, \cdots, i_k)$, $(X_{i_1}, \cdots, X_{i_k})$ and $(X_{i_1+t}, \cdots, X_{i_k+t})$ have the same distribution. $\{X_i\}_{i=1}^{\infty}$ are called m-dependent if for any k, $(X_1, \cdots, X_k)$ and $(X_{k+m+1}, \cdots)$ are independent.

In the following we assume $EX_1=0$ for convenience. We have the following conclusions.

Lemma 2. 1. *Let $X_1, \cdots, X_n$ be a sequence of strictly stationary and m-dependent random variables. If $E|X_1|^p<\infty (p\geqslant 2)$, then*

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant Cn^{p/2}m^{p/2}. \tag{2.1}$$

Proof It is easy to see that we can only consider the $n=km$ case. Write

$$\sum_{i=1}^{n} X_i = \{X_1+X_{m+1}+\cdots+X_{(k-1)m+1}\}+\{X_2+X_{m+2}+\cdots+X_{(k-1)m+2}\}$$
$$+\cdots+\{X_m+X_{2m}+\cdots+X_{km}\}=\sum_{i=1}^{n} R_i. \tag{2.2}$$

Clearly $E|R_i|^p<\infty$. Note that each R_i is the sum of k i. i. d. random variables. By Marcinkewicz-Zygmund inequality, we have

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant m^p E|R_1|^p \leqslant Cm^p k^{p/2} E|X_1|^p = Cn^{p/2}m^{p/2}.$$

This proves Lemma 2.1.

Remark. In general, m is a fixed positive integer, and yet we know the lemma still holds if $m=n^\alpha$ $(0<\alpha<1)$ from the above proof.

At first, we discuss the jackknife case. Let $\bar{X}_{n,-i}$ denote the average of observations except B_i. Define the pseudo-values

$$J_i=b^{-1}[n\bar{X}_n-(n-b)\bar{X}_{n,-i}]=b^{-1}\sum_{j=1}^{i+b-1} X_j (i=1, \cdots, n-b+1) \tag{2.3}$$

and the jackknife variance estimator of $n^{1/2}\bar{X}_n$

$$\hat{V}_{J,b}(\bar{X}_n)=b\left\{(n-b+1)^{-1}\sum_{i=1}^{n-b+1}\left(b^{-1}\sum_{j=1}^{i+b-1} X_j-\bar{X}_n\right)^2\right\}$$
$$=(n-b+1)^{-1}\sum_{i=1}^{n-b+1}\left[b^{-1}\left(\sum_{j=1}^{i+b-1} X_j\right)^2\right]+b\bar{X}_n^2$$
$$-2b^{1/2}\bar{X}_n(n-b+1)^{-1}\sum_{i=1}^{n-b+1}\left[b^{-1/2}\sum_{j=1}^{i+b-1} X_j\right]\triangleq A_n+B_n+C_n. \tag{2.4}$$

Theorem 2. 1. *Let $X_1, X_2, \cdots, X_n$ be a sequence of strictly stationary and m-dependent variables. If $E|X_1|^{4+s}<\infty$ and $b\sim n^\alpha$ for any $s>0$ and $0<\alpha<s/(4+s)$, then*

$$\hat{V}_{J,b}(\bar{X}_n)\to \mathrm{Var}(X_1)+2\sum_{j=1}^{m}\mathrm{Cov}(X_1, X_{1+j}) \text{ a. s.} \tag{2.5}$$

Proof Since m is fixed, using (2.2) and $k^{-1}R_i=O(k^{-1/2}(\log\log k)^{1/2})$ a. s. which is the property of the sum of i. i. d. random variables, we get

$$\bar{X}_n=O(n^{-1/2}(\log\log n)^{1/2}) \text{ a. s.}$$

Hence $b^{1/2}\bar{x}_n\to 0$ a. s. So $D_n\to 0$ a. s. Clearly

$$\mathrm{Var}\left(b^{-1/2}\sum_{j=1}^{b} X_j\right)=\mathrm{Var}(X_1)+2\sum_{j=1}^{m}\left(1-\frac{j}{b}\right)\mathrm{Cov}(X_1, X_{1+j})$$
$$\to \mathrm{Var}(X_1)+2\sum_{j=1}^{m}\mathrm{Cov}(X_1, X_{1+j}) \quad (\text{as } b\sim n^\alpha\to\infty). \tag{2.7}$$

Note that $\left(b^{-1/2}\sum_{j=1}^{i+b-1} X_j\right)^2-\mathrm{Var}\left(b^{-1/2}\sum_{j=1}^{b} X_j\right)$ $(i=1, 2, \cdots, n-b+1)$

are strictltl stationary and $(b+m)$-dependent. The applications of the remark of Lemma 2.1 and Markov inequality yield that for any $\delta>0$

$$P\left\{\left|A_n-\operatorname{Var}\left(b^{-1/2}\sum_{j=1}^{b}X_j\right)\right|>\delta\right\}\sim Cn^{-(1+s/4)(1-\alpha)}. \tag{2.8}$$

Hence by the Borel-Cantelli Lemma, we have

$$A_n\to\operatorname{Var}(X_1)+2\sum_{j=1}^{m}\operatorname{Cov}(X_1, X_{1+j}) \quad \text{a. s.} \tag{2.9}$$

The results of A_n and D_n imply that $C_n\to 0$ a. s. This completes the proof.

The moving blocks jackknife estimator of $\bar{X}_n$ is the average of $(n-b+1)$ pseudo-values, that is,

$$\begin{aligned} J_b(\bar{X}_n)&=\frac{1}{b(n-b+1)}\sum_{i=1}^{n-b+1}(X_i+\cdots+X_{i+b-1})\\ &=\bar{X}_n+\left[\frac{n}{(n-b+1)}-1\right]\bar{X}_n\\ &\quad-\frac{1}{b(n-b+1)}\{[(b-1)X_1+(b-2)X_2+\cdots+2X_{b-2}+X_{b-1}]\\ &\quad+[(b-1)X_n+\cdots+2X_{n-(b-3)}+X_{n-(b-2)}]\}. \end{aligned} \tag{2.10}$$

Lemma 2. 2. *Under the conditions of Theorem* 2.1, *for* $0<\beta<\frac{2+s}{2(4+s)}$, *we have*

$$n^{\beta}J_b(\bar{X}_n)\to 0 \quad a.\ s. \tag{2.11}$$

Proof Following the proof of (2.8), for $\delta>0$, we have

$$P\{|n^{\beta}J_b(\bar{X}_n)|>\delta\}\leqslant Cn^{-(2+s/2)+(4+s)\beta}. \tag{2.12}$$

Thus

$$\sum_{n=1}^{\infty}P\{|n^{\beta}J_b(\bar{X}_n)|>\delta\}<\infty. \tag{2.13}$$

So (2.2) follows.

In view of the classical jackknife, the jackknife variance estimator of $n^{1/2}\bar{X}_n$ can also be defined as

$$\tilde{V}_{J,b}(\bar{X}_n)=b\left\{\frac{1}{n-b+1}\sum_{i=1}^{n-b+1}\left(b^{-1}\sum_{j=1}^{i+b-1}X_j-J_b(\bar{X}_n)\right)^2\right\}. \tag{2.14}$$

It can be seen that the difference between $J_b(\bar{X}_n)$ and $\bar{X}_n$ yields the difference between $\hat{V}_{J,b}(\bar{X}_n)$ and $\tilde{V}_{J,b}(\bar{X}_n)$. Since

$$\begin{aligned} &\left|\frac{1}{b(n-b+1)}\{(b-1)X_1+(b-2)X_2+\cdots 2X_{b-2}+X_{b-1}\}\right|\\ &\leqslant\frac{b-1}{n-b+1}\,\frac{|X_1|+\cdots+|X_{b-1}|}{b-1} \end{aligned} \tag{2.15}$$

and since

$$\frac{|X_1|+\cdots+|X_{b-1}|}{b-1}\to E|X_1| \quad \text{a. s. (as } b\to\infty)$$

which can be obtained under the condition $E|X_1|^2<\infty$, we arrive at

Lemma 2. 3. *Under the conditions of Theorem* 2.1,

$$J_b(\bar{X}_n)-\bar{X}_n=O(b/n) \quad \text{a. s.} \tag{2.16}$$

Using the above result and the central limit theorem for the strictly stationary and m-dependent sequences, we have

Theorem 2. 2. *Let $X_1, \cdots, X_n$ be strictly stationary and m-dependent random variables. If $E|X_1|^{4+s}<\infty$ and $b\sim n^\alpha$ for any $s>0$ and $0<\alpha<\min\left[\frac{s}{4+s}, \frac{1}{2}\right]$, then*

1. $\hat{V}_{J,b}(\bar{X}_n)\to \mathrm{Var}(X_1)+2\sum_{j=1}^{m}\mathrm{Cov}(X_1, X_{1+j})$ a. s., (2.18)

2. $\frac{n^{1/2}(J_b(\bar{X}_n)-EX_1)}{\hat{\sigma}}\xrightarrow[\mathscr{L}]{} N(0, 1)$, (2.19)

where $\hat{\sigma}^2=\hat{V}_{J,b}(\bar{X}_n)$ *or* $\tilde{V}_{J,b}(\bar{X}_n)$.

We continue to study the bootstrap behavior. Let $\xi_1, \xi_2, \cdots, \xi_n$ be i. i. d. sampled blocks from $B_1, B_2, \cdots, B_{n-b+1}$, where $\xi_i=(\xi_{i1}, \xi_{i2}, \cdots\xi_{ib})(i=1, \cdots, k)$. Let $(Y_1, Y_2, \cdots Y_l)=(\xi_1, \cdots, \xi_k)$ stand for the whole $l=bk$ resamples. Let F_l^* be the empirical distribution based on these l resamples. Following the idea of the bootstrap, we use the diagram of $l^{1/2}(\bar{Y}_l-\bar{X}_n)$ to estimate the distribution of $n^{1/2}(\bar{X}_n-EX)$ and the variance of $l^{1/2}\bar{Y}_l$ to estimate the variance of $n^{1/2}\bar{X}_n$. The following two theorems show the a. s. convergence of the bootstrap estimators. Notations such as Var^* and P^* refer to probability calculations under the resampling model.

Theorem 2. 3. *Let $X_1, X_2, \cdots, X_n$ be a sequence of strictly stationary and m-dependent random variables. If $E|X_1|^{4+s}<\infty$ and $b\sim n^\alpha$ for any fixed $s>0$ and $0<\alpha<\frac{s}{4+s}$, we have*

$$\mathrm{Var}^*(l^{1/2}\bar{Y}_l)\to \mathrm{Var}(X_1)+2\sum_{j=1}^{m}\mathrm{Cov}(X_1, X_{1+j}) \quad \text{a. s.} \tag{2.19}$$

Proof. Let $\tilde{\xi}_i=(\xi_{i1}+\cdots+\xi_{ib})/b^{1/2}(i=1, 2, \cdots, k)$. Noting that $\tilde{\xi}_1, \cdots, \tilde{\xi}_k$ are i. i. d. under the MBB scheme, we get $l^{1/2}\bar{Y}_l=k^{-1/2}\sum_{i=1}^{k}\tilde{\xi}_i$ and $\mathrm{Var}^*(l^{1/2}\bar{Y}_l)=\mathrm{Var}^*(\tilde{\xi}_1)$. Let $\tilde{B}_i=b^{-1/2}(X_i+X_{i+1}+\cdots X_{i+b-1})$. Then

$$\begin{aligned}\mathrm{Var}^*(\tilde{\xi}_1)&=\frac{1}{n-b+1}\sum_{i=1}^{n-b+1}(\tilde{B}_i-E^*\tilde{\xi}_1)^2\\&=\frac{1}{n-b+1}\sum_{i=1}^{n-b+1}\tilde{B}_i^2-\left(\frac{1}{n-b+1}\sum_{i=1}^{n-b+1}\tilde{B}_i\right)^2\\&=A_n-(b^{1/2}J_b(\bar{X}_n))^2.\end{aligned} \tag{2.20}$$

Note that $b^{1/2}\sim n^{\alpha/2}$ and $0<\alpha/2<\frac{s}{2(4+s)}<\frac{2+s}{2(4+s)}$. By Lemma 2.2 and Theorem 2.1, we get $n^{1/2}J_b(\bar{X}_n)\to 0$ a. s. and $A_n\to \mathrm{Var}(X_1)+2\sum_{j=1}^{m}\mathrm{Cov}(X_1, X_{1+j})$ a. s. This completes the proof.

Theorem 2. 4. *Under the conditions of Theorem 2.3, as k and $n\to\infty$, we have*

$$\sup_t |P^*\{l^{1/2}(\overline{Y}_l - E^*\overline{Y}_l) \leqslant t\} - P\{n^{1/2}\overline{X}_n \leqslant t\}| \to 0 \quad \text{a. s.} \tag{2.21}$$

Proof Clearly $l^{1/2}(\overline{Y}_l - E^*\overline{Y}_l) = k^{-1/2}\sum_{i=1}^{k}(\tilde{\xi}_i - E^*\tilde{\xi}_1)$. Recall the result of Katz[4]: Let $Z_1, \cdots, Z_n$ be i. i. d. random variables with $EZ_1 = 0$ and $\text{Var } Z_1 = 1$. Let $g(z)$ satisfy: (1) it is a nonnegative, even and nondecreasing function and $\lim_{z\to\infty} g(z) = \infty$. (2) $z/g(z)$ is nondecreasing on $(0, \infty)$. If $Ez_1^2 g(z_1) < \infty$, then there exists a constant C such that

$$\sup_t \left| P\left\{\sum_{i=1}^{n} z_i/n^{1/2} \leqslant t\right\} - \Phi(t) \right| \leqslant C E z_1^2 g(z_1)/g(n^{1/2}).$$

Let $z_i = (\tilde{\xi}_i - E^*\tilde{\xi}_1)/(\text{Var}^*\tilde{\xi}_1)^{1/2}$ and $g(z) = |z|^{s/p}$, where $p \geqslant \max(s, 2)$ and is a constant. Then

$$\sup_t |P^*\{l^{1/2}(\overline{Y}_l - E^*\overline{Y}_l)/(\text{Var}^*\tilde{\xi}_1)^{1/2} \leqslant t\} - \Phi(t)|$$
$$\leqslant C \frac{E^*|\tilde{\xi}_1 - E^*\tilde{\xi}_1|^{2+s/p}}{k^{s/2p}(\text{Var}^*\tilde{\xi}_1)^{1+s/2p}}. \tag{2.22}$$

By C_r-inequality, $E^*|\tilde{\xi}_1 - E^*\tilde{\xi}_1|^{2+s/p} \leqslant C(E^*|\tilde{\xi}_1|^{2+s/p} + |E^*\tilde{\xi}_1|^{2+s/p})$. Using the same method as in Theorem 2.1, we can prove that for any $\delta > 0$

$$P\{|E^*|\tilde{\xi}_1|^{2+s/p} - |E^*\tilde{\xi}_1|^{2+s/p}| > \delta\} \sim Cn^{-(1-\alpha)\frac{(4+s)p}{2(2p+s)}}. \tag{2.23}$$

In view of the conditions on α, p and the Borel-Cantelli Lemma, we have

$$E^*|\tilde{\xi}_1|^{2+s/p} - E|\tilde{B}_1|^{2+s/p} \to 0 \quad \text{a. s.} \tag{2.24}$$

By Lemma 2.1 we have $E|\tilde{B}|^{2+s/p} \leqslant C$. Thus

$$E^*|\tilde{\xi}_1|^{2+s/p} \leqslant C \quad \text{a. s.} \tag{2.25}$$

It has been proved that $\text{Var}^*(\tilde{\xi}_1) \to \text{Var}(X_1) + 2\sum_{j=1}^{m}\text{Cov}(X_1, X_{1+j})$ a. s. and $E^*\tilde{\xi}_1 \to 0$ a. s. . So

$$\sup_t |P^*\{l^{1/2}(\overline{Y}_l - E^*\overline{Y}_l) \leqslant t\} - P\{n^{1/2}\overline{X}_n \leqslant t\}| \to 0 \text{ a. s.} \tag{2.26}$$

The theorem follows from the above result and the central limit theorem for the strictly stationary and m-dependent sequences.

§ 3. Strictly Stationary and φ-Mixing Case

$\{X_i\}_{i=1}^{\infty}$ are called φ-mixing if there exits a sequence $\{\varphi(n)\}$ such that $1 \geqslant \varphi(1) \geqslant \varphi(2) \geqslant \cdots$, $\lim_{n\to\infty}\varphi(n) = 0$ and

$$\sup_{k\geqslant 1}\ \sup_{\substack{B\in M_1^k \\ p(B)>0}}\ \sup_{A\in M_{k+n}^{\infty}} |P(A|B) - P(A)| \leqslant \varphi(n)$$

where M_a^b denotes the σ-field generated by $\{X_i(a \leqslant i \leqslant b)\}$.

Throughout this section, we assume $EX_1 = 0$, $EX_1^2 < \infty$, $\sigma^2 = EX_1^2 + 2\sum_{k=2}^{\infty} EX_1 X_k$

converges absolutely and $\sigma^2>0$.

We start with two lemmas which can be found in Billingsley [5] and Schneider [6] respectively.

Lemma 3. 1. *Let $X_1, X_2, \cdots$ be a sequence of strictly stationary and φ-mixing random variables. Let ξ and η be two random variables measurable with respect to the σ-fields M_1^k and M_{k+n}^∞, respectively. If $E|\xi|^r<\infty$ and $E|\eta|^s<\infty$, where $r, s>1$ and $r^{-1}+s^{-1}=1$, then*

$$|E\xi\eta-E\xi E\eta|\leqslant 2\varphi^{1/r}(n)E^{1/r}\{|\xi|^r\}E^{1/s}\{|\eta|^s\}. \tag{3.1}$$

Lemma 3. 2. *Let $X_1, X_2, \cdots$ be φ-mixing random variables with $EX_n=0$ $(n=1, 2, \cdots)$ and $\sum_{n=1}^{\infty}\{\varphi(n)\}^{1/2}<\infty$. If $\sup_n E|X_n|^s\leqslant N$ for some $s>2$ and $N>1$, then for $v\in[2, s]$, we have*

$$E\left|\sum_{i=1}^{n}X_i\right|^v\leqslant 2C(v)n^{v/2} \tag{3.2}$$

where $C(v)$ depends only on φ, v, s and N.

The following theorems present the conditions under which the weak and strong convergence of MBJ and MBB variance estimators are obtained. Like the discussion in Section 2, the proofs of the two variance estimators are similar. In order to save space, we only prove the weak convergence of MBJ variance estimator and the strong convergence of MBB variance estimator.

Theorem 3. 1. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of strictly stationary and φ-mixing random variables with $\sum_{n=1}^{\infty}\varphi^{1/2}(n)<\infty$ and $EX_1^4<\infty$, as $b\to\infty$ and $b/n\to 0$. We have*

$$\hat{V}_{J,b}(\overline{X}_n)\xrightarrow{p}\varphi^2. \tag{3.3}$$

Proof Following (2.4), we write

$$\hat{V}_{J,b}(\overline{X}_n)=A_n+B_n+C_n.$$

For any $\delta>0$, by Lemma 3.2 and Markov inequality,

$$P\{|b^{1/2}\overline{X}_n|>\delta\}\leqslant C\frac{b}{n}\longrightarrow 0. \tag{3.4}$$

So $b^{1/2}\overline{X}_n\to 0$. That means $D_n\xrightarrow{p}0$. It remains to show $A_n\xrightarrow{p}\sigma^2$. We use the same symbles as in section 2. Let $\eta_i=\widetilde{B}_i^2-E\widetilde{B}_1^2$ $(i=1, 2, \cdots, n-b+1)$. Clearly $\{\eta_i\}$ are strictly stationary. Note that

$$E\left(\sum_{i=1}^{n-b+1}\eta_i\right)^2\leqslant(n-b+1)E\eta_1^2+2(n-b)\sum_{j=1}^{b}|E(\eta_1\eta_{1+j})|$$
$$+2(n-b)\sum_{j=b+1}^{n-b}|E(\eta_1\eta_{1+j})|. \tag{3.5}$$

By Lemma 3.1 we have, for $b+1\leqslant j\leqslant n-b$,

$$|E(\eta_1\eta_{1+j})|\leqslant 2\varphi^{1/2}(j-b+1)E\eta_1^2. \tag{3.6}$$

So

$$E\left(\sum_{i=1}^{n-b+1}\eta_i\right)^2\leqslant\left\{n+2bn+4(n-b)\sum_{j=1}^{\infty}\varphi^{1/2}(j)\right\}E\eta_1^2\leqslant Cnb \tag{3.7}$$

Therefore, for any $\delta>0$,

$$P\{|A_n-\operatorname{Var}\widetilde{B}_1|>\delta\}\leqslant\delta^{-2}E\left(\sum_{i=1}^{n-b+1}\eta_i\right)^2/(n-b+1)^2$$
$$\leqslant Cnb/(n-b+1)^2\to 0. \tag{3.8}$$

Using the Lemma 3 of Billingsley [5] (p. 172), we get $\operatorname{Var}(\widetilde{B}_1)\to\sigma^2$. Thus $A_n\xrightarrow{p}\sigma^2$. The proof is completed.

Lemma 3. 3. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of strictly stationary and φ-mixin random variables with $EX_1=0$, $E|X_1|^s<\infty (s\geqslant 8)$ and $\sum_{n=1}^{\infty}\varphi^{(s-4)/2s}(n)<\infty$. Let $\eta_i=\widetilde{B}_i^2-E\widetilde{B}_1^2 (i=1, \cdots, n-b+1)$. If $b\sim n^\alpha$, where $\alpha\in(0, 2^{-1})$, then*

$$E\left(\sum_{j=1}^{n-b+1}\eta_j\right)^4\leqslant C(n-b+1)^2b^2. \tag{3.9}$$

Proof Since $\{\eta_j\}_{j=1}^{n-b+1}$ are strictly stationary, we arrive at

$$E\left(\sum_{j=1}^{n-b+1}\eta_j\right)^4\leqslant 4!(n-b+1)\sum|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|, \tag{3.10}$$

where $\sum$ denotes the sum over

$$\{i, j, k\geqslant 0, i+j+k\leqslant n-b\}. \tag{*}$$

We now divide the above set into

$$(*)=\{i<b, j<b, k<b, (*)\}\cup\{i<b, j<b, k\geqslant b, (*)\}$$
$$\cup\{i<b, j\geqslant b, k<b, (*)\}\cup\{i<b, j\geqslant b, k\geqslant b, (*)\}$$
$$\cup\{i\geqslant b, j<b, k<b, (*)\}\cup\{i\geqslant b, j<b, k\geqslant b, (*)\}$$
$$\cup\{i\geqslant b, j\geqslant b, k<b, (*)\}\cup\{i\geqslant b, j\geqslant b, k\geqslant b, (*)\}$$
$$=\sum_{h=1}^{8}I_h. \tag{3.11}$$

By Lemma3.2 we have, for any $(i, j, k)\in(*)$,

$$|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|\leqslant E\eta_1^4<\infty. \tag{3.12}$$

Hence

$$\sum_{I_1}|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|\leqslant b^3E\eta_1^4<C(n-b+1)b^2, \tag{3.13}$$

$$\sum_{I_2}|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|\leqslant b^2(n-b+1)E\eta_1^4<C(n-b+1)b^2. \tag{3.14}$$

The situation of I_3 and I_5 is similar to that of I_2. And the proofs of I_4, I_6 and I_7 are similar, so we only take I_6 for example. From Lemma 3.1,

$$|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|$$
$$\leqslant 2\varphi^{(s-2)/s}(k-b+1)[E|\eta_1\eta_{1+i}\eta_{1+i+j}|^{s/(s-2)}]^{(s-2)/s}[E|\eta_{1+i+j+k}|^{s/2}]^{2/s}$$
$$\leqslant 2\varphi^{(s-2)/s}(k-b+1)[E|\eta_1\eta_{1+i}\eta_{1+i+j}|^{s/6}]^{6/s}[E|\eta_{1+i+j+k}|^{s/2}]^{2/s}$$
$$\leqslant 2\varphi^{(s-2)/s}(k-b+1)[E|\eta_1|^{s/2}]^{8/s}. \tag{3.15}$$

Hence

$$\sum_{I_6}|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|\leqslant C(n-b+1)b. \tag{3.16}$$

Also using Lemma 3.1, for $(i, j, k)\in I_8$, we have

$$\begin{aligned}&|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|\\&\leqslant 2\varphi^{(s-4)/s}(j-b+1)[E|\eta_1\eta_{1+i}|^{s/(s-4)}]^{(s-4)/s}[E|\eta_{1+i+j}\eta_{1+i+j+k}|^{s/4}]^{4/s}\\&\leqslant 2\varphi^{(s-4)/s}(j-b+1)[E|\eta_1|^{s/2}]^{8/s}.\end{aligned} \tag{3.17}$$

Combining (3.15) with (3.17), we have

$$\begin{aligned}&\sum_{I_8}|E(\eta_1\eta_{1+i}\eta_{1+i+j}\eta_{1+i+j+k})|\\&\leqslant \sum_{(I_8, j<k)}2\varphi^{(s-2)/s}(k-b+1)[E|\eta_1|^{s/2}]^{8/s}\\&\quad+\sum_{(I_8, k<j)}2\varphi^{(s-4)/s}(j-b+1)[F|\eta_1|^{s/2}]^{8/s}\\&\leqslant 2[E|\eta_1|^{s/2}]^{8/s}\Big\{(n-b+1[\sum_{k=b}^{n-b+1}\sum_{j=b}^{k}\varphi^{(s-2)/s}(k-b+1)\\&\quad+(n-b+1)\sum_{j=b}^{n-b+1}\sum_{k=b}^{j}\varphi^{(s-4)/s}(j-b+1)\\&\leqslant C(n-b+1)\sum_{j=b}^{n-b+1}\sum_{k=j}^{n-b+1}\varphi^{(s-4)/2s}(k-b+1)\varphi^{(s-4)/2s}(j-b+1)\\&<C(n-b+1).\end{aligned} \tag{3.18}$$

The conclusion follows by substituting (3.13), (3.14), (3.16) and (3.18) into the left hand side of (3.10).

Theorem 3. 2. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a squence of strictly stationary and φ-mixing random variables with $EX_1=0$, $E|X_1|^s<\infty (s\geqslant 8)$ and $\sum_{n=1}^{\infty}\varphi^{(s-4)/2s}(n)<\infty$. If $b\sim n^{\alpha}$, where $\alpha\in(0, 2^{-1})$, then*

$$\mathrm{Var}^*(l^{1/2}\overline{Y}_l)\to\sigma^2 \quad \text{a. s.} \tag{3.19}$$

Proof It is enough to prove that $\mathrm{Var}^*(\tilde{\xi}_1)\to\sigma^2$ a. s. By Lemma 3.3, for any $\delta>0$, we have

$$P\left\{\left|\frac{1}{n-b+1}\sum_{j=1}^{n-b+1}\eta_j\right|>\delta\right\}\leqslant\frac{1}{\delta^4}\frac{1}{(n-b+1)^4}Cb^2(n-b+1)^2. \tag{3.20}$$

Since $b\sim n^{\alpha}$, where $\alpha\in(0, 2^{-1})$, by the Borel-Cantelli Lemma, we have

$$\frac{1}{n-b+1}\sum_{i=1}^{n-b+1}\tilde{B}_i^2\to\sigma^2 \quad \text{a. s.}$$

So
$$\mathrm{Var}^*(\tilde{\xi}_1)\to\sigma^2 \quad \text{a. s.} \tag{3.21}$$

The proof is completed.

We continue to study the a. s. convergence of the bootstrap distribution.

Theorem 3. 3. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a squence of strictly stationary and φ-mixing random variables with $EX_1=0$, $E|X_1|^{s+\varepsilon}<\infty$ for some $s\geqslant 8$ and $\varepsilon>0$, and $\sum_{n=1}^{\infty}\varphi^{(s-4)/2s}(n)<\infty$. If $b\sim n^{\alpha}$, where $\alpha\in(0, 2^{-1})$, then as k, $n\to\infty$*

$$\sup_t|P^*\{l^{1/2}(\overline{Y}_l-E^*\overline{Y}_l)\leqslant t\}-P\{n^{1/2}\overline{X}_n\leqslant t\}|\to 0 \quad a. s. \tag{3.22}$$

Proof The proof is similar to that of Theorem 2.4. Applying (2.22) with $p=4$ we get

$$\sup_t |P^*\{l^{1/2}(\overline{Y}_l - E^*\overline{Y}_l)/(\mathrm{Var}^*\tilde{\xi}_1)^{1/2} \leqslant t\} - \Phi(t)|$$

$$\leqslant C\frac{E^*|\tilde{\xi}_1|^{2+\frac{\varepsilon}{4}} + |E^*\tilde{\xi}_1|^{2+\frac{\varepsilon}{4}}}{k^{\varepsilon/8}(\mathrm{Var}^*\tilde{\xi}_1)^{1+\frac{\varepsilon}{8}}}. \tag{3.24}$$

To prove that the above expression convergences a. s. to 0, it is enough to prove $E^*|\tilde{\xi}_1|^{2+\varepsilon/4} - E|\tilde{B}_1|^{2+\varepsilon/4} \to 0$ a. s. and $E|\tilde{B}_1|^{2+\varepsilon/4} \leqslant C$. But the latter is easy to prove. Applying the method of Lemma 3.3 and Theorem 3.2 with η_i been replaced by $(|\tilde{B}_i|^{2+\varepsilon/4} - E|\tilde{B}_i|^{2+\varepsilon/4})$, we can prove $E^*|\tilde{\xi}_1|^{2+\varepsilon/4} - E|\tilde{B}_1|^{2+\varepsilon/4} \to 0$ a. s. Thus we obtain (3.23). The conclusion follows from (3.23) and the central limit theorem for the strictly stationary and φ-mixing sequences.

Remark. It is easy to extend our results for $\overline{X}_n$ to the regular function of $\overline{X}_n$. The proof is trivial, so we omitted the details.

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