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ON HILBERT'S INEQUALITY

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Abstract

This paper gives some improvements of Hilbert's inequality. The main results are:

(i) $$\left|\int_0^\infty\int_0^\infty \frac{f(x)g(x)}{x+y}\,dx\,dy\right|^4 \leqslant \pi^4\left\{\left(\int_0^\infty f^2(x)dx\right)^2 - \left(\int_0^\infty f^2(x)\,k(x)\,dx\right)^2\right\}\left\{\left(\int_0^\infty g^2(x)dx\right)^2 - \left(\int_0^\infty g^2(x)k(x)dx\right)^2\right\}$$

where $k(x)=\frac{2}{\pi}\int_0^\infty \frac{1}{1+t^2}c(t^2x)dt - c(x)$, $1-c(x)+c(y)\geqslant 0$ and $f, g\geqslant 0$. The case $k(x)=0$ is Hilbert's integral form.

(ii) $$\left|\sum_{r,s=1}^{n}{}' \frac{a_r b_r}{r+s}\right|^2 + \left|\sum_{\substack{r,s=1\\ r\neq s}}^{n} \frac{a_r b_s}{r-s}\right|^2 \leqslant \pi^2\left(\sum_{r=1}^{n}|a_r|^2\sum_{s=1}^{n}|b_s|^2\right).$$

Take zero in place of the second term of the left part, then it reduces to Hilbert's discrete form. And

(iii) An improvment of Hardy-Hilbert Inequality.

Hilbert's inequality is an important theorem for analytic function which may be put into two forms: integral and discrete

(A) $$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy \leqslant \pi\sqrt{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx},\quad f,\ g\geqslant 0$$

(B) $$\left|\sum_{r,s=1}^{n}\frac{a_r b_s}{r+s}\right| \leqslant \pi\sqrt{\sum_{r=1}^{n}|a_r|^2\sum_{s=1}^{n}|b_s|^2}.$$

The following inequality is in the name of Hilbert:

(C) $$\left|\sum_{\substack{r,s=1\\ r\neq s}}^{n}\frac{a_r b_s}{r-s}\right|^2 \leqslant A\sqrt{\sum_{r=1}^{n}|a_r|^2\sum_{s=1}^{n}|b_s|^2},$$

where $A=3\pi$. If a's and b's are real numbers, then $A=2\pi$; if $b=\bar{a}$, then $A=\pi$ (see [1, 3]).

Theorem 1. *Let positive functions $f(x), g(x)\in L^2(0,\infty)$. Then*

$$\left(\int_0^\infty\int_0^\infty f(x)g(x)dx\,dy\right)^2 \leqslant \pi^4\left\{\left(\int_0^\infty f^2(x)dx\right)^2 - \left(\int_0^\infty (f^2(x)k(x)dx\right)^2\right\}\left\{\left(\int_0^\infty g^2(x)dx\right)^2 - \left(\int_0^\infty g^2(x)k(x)dx\right)^2\right\},$$

where $k(x)=\frac{2}{\pi}\int_0^\infty \frac{c(xt^2)}{1+t^2}dt - c(x)$ *and* $1-c(x)+c(y)\geqslant 0$.

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Proof First, suppose that $g=f$ and let $c(x)$ be a real function such that $1-c(x)+c(y)\geqslant 0$. By Cauchy-Schwarz inequality, we have

$$J=\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}\,dx\,dy=\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}[1-c(x)+c(y)]\,dx\,dy\leqslant\sqrt{J_1J_2}, \tag{2.2}$$

where

$$J_1=\int_0^\infty\int_0^\infty \frac{f(y)}{x+y}\left(\frac{y}{x}\right)^{\frac{1}{2}}(1-c(x)+c(y))\,dx\,dy,$$

$$J_2=\int_0^\infty\int_0^\infty \frac{f^2(x)}{x+y}\left(\frac{x}{y}\right)^{\frac{1}{2}}(1-c(x)+c(y))\,dx\,dy.$$

Since

$$J_1=\pi\int_0^\infty f^2(y)dy-\pi\int_0^\infty k(y)f^2(y)dy,\ J_2=\pi\int_0^\infty f^2(x)dx+\int_0^\infty k(x)f^2(x)dx, \tag{2.3}$$

the theorem follows from (2.3) and (2.2).

If $g\neq f$, note that

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy=\int_0^\infty\left(\int_0^\infty e^{-xw}f(x)\,dx\int_0^\infty e^{-yw}g(y)\,dy\right)dw$$

$$\leqslant\sqrt{\int_0^\infty\left(\int_0^\infty e^{-xw}f(x)dx\right)^2dw\int_0^\infty\left(\int_0^\infty e^{-yw}g(y)dy\right)^2dw}$$

$$=\sqrt{\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y}\,dx\,dy\int_0^\infty\int_0^\infty \frac{g(x)g(y)}{x+y}\,dx\,dy}.$$

Using Theorem 1 for $g=f$, the theorem follows at once.

Example 1. If $c(x)=\cos\sqrt{x}$, then $k(x)=\frac{1}{2}(e^{-\sqrt{x}}-\cos\sqrt{x})$.

2. If $c(x)=1$, then $k(x)=0$.

Theorem 2. *Let* $a_n\geqslant 0$ *and*

$$A(x)=\sum_{n=0}^\infty a_nx^n,\quad A^*(x)=\sum_{n=0}^\infty \frac{a_nx^n}{n!}.$$

Then

$$\left\{\int_0^1 A^2(x)dx\right\}^2\leqslant\pi^2\left\{\left[\int_0^\infty(e^{-x}A^*(x))^2\,dx\right]^2-\left[\int_0^\infty(e^{-x}A^*(x))^2k(x)\,dx\right]^2\right\},$$

where $k(x)$ *is defined by* (2.1).

The case $k(x)=0$ *is Widder's Theorem.*[2]

Proof Since

$$A(x)=\int_0^\infty e^{-t}A^*(xt)dt=\frac{1}{x}\int_0^\infty e^{-\frac{u}{x}}A^*(u)du$$

and

$$\int_0^1 A^2(x)\mathrm{d}x=\int_0^1\frac{1}{x^2}\,\mathrm{d}x\left\{\int_0^\infty e^{-\frac{u}{x}}A^*(u)du\right\}^2$$

$$=\int_1^\infty dy\left\{\int_0^\infty e^{-uy}A^*(u)du\right\}^2=\int_0^\infty dw\left\{\int_0^\infty e^{-uw}a(u)du\right\}^2$$

where $\alpha(u)=e^{-u}A^*(u)$, by Theorem 1 we have

$$\int_0^1 A^2(x)dx=\int_0^\infty\int_0^\infty \frac{\alpha(u)\alpha(v)}{u+v}du\,dv\leqslant\pi\sqrt{\left(\int_0^\infty \alpha^2(u)du\right)^2-\left(\int_0^\infty \alpha^2(u)k(u)du\right)^2}.$$

Thus the theorem is proved.

Theorem 3. *Let a's and b's be arbitrary complex numbers. Then*

$$\left|\sum_{r,s=1}^n \frac{a_r b_s}{r+s}\right|^2+\left|\sum_{\substack{r,s=1\\ r\neq s}}^n \frac{a_r b_s}{r-s}\right|^2\leqslant\pi^2\sum_{r=1}^n|a_r|^2\sum_{s=1}^n|b_s|^2. \tag{3.1}$$

Proof It is easy to deduce tnat

$$\int_{-\pi}^{\pi} t\left[\sum_{r=1}^n(-1)^r(a_r\cos(rt)-b_r\sin(rt))\right]^2 dt=2\pi(S-T), \tag{3.2}$$

where

$$S=\sum_{r,s=1}^n\frac{a_r b_s}{r+s},\quad T=\sum_{\substack{r,s=1\\ r\neq s}}^n\frac{a_r b_s}{r-s}.$$

Therefore

$$2\pi|S-T|\leqslant\pi\int_{-\pi}^{\pi}\left|\sum_{r=1}^n(-1)^r(a_r\cos(rt)-b_r\sin(rt))\right|^2dt$$

$$=\pi^2\sum_{r=1}^n(|a_r|^2+|b_r|^2). \tag{3.3}$$

It is important to notice that: (i) $\sum_{r,s=1}^n\frac{a_r b_s}{r+s}=\sum_{r,s=1}^n\frac{b_r a_s}{r+s}$, $\sum_{\substack{r,s=1\\ r\neq s}}^n\frac{a_r b_s}{r-s}=-\sum_{r\neq s}^n\frac{a_s b_r}{r-s}$,
(ii) Interchange a's and b's in (3.3), we obtain

$$2\pi|S+T|\leqslant\pi^2\left(\sum_{r=1}^n|a_r|^2+\sum_{r=1}^n|b_r|^2\right). \tag{2.4}$$

Squaring (3.3) and (3.4) and then adding them, we have

$$|S|^2+|T|^2\leqslant\frac{\pi^2}{4}\left\{\sum_{r=1}^n(|a_r|^2+|b_r|^2)\right\}^2. \tag{3.5}$$

Take $a_r\Big/\sqrt{\sum_{k=1}^n|a_k|^2}$, $b_r\Big/\sqrt{\sum_{k=1}^n|b_k|^2}$ in place of a_r and b_r respectively in (3.5), then the result follows.

Corollary. *If a's and b's are real numbers, then*

$$|S|+|T|\leqslant\pi\sqrt{\sum_{r=1}^n a_r^2\sum_{x=1}^n b_k^2}.$$

(3.3) and (3.4) in combination yield the assertionof (3.6).

Theorem 4. *Let $K(x, y)$ be a homogenous form of degree -1 and $K(x, y)\geqslant0$. Let $f(x)$, $g(x)\geqslant0$, $p\geqslant q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If $1-b(x)c(y)+c(x)b(y)\geqslant0$ and*

$$\int_0^\infty K(1, y)y^{-\frac{1}{q}}dy=\int_0^\infty K(x, 1)x^{-\frac{1}{p}}dx=k,$$

then

$$\int_0^\infty\int_0^\infty K(x, y)f(x)g(y)dx\,dy$$

$$\leqslant k^{\frac{1}{q}}\left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}-\frac{1}{p}}\left\{\left(k\int_0^\infty f^p(x)dx\int_0^\infty g^q(x)dx\right)^2\right.$$

$$-\left(\int_0^\infty f^p(x)C(x)dx\int_0^\infty g^q(x)b(x)dx\right.$$

$$\left.-\int_0^\infty f^p(x)B(x)\,dx\int_0^\infty g^q(x)c(x)dx\right)^2\Big\}^{\frac{1}{2p}}, \qquad (4.1)$$

where

$$B(x)=\int_0^\infty K(1,\,w)w^{-\frac{1}{q}}b(xw)dw,\ C(x)=\int_0^\infty K(1,\,w)w^{-\frac{1}{q}}c(xw)dw,$$

$b(x)=c(x)$ *is Hardy-Hilbert Theorem.*

The proof of this theorem is based on the following inequality[4]:

$$\int_0^\infty F(x)G(x)\,dx\leqslant\left(\int_0^\infty G^q(x)\,dx\right)^{\frac{1}{q}-\frac{1}{p}}\Big\{\left(\int_0^\infty F^p(x)dx\int_0^\infty G^q(x)\,dx\right)^2$$

$$-\left(\int_0^\infty F^p(x)c(wx)dx\int_0^\infty G^q(x)b(wx)dx-\int_0^\infty F^p(x)b(wx)dx\int_0^\infty G^q(x)c(wx)dx\right)^2\Big\}^{\frac{1}{2p}}, \qquad (4.2)$$

where $F,\ G\geqslant 0$.

Now we come to prove the theorem. By (4.2) we have

$$I=\int_0^\infty\int_0^\infty K(x,\,y)f(x)g(y)dx\,dy=\int_0^\infty K(1,\,w)dw\int_0^\infty f(x)g(wx)dx$$

$$\leqslant\int_0^\infty K(1,\,w)dw\left(\int_0^\infty g^q(wx)dx\right)^{\frac{1}{q}-\frac{1}{p}}\Big\{\left(\int_0^\infty f^p(x)dx\int_0^\infty g^q(wx)dx\right)^2$$

$$-\left(\int_0^\infty f^p(x)c(wx)dx\int_0^\infty g^q(wx)b(wx)dx-\int_0^\infty f^p(x)b(wx)dx\int_0^\infty g^q(wx)c(wx)dx\right)^2\Big\}^{\frac{1}{2p}}$$

$$=\int_0^\infty K(1,\,w)w^{-\frac{1}{q}}dw\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}}\Big\{\left(\int_0^\infty f^p(x)dx\int_0^\infty g^q(x)dx\right)^2$$

$$-\left(\int_0^\infty f^p(x)c(wx)dx\int_0^\infty g^q(x)b(x)dx-\int_0^\infty f^p(x)b(wx)dx\int_0^\infty g^q(x)c(x)dx\right)^2\Big\}^{\frac{1}{2p}}$$

$$=\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}-\frac{1}{p}}\int_0^\infty K(1,\,w)w^{-\frac{1}{q}}[J_1(w)J_2(w)]^{\frac{1}{2p}}dw, \qquad (4.3)$$

where

$$J_1(w)=\int_0^\infty f^p(x)dx\int_0^\infty g^q(x)dx-\int_0^\infty f^p(x)c(wx)dx\int_0^\infty g^q(x)b(x)dx$$

$$+\int_0^\infty f^p(x)b(wx)dx\int_0^\infty g^q(x)c(x)dx,$$

$$J_2(w)=\int_0^\infty f^p(x)dx\int_0^\infty g^q(x)dx+\int_0^\infty f^p(x)c(wx)dx\int_0^\infty g^q(x)b(x)dx$$

$$-\int_0^\infty f^p(x)b(wx)\,dx\int_0^\infty g^q(x)c(x)dx.$$

Applying Holder inequality to (4.3), we have

$$\int_0^\infty K(1, w) w^{-\frac{1}{q}} J_1^{\frac{1}{2p}}(w) J_2^{\frac{1}{2p}}(w) dw$$

$$\leqslant \left(\int_0^\infty K(1, w) w^{-\frac{1}{q}} dw\right)^{\frac{1}{q}} \left\{\int_0^\infty K(1, w) w^{-\frac{1}{q}} J_1(w) dw \int_0^\infty K(1, w) w^{-\frac{1}{q}} J_2(w) dw\right\}^{\frac{1}{2p}}$$

$$= k^{\frac{1}{q}} \left\{\left(k \int_0^\infty f^p(x) dx \int_0^\infty g^q(x) dx\right)^2 - \left(\int_0^\infty f^p(x) C(x) dx \int_0^\infty g^q(x) b(x) dx\right.\right.$$

$$\left.\left. - \int_0^\infty f^p(x) B(x)\, dx \int_0^\infty g^q(x) c(x) dx\right)^2\right\}^{\frac{1}{2p}}. \tag{4.4}$$

In virtue of (4.3) and (4.4), the proof is complete.

Refferences

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[2] Widder, D. V., An inequality related to one of Hilbert's, *Journ. London Math. Soc.*, 4(1924), 194—198.

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[4] Hu Ke, On an inequality and it's applications, *Scientia*, (1981), 141—148.