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THE NECESSARY AND SUFFICIENT CONDITIONS FOR LIPSCHITZ LOCAL HOMEOMORPHISM

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Abstract

For a locally Lipschitz map $f: R^n \to R^n$, the well known inverse function theorem gives a sufficient condition for f to be a Lipschitz local homeomorphims at a point x_0, that is, $\partial f(x_0)$ is invertible. In this paper, it is showed that this condition is not necessary and some necessary and sufficient conditions are given.

Throughout this paper, R^n denotes the n-dimensional Euclidean space, C^1 and C^{1-} denote the class of continuously differentiable maps and of locally Lipschitz maps respectively.

Let U be an open subset of R^n and $f: U \to R^n$ a map. If $f: U \to f(U) = V$ is a homeomorphism, $f \in C^1$ (C^{1-}) and its inverse $f^{-1} \in C^1$ (correspondently, C^{1-}), then we say that f is a C^1 (correspondently, C^{1-}) homeomorphism on U. Let $x_0 \in U$. If there exists a neighborhood $\Omega \subset U$ of x_0 such that f is a C^1 (C^{1-}) homeomorphism on Ω, then we say that f is a C^1 (correspondently, C^{-1}) local homeomorphism at x_0.

The following two inverse function theorems are well known.

Proposition 1.[1] *Let $f: U \to R^n \in C^1$, $x_0 \in U$. Then f is a C^1 local homeomorphism at x_0 if and only if $f'(x_0)$ is invertible.*

Proposition 2[2]. *Let $f: U \to R^n \in C^{1-}$, $x_0 \in U$. If the generalized Jacobian $\partial f(x_0)$ is invertible (i. e., each element of $\partial f(x_0)$ is invertible), then f is a C^{1-} local homeomorphism at x_0.*

The following example shows that in Proposition 2, the condition that $\partial f(x_0)$ is invertible is not necessary.

Example 1. We identify R^2 with complex plane and using complex representation: $z = re^{i\theta}$. Define $f: R^2 \to R^2$ by

$$f(re^{i\theta}) = \begin{cases} 0, & r = 0, \\ re^{i\varphi(\theta)}, & r > 0,\ \theta \in [0, 2\pi], \end{cases}$$

where

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$$\varphi(\theta)=\begin{cases}\theta, & \theta\in\left[0,\frac{\pi}{6}\right],\\ 7\theta-\pi, & \theta\in\left[\frac{\pi}{6},\frac{\pi}{3}\right],\\ \theta+\pi, & \theta\in\left[\frac{\pi}{3},\frac{2\pi}{3}\right],\\ \frac{1}{4}\theta+\frac{3}{2}\pi, & \theta\in\left[\frac{2\pi}{3},2\pi\right].\end{cases}$$

Then $f:R^2\to R^2$ is a C^{1-} homeomorphism, but $0\in\partial f(0)$, since $I\in\partial f(0)$, $-I\in\partial f(0)$, where I is the identify operator.

In this paper, we give a necessary and sufficient condition for f to be a C^{1-} local homeomorphism. We always suppose that $f:U\to R^n\in C^{1-}$. Denote

$$U_f=\{x\in U\,|\,f \text{ is not differentiable at } x\}.$$

Then $\mu(U_f)$, the Lebesgue measure of U_f, is zero. Let S be the unit sphere in R^n and $B(x, s)$ the s-neighborhood of x.

Definition 1. i) *Let $x\in U$. x is said to be a regular point of f if f is differentiable at x and $f'(x)$ is invertible. Otherwise we say that x is a singular point of f.*

ii) *Let $y\in R^n$. y is said to be a singular value of f if there is a singular point x of f such that $f(x)=y$. Otherwise y is said to be a regular value of f.*

For C^{1-} map, as in the C^1 case, we have

Proposition 3. *If $E\subset U$ and $\mu(E)=0$, then $\mu(f(E))=0$.*

Proposition 4. *The set of singular values of f has measure zero in R^n.*

The proofs of above two propositions are similar to that in the C^1 case. The only difference is that the set of singular points of C^{1-} map f contains the set U_f. Since $\mu(U_f)=0$, we have $\mu(f(U_f))=0$. So the Sard's theorem is also valid for C^{1-} case.

Let $x_0\in U$. If there exists a neighborhood Ω of x_0 such that $\bar{\Omega}\subset U$ and $f(x)\neq f(x_0)$ for all $x\in\bar{\Omega}\setminus\{x_0\}$, then we can define the index of f at x_0 by

$$\operatorname{index}[f, x_0]=\deg(f, \Omega, f(x_0)).$$

The following two results (see [3]) are useful in the sequel.

Proposition 5. *If f is differentiable at x_0 and $f'(x_0)$ is invertible, then* $\operatorname{index}[f, x_0]=\operatorname{sgn}\det f'(x_0)$.

Proposition 6. *Suppose that Ω is a bounded open subset of U and $\bar{\Omega}\subset U$. If $y\in R^n\setminus f(\partial\Omega)$ is a regular value of f, then*

$$\deg(f, \Omega, y)=\sum_{x_i\in f^{-1}(y)}\operatorname{index}[f, x_i].$$

Definition 2. *Let $x_0\in U$ and $h\in R^n$. We define the generalized right directional derivative of f at x_0 in the direction h by*

$$\delta f(x_0, h)=\Big\{y\in R^n \mid \text{There is a sequence } t_n\to 0^+ \text{ such that } \lim_{n\to\infty}\frac{1}{t_n}\Big(f(x_0+t_nh)-f(x_0)\Big)=y\Big\}.$$

It is clear that, for any $x\in U$ and $h\in R^n$, $\delta f(x, h)$ is a nonempty compact subset of R^n, $\delta f(x, \lambda h)=\lambda\delta f(x, h)$ when $\lambda>0$. If L is the Lipschitz constant of f near x_0, then $\|\delta f(x_0, h)\|\leqslant L\|h\|$ (i.e., $\|y\|\leqslant L\|h\|$ for all $y\in\delta f(x_0, h)$). If f has the usual right directional derivative at x_0 in the direction h,

$$f'(x_0, h)=\lim_{t\to 0^+}\frac{1}{t}(f(x_0+th)-f(x_0)),$$

then $\delta f(x_0, h)=\{f'(x_0, h)\}$ is a singleton. If f is differentiable at x_0, then $\delta f(x_0, h)=\{f'(x_0)h\}$ for erery $h\in R^n$.

Our main result is the following

Theorem 1. *Suppose that $f:U\to R^n\in C^{1-}$ and $x_0\in U$. Then, f is a C^{1-} local homeomorphism at x_0 if and only if the following three conditions are satisfied:*

i) *There exist a neighborhood Ω of x_0 and a positive constant L_1 subh that*

$$\|\delta f(x, h)\|\geqslant L_1,\ \forall x\in\Omega,\ \forall h\in S.$$

ii) *There exists a neighborhood Ω of x_0 such that*

$$\det f'(x)>0 \text{ (or } <0),\ \forall x\in\Omega\backslash\Omega_f.$$

iii) index$[f, x_0]=1$, (*correspondently*, -1).

Proof At first we prove the necessity.

Suppose that V is a neighborhood of x_0, W is a neighborhood of $f(x_0)$, $f:V\to W$ is a homeomorphism, both f and its inverse $f^{-1}:W\to V$ are Lipschitzian.

Since $f^{-1}f=I:V\to V$, by the product formula for the index, we have

$$\operatorname{index}[f, x_0]\cdot\operatorname{index}[f^{-1}, f(x_0)]=\operatorname{index}[I, x_0]=1.$$

It follows that index$[f, x_0]=1$ (or -1). Thus the condition iii) is satisfied.

Because of the Lipschitzness of f^{-1} there is a positive constant L_1 such that

$$\|f(x)-f(y)\|\geqslant L_1\|x-y\|,\ \forall x, y\in V.$$

By the definition of $\delta f(x, h)$ it follows that the condition i) with $\Omega=V$ is satisfied.

Let $x\in V\backslash V_f$. The condition i) implies that

$$f'(x)h\neq 0,\ \forall h\in S.$$

It follows that $f'(x)$ is invertible.

Take a ball neighborhood Ω of x_0 such that $\bar\Omega\subset V$ and $\deg(f, \Omega, f(x_0))=\operatorname{index}[f, x_0]$. By the connected domain property of degree, for each $x\in\Omega\backslash\Omega_f$, we have

$$\operatorname{sgn}\det f'(x)=\deg(f, \Omega, f(x))=\deg(f, \Omega, f(x_0))=\operatorname{index}[f, x_0].$$

This shows that the condition ii) is satisfied. So the proof of the necessity is completed.

Before proving the sufficiency we give the following

Lemma 1. *If the condition i) in Theorem 1 holds and L_2 is a positive constant* *with $L_2<L_1$, then for each $\bar{x}\in\Omega$ there is a $\rho>0$ such that $B(\bar{x},\rho)\subset\Omega$ and*

$$\|f(x)-f(\bar{x})\|\geqslant L_2\|x-\bar{x}\|,\ \forall x\in B(\bar{x},\rho).$$

Proof of Lemma 1 Suppose that the conclusion is not valid. Then there are a point $\bar{x}\in\Omega$ and a sequence $x_n\to\bar{x}$ such that $\|f(x_n)-f(\bar{x})\|<L_2\|x_n-\bar{x}\|$ for all x_n. By the compactness of S we may assume that

$$\frac{x_n-\bar{x}}{\|x_n-x\|}\to h,\ \text{i. e.}\ \frac{x_n-\bar{x}-\|x_n-x\|h}{\|x_n-\bar{x}\|}\to 0.$$

Let L be the Lipschitz constant of f near x. Then

$$\|f(\bar{x}+\|x_n-\bar{x}\|h)-f(\bar{x})\|\leqslant\|f(\bar{x}+\|x_n-\bar{x}\|h)-f(x_n)\|+\|f(x_n)-f(\bar{x})\|$$
$$\leqslant L\|x_n-\bar{x}-\|x_n-\bar{x}\|h\|+L_2\|x_n-\bar{x}\|.$$

It follows that there is a $y\in\delta f(x,h)$ such that $\|y\|\leqslant L_2<L_1$. This contradicts the condition i). Lemma 1 is proved.

We now give the proof of the sufficiency.

Suppose that the conditions i)–iii) are satisfied. Without loss of generality we may assume that $\bar{\Omega}$ is a bounded subset of U and $f(x)\neq f(x_0)$ for all $x\in\bar{\Omega}\setminus\{x_0\}$. Then

$$\deg(f,\Omega,f(x_0))=\text{index}[f,x_0]=1.$$

Let W be the connected component of $R^n\setminus f(\partial\Omega)$ containing $f(x_0)$. Then

$$\deg(f,\Omega,y)=1,\ \forall y\in W. \tag{1}$$

Let $V=f^{-1}(W)\cap\Omega$. Then V is an open subset of Ω and $f^{-1}(y)\cap\Omega$ is a nonempty compact subset of V for each $y\in W$. By Lemma 1, $f^{-1}(y)\cap\Omega$ is a finite set in V for each y in W. The condition ii) implies that

$$\text{index}[f,x]=1,\ \forall x\in V\setminus V_f. \tag{2}$$

By Proposition 6 it follows that $f^{-1}(y)\cap\Omega$ is a singleton in V for each regular value $y\in W$ of $f|_\Omega$.

By Lemma 1 index $[f,x]$ is well defined for each x in V.

We claim that

$$\text{index}[f,x]=1,\ \forall x\in V. \tag{3}$$

Indeed, let $\bar{x}$ in V be given arbitrarily and $f(\bar{x})=\bar{y}$. Choose a neighborhood G of $\bar{x}$ such that $\bar{G}\subset V$ and $f(x)\neq\bar{y}$ for all $x\in\bar{G}\setminus\{\bar{x}\}$. Thus index$[f,\bar{x}]=\deg(f,G,\bar{y})$. Choose $\varepsilon>0$ sufficiently small such that $B(\bar{y},\varepsilon)\subset W\setminus f(\partial G)$. Then

$$\deg(f,G,y)=\deg(f,G,\bar{y}),\ y\in B(\bar{y},\varepsilon). \tag{4}$$

By the continuity of f there is a $\rho>0$ such that $f(B(\bar{x},\rho))\subset B(\bar{y},\varepsilon)$. Since $\mu(V_f)=0$, there is an $x_1\in B(\bar{x},\rho)$ such that $x_1\notin V_f$. Take $\rho_1>0$ sufficiently small such that $B(x_1,\rho_1)\subset B(\bar{x},\rho)$ and $\deg(f,B(x_1,\rho_1),f(x_1))=\text{index}[f,x_1]=1$. By the property of degree $f(B(x_1,\rho_1))$ contains an open subset D of $B(\bar{y},\varepsilon)$. By Proposition 4 there is a y' in D which is a regular value of $f|_\Omega$. Noting that $f^{-1}(y')\cap\Omega$ is a singleton in $B(x_1,\rho_1)$ we have

$$\text{index}[f, \bar{x}] = \deg(f, G, \bar{y}) = \deg(f, G, y') = 1.$$

The formula (3) is proved.

From (1) and (3) it follows that $f^{-1}(y) \cap \Omega$ is a singleton in V for each $y \in W$. Thus $f: V \to W$ is a bijection. Consequently $f: V \to W$ is a homeomorphism since f is also an open map.

The following lemma will complete the proof of Theorem 1.

Lemma 2. *Suppose that V and W are two open subsets of R^n, $f: V \to W$ is a homeomorphism, $f \in C^{1-}$ and satisfies the condition i) on V. Then $f^{-1}: W \to V$, the inverse map of f, belongs to C^{1-} and has locally Lipschitz constant L_1^{-1}.*

Proof of Lemma 2 Without loss of generality we may assume that W is convex. Let L_2 be any positive constant with $L_2 < L_1$. Take y and y' in W arbitrarily. Let

$$\varphi(t) = f^{-1}(y + t(y' - y)), \ \forall t \in [0, 1].$$

Then φ is a path in V from $f^{-1}(y)$ to $f^{-1}(y')$. By Lemma 1, for each $t \in [0, 1]$, there is a ball $B(\varphi(t), \rho(t))$ having the property indicated in Lemma 1. By the compactness of $\varphi([0, 1])$ we may take $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ such that

$$\bigcup_{i=0}^{m} B_i \supset \varphi([0, 1]); \ B_i \cap B_{i+1} \neq \varnothing, \ (i = 0, 1, \cdots, m-1),$$

where $B_i = B(\varphi(t_i), \rho(t_i))$. Take again $s_i \in [0, 1]$ $(i = 1, 2, \cdots, m)$ such that

$$0 = t_0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_m < t_m = 1$$

and
$$\varphi(s_i) \in B_{i-1} \cap B_i, \ i = 1, 2, \cdots, m.$$

For shortness we denote $y(t) = y + t(y' - y)$. Then

$$\|y(s_i) - y(t_{i-1})\| \geqslant L_2 \|\varphi(s_i) - \varphi(t_{i-1})\|, \ i = 1, 2, \cdots, m,$$
$$\|y(t_i) - y(s_i)\| \geqslant L_2 \|\varphi(t_i) - \varphi(s_i)\|, \quad i = 1, 2, \cdots, m.$$

Consequently, we have

$$\begin{aligned} \|y' - y\| &= \sum_{i=1}^{m} (\|y(s_i) - y(t_{i-1})\| + \|y(t_i) - y(s_i)\|) \\ &\geqslant L_2 \sum_{i=1}^{m} (\|\varphi(s_i) - \varphi(t_{i-1})\| + \|\varphi(t_i) - \varphi(s_i)\|) \\ &\geqslant L_2 \|f^{-1}(y') - f^{-1}(y)\|, \end{aligned}$$

i. e.,
$$\|f^{-1}(y') - f^{-1}(y)\| \leqslant \frac{1}{L_2} \|y' - y\|.$$

By the arbitrarity of positive constant $L_2 < L_1$ we obtain

$$\|f^{-1}(y') - f^{-1}(y)\| \leqslant \frac{1}{L_1} \|y' - y\|.$$

This completes the proof of Lemma 2 and then Theorem 1 is proved.

Remark 1. Proposition 2 is a corollary to our Theorem 1. Indeed, when $\partial f(x_0)$ is invertible, by the upper semicontinuity of ∂f it is not difficult to see that the conditions i) and ii) are satisfied. From Pourciau [4] the condition iii) is also satisfied.

Finally we give some examples which show that any one of the conditions i) —iii) in Theorem 1 can not be dropped.

Example 2. The function $f: R^1 \to R^1$ defined by

$$f(x)=\begin{cases} x^3, & x<0, \\ x, & x \geqslant 0 \end{cases}$$

satisfies the conditions ii) and iii) but does not satisfy the condition i) at 0. $f: R^1 \to R^1$ is a homeomorphism and $f \in C^{1-}$ but $f^{-1} \notin C^{1-}$.

Example 3. The function $f: R^1 \to R^1$ defined by

$$f(0)=0,\ f(x)=x+2x^2 \sin \frac{1}{x} \quad (x \neq 0)$$

belongs to C^{1-}, $f'(0)=1$, index$[f, 0]=1$, but f is not a local homeomorphism at 0. f does not satisfy the conditions i) and ii).

Example 4. The map $f: R^2 \to R^2$ defined by

$$f(0)=0,\ f(re^{i\theta})=re^{i2\theta} \quad (r>0,\ \theta \in [0, 2\pi])$$

satisfies the conditions i) and ii) at 0, but f does not satisfy the condition iii) since index $[f, 0]=2$. f is not a local homeomorphism at 0.

Example 5. Define $f: R^2 \to R^2$ by $f(0)=0$, $f(re^{i\theta})=re^{i\varphi(\theta)}$, where

$$\varphi(\theta)=\begin{cases} \theta, & \theta \in \left[0, \frac{\pi}{6}\right], \\ 9\theta-\frac{4\pi}{3}, & \theta \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right], \\ -\theta+2\pi, & \theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right], \\ \frac{1}{2}\theta+\pi, & \theta \in \left[\frac{2\pi}{3}, 2\pi\right]. \end{cases}$$

Then, $f \in C^{1-}$, f satisfies the conditions i) and iii) at 0, index$[f, 0]=1$, but f does not satisfy the condition ii). Although $f'(x)$ is invertible for all $x \in R^2 \backslash R_f^2$, the $\det f'(x)$ have different signs. f is not a local homeomorphism at 0.

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