Chin. Ann. of Math.
13B:4(1991), 46—53.

HECKE OPERATOR AND PELLIAN EQUATION CONJECTURE (II) **

LU HONGWEN (陆洪文)*

Abstract

It is proved that the famous Ankeny–Artin–Chowla conjecture on Pellian equation is equivalent to a conjecture on Hecke operator.

§ 1. Introduction and Results

For a prime $p\equiv 1 (\mathrm{mod}\ 4)$, there exists a famous conjecture due to N. C. Ankeny, E. Artin and S. Chowla [1] as follows.

Conjecture 1.1. *Let $p\equiv 1(\mathrm{mod}\ 4)$ be a prime. Then for the least solution*

$$\varepsilon=\frac{t+u\sqrt{p}}{2}$$

of the Pellian equation

$$x^2-py^2=-4,$$

we have that $p\nmid u$.

Let T_N be the Hecke operator, i. e.

$$(T_N f)(\alpha)=\sum_{\substack{mn=N\\ m>0\\ w(\mathrm{mod}\, m)}} f\left(\frac{n\alpha+w}{m}\right),$$

for a positive integer N and a function f of α.

For a real quadratic irrational number β, let

$$\Psi(\beta)=\begin{cases}\sum_{j=1}^{k}(-1)^{j+s}a_j, & \text{if } k \text{ is even},\\ 0, & \text{if } k \text{ is odd},\end{cases}$$

where β has a development of the simple continued fractions

$$\beta=[\hat{a}_0, \cdots, \hat{a}_0, \overline{a_1, \cdots, a_k}],$$

with the basic period $\overline{a_1, \cdots, a_k}$.

We call $\Psi(\beta)$ the Hirzebruch sum of β.

We have proved the following

Manuscript received November 12, 1989.

* Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, China

** Projects supported by the National Natural Science Foundation of China and SF of Chinese Educational Ministry.

Theorem 1.2. *let $p\equiv 1 \pmod 4$ be a prime. For the least solution $\epsilon=\frac{t+u\sqrt{p}}{2}$ of the Pellian equation $x^2-py^2=-4$, the nonnegative integer ω is defined by*

$$p^{\omega}\|u.$$

Then we have

(1) *For any nonnegative integer k, we have*

$$(T_{p^k}\Psi)(2\sqrt{p})=\lambda_k\Psi(2\sqrt{p}),$$

where $\lambda_0=1$, and for $k\geqslant 1$

$$\lambda_k=\sum_{n=0}^{k}p^n g_p^{(n)},\ k\geqslant 1,$$

with

$$g_p^{(n)}=\begin{cases}1, & if\ 0\leqslant n\leqslant\omega,\\ p^{n-\omega}, & if\ n\geqslant\omega,\end{cases}$$

(2) *For the Dirichlet series*

$$J_p(s)=\sum_{k=0}^{\infty}\frac{\lambda_k}{p^{ks}},\ \mathrm{Re}(s)>3,$$

we have

$$J_p(s)=(1-\lambda)^{-1}(1-p^2\lambda)^{-1}\begin{cases}1, & if\ \omega=0,\\ 1-(p-1)\sum_{n=1}^{\omega}(p\lambda)^n, & if\ \omega\geqslant 1,\end{cases}$$

with $\lambda=p^{-s}$.

Conjecture 1.3. *$J_p(s)$ is the Zeta function of $P^1(F_{p^s})$.*

Proposition 1.4. *Conjecture 1.1$\Leftrightarrow$Conjecture 1.3.*

§ 2. The Action of Hecke Operator on Hirzebruch Sum

In our paper [4], the following results have been proved.

Theorem 2.1. *Let the positive integer $\Delta=B^2+4AC$ be a non-square with three integers A, B and C satisfying the following conditions:*

$$A\geqslant 1 \text{ and } g.c.d.(A, B, C)=1.$$

Then, for a positive integer N and the real quadratic irrational number $\alpha=\frac{B+\sqrt{\Delta}}{2A}$, we have

$$\sum_{\substack{mn=N\\ m>0\\ w(\mathrm{mod}\, m)}}J(n,m,w,\alpha)\Psi\left(\frac{n\alpha+w}{m}\right)=\sigma(N)J(N,\Delta)\Psi(\alpha),$$

where $\sigma(N)$ is the sum of the positive divisors of N, and the positive integers $J(N,\Delta)$ and $J(n, m, w, \alpha)$ are defined as follows.

Let $\frac{t_0+u_0\sqrt{\Delta}}{2}$ and $\frac{t+u\sqrt{\Delta}}{2}$ be the least solutions of the Pell's equations $x^2-\Delta y^2=4$ and $x^2-\Delta N^2y^2=4$, respectively. Then $J(N,\Delta)$ is defined by the following equation

$$\frac{t+uN\sqrt{\Delta}}{2}=\left(\frac{t_0+u_0\sqrt{\Delta}}{2}\right)^{J(N,\Delta)}$$

Let

$$V=V(n,\ m,\ w,\ \alpha)=u.\ g.\ c.\ d.\ (Am^2,\ BN+2Amw,\ Aw^2+Bnw-Cn^2).$$

Then V is a positive integer and it is easily seen that $V^2\mid \Delta N^2u^2$. Let

$$D=D(n,\ m,\ w,\ \alpha)=(t^2-4)V^{-2}.$$

Then D is a non-square positive integer, and we define the positive integers $J(n,\ m,\ w,\ \alpha)$ by the following equations

$$\frac{t+V\sqrt{D}}{2}=\left(\frac{\hat{t}+\hat{u}\sqrt{D}}{2}\right)^{J(n,m,w,\alpha)}$$

where $\frac{\hat{t}+\hat{u}\sqrt{D}}{2}$ is the least solution of the Pell's equation $x^2-Dy^2=4$.

Proposition 2.2. *Let $p\equiv 1 \pmod 4$ be a prime. Then we have*

$$(T_p\Psi)(2\sqrt{p})=(pg_p^{(1)}+1)\Psi(2\sqrt{p}),$$

where $g_p^{(1)}$ is the same as in Theorem 1.2.

Theorem 2.1 and Proposition 2.2 are the Theorem 7 and Theorem 1 in [4] respectively.

Definition 2.3. *For a non-square positive integer D and a positive integer N, we define a positive integer $J=J(DN^2;\ D)$ as follows. Let the least solutions of the Pellian equations $x^2-DN^2y^2=4$ and $x^2-Dy^2=4$ be ϵ_{DN^2} and ϵ_D, respectively, then $\epsilon_{DN^2}=\epsilon_D^J$, i. e. let $\epsilon_D=\frac{t+u\sqrt{D}}{2}$, then J is the least positive integer such that N divides u_J of $\epsilon_D^J=\frac{t_J+u_J\sqrt{D}}{2}$.*

Therefore, Theorem 2.1 can be rewritten as follows:

Theorem 2.4. *Let the positive integer $\Delta=B^2+4AC$ be a non-square with three integers A,B and C satisfying the following conditions*

$$A\geq 1 \text{ and } g.\ c.\ d.\ (A,\ B,\ C)=1.$$

Then, for a positive integer N and the real quadratic irrational number $\alpha=\frac{B+\sqrt{\Delta}}{2A}$, we have

$$\sum_{\substack{mn=N\\ m,n\geq 1\\ w(\mathrm{mod}\,m)}} J(\Delta N^2;\ \Delta N^2.\ g.\ c.\ d.\ (Am^2,\ BN+2Amw,\ Aw^2+Bnw-Cn^2)^{-2})$$

$$\times\Psi\left(\frac{n\alpha+w}{m}\right)=\sigma(N)J(\Delta N^2;\ \Delta)\Psi(\alpha).$$

§ 3. The Proof of Theorem 1.2

Late on, we always assume that the prime $p\equiv 1 \pmod 4$. For a nonnegative integer k, we define that

$$T\left(p^k, \frac{2\sqrt{p}}{p}\right) = \sum_{w(\mathrm{mod}\, p^k)} \Psi\left(\frac{w + \frac{2\sqrt{p}}{p}}{p^k}\right).$$

We are going to prove

Proposition 3.1. *For a nonnegative integer k, we have*

$$T\left(p^k, \frac{2\sqrt{p}}{p}\right) = \mu_k \Psi(2\sqrt{p}),$$

and

$$(T_{p^k}\Psi)(2\sqrt{p}) = \lambda_k \Psi(2\sqrt{p}),$$

where

$$\mu_k = p^k g_p^{(k)} \text{ and } \lambda_k = \sum_{n=0}^{k} \mu_n = \sum_{n=0}^{k} p^n g_p^{(n)},\ k \geq 0,$$

with $g_p^{(n)}$ *defined as in Theorem* 1.2.

Proof At first, it is easy to see that

$$\lambda_0 = 1 \text{ and } \lambda_1 = p g_p^{(1)} + 1,$$

by using Proposition 2.2.

Secondly, putting $A=1$, $B=0$, $C=p$ and $N=2$ in Theorem 2.4, we have $\Delta = 4p$, and $\alpha = \sqrt{p}$.

Since

$$g.c.d\,(Am^2, BN + 2Amw, Aw^2 + Bwn - Cn^2) = \begin{cases} 1, \text{ if } n=2, m=1, \\ 1, \text{ if } n=1, m=2, \text{ and } w=0, \\ 4, \text{ if } n=1, m=2, \text{ and } w=1, \end{cases}$$

we get

$$\Psi(2\sqrt{p}) + \Psi\left(\frac{\sqrt{p}}{2}\right) + J(16p;\, p)\Psi\left(\frac{1+\sqrt{p}}{2}\right) = 3 \cdot J(16p;\, 4p)\Psi(\sqrt{p}). \quad (1)$$

It is easy to see that the length of the basic period for the development of the simple continued fractions for $\sqrt{p}$ is an odd, so does for $\frac{1+\sqrt{p}}{2}$. So we have

$$\Psi(\sqrt{p}) = \Psi\left(\frac{1+\sqrt{p}}{2}\right) = 0. \quad (2)$$

From (1) and (2), we get

$$\Psi\left(\frac{\sqrt{p}}{2}\right) = -\Psi(2\sqrt{p}). \quad (3)$$

It is not difficult to see that

$$\Psi\left(\frac{2\sqrt{p}}{p}\right) = \Psi\left(\frac{2}{\sqrt{p}}\right) = -\Psi\left(\frac{\sqrt{p}}{2}\right). \quad (4)$$

Therefore, according to (3) and (4), we have

$$\Psi\left(\frac{2\sqrt{p}}{p}\right) = \Psi(2\sqrt{p}), \quad (5)$$

which shows that

$$\mu_0 = 1,$$

Let

$$g_p^{(k)}=J(16p^{2k+1};\ 16p),\ k\geqslant 0. \tag{6}$$

We need to prove that

$$g_p^{(k)}=\begin{cases}1, & \text{if } 0\leqslant k\leqslant\omega;\\ p^{k-\omega}, & \text{if } k\geqslant\omega.\end{cases} \tag{7}$$

If $p\equiv 1\ (mod\ 8)$, then the solution $\frac{x+\sqrt{p}\,y}{2}$ for the Pellian equation $x^2-py^2=-4$ satisfies $x\equiv y\equiv 0\ (mod\ 2)$. Hence, in this case, the least solution of the Pellian equation $x^2-16py^2=4$ is ϵ^2, where $\epsilon=\frac{t+u\sqrt{p}}{2}$ is the least solution of the Pellian equation $x^2-py^2=-4$. When $p\equiv 5(mod\ 8)$, it is easy to see that the least solution of the Pellian equation $x^2-16py^2=4$ is ϵ^3. Therefore we have proved (7).

There exists no difficulty in showing that

$$J(D(MN)^2;\ D)=J(D(MN)^2;\ DN^2)\cdot J(DN^2;\ D),$$

for two positive integers M and N, so we get

$$J(16p^{2k+1};\ 16p^{2l+1})=\frac{g_p^{(k)}}{g_p^{(l)}},\ \text{for } 0\leqslant l\leqslant k. \tag{8}$$

Next, putting $A=1$, $B=0$, $C=4p$ and $N=p^k$, $k\geqslant 2$, in Theorem 2.4. Then we have $\Delta=16p$ and $\alpha=2\sqrt{p}$.

For two integers m and n such that $m+n=k$ and $m,\ n\geqslant 0$, we have

$$g.c.d.\ (p^{2m},\ 2p^m w,\ w^2-4p^{2n+1})$$
$$=\begin{cases}1, & \text{if } m=0 \text{ or } p\nmid w;\\ p, & \text{if } p\mid w \text{ and } n=0,\ m=k;\\ p^2\cdot g.c.d.\ (p^{2(m-1)},\ 2p^{m-1}w_1,\ w_1^2-4p^{2(n-1)+1}), & \text{if } p\mid w,\ w=pw_1 \text{ and } n,\ m\geqslant 1.\end{cases}$$

Therefore we get

$$\sigma(p^k)g_p^{(k)}\Psi(2\sqrt{p})=(T_{p^k}\Psi)(2\sqrt{p})+(J(16p^{2k+1};\ 16p^{2k-1})-1)\sum_{w(\mathrm{mod}\,p^{k-1})}\Psi\left(\frac{\frac{2\sqrt{p}}{p}+w}{p^{k-1}}\right)$$
$$+\sum_{\substack{u+m=k\\ n,m\geqslant 1\\ w(\mathrm{mod}\,p^{m-1})}}(J(16p^{2k+1};\ 16p^{2k-3}\,g.c.d.(p^{2m-2},\ 2p^{m-1}w,\ w^2-4p^{2n-1})^{-2})-1)$$
$$\times\Psi\left(\frac{p^{n-1}2\sqrt{p}+w}{p^{m-1}}\right)=(T_{p^k}\Psi)(2\sqrt{p})+\left(\frac{g_p^{(k)}}{g_p^{k-1}}-1\right)T\left(p^{k-1},\ \frac{2\sqrt{p}}{p}\right)$$
$$-\sum_{\substack{n+m=k-2\\ n,m\geqslant 0\\ w(\mathrm{mod}\,p^m)}}\Psi\left(\frac{p^n2\sqrt{p}+w}{p^m}\right)+J(16^{2k+1};\ 16p^{2k-3})\sum_{\substack{n+m=k-2\\ n,m\geqslant 0\\ w(\mathrm{mod}\,p^m)}}\Psi\left(\frac{p^n2\sqrt{p}+w}{p^m}\right)$$
$$\times J(16p^{2(k-2)+1};\ 16p^{2(k-2)+1}g.c.d.\ (p^{2m},\ 2p^m w,\ w^2-4p^{2n+1})^{-2})$$
$$=(T_{p^k}\Psi)(2\sqrt{p})+\left(\frac{g_p^{(k)}}{g_p^{(k-1)}}-1\right)T\left(p^{k-1},\ \frac{2\sqrt{p}}{p}\right)$$
$$+J(16p^{2k+1};\ 16p^{2k-3})\sigma(p^{k-2})J(16p^{2k-3};\ 16p)\Psi(2\sqrt{p})-(T_{p^{k-2}}\Psi)(2\sqrt{p}),$$

by using (6), (8), etc., So we have

$$(T_{p^k}\Psi)(2\sqrt{p})=(T_{p^{k-1}}\Psi)(2\sqrt{p})+(p^k+p^{k-1})g_p^{(k)}\Psi(2\sqrt{p})$$
$$+\left(1-\frac{g_p^{(k)}}{g_p^{(k-1)}}\right)T\left(p^{k-1},\ \frac{2\sqrt{p}}{p}\right),\ k\geqslant 2. \tag{9}$$

Next, putting $A=p$, $B=0$, $C=4$ and $N=p$ in Theorem 2.4, we have $\Delta=16p$ and $\alpha=\frac{2\sqrt{p}}{p}$.

Since

$$g.\ c.\ d.\ (p,\ 0,\ 4p^2)=p \text{ and } g.\ c.\ d.\ (p^3,\ 2p^2w,\ pw^2-4)=1,$$

we get

$$J(16p^3;\ 16p)\Psi(2\sqrt{p})+\sum_{w(\bmod p)}\Psi\left(\frac{w+\frac{2\sqrt{p}}{p}}{p}\right)=(p+1)J(16p^3;\ 16p)\Psi\left(\frac{2\sqrt{p}}{p}\right),$$

which combinied with (5) implies

$$T\left(p,\frac{2\sqrt{p}}{p}\right)=pg_p^{(1)}\Psi(2\sqrt{p}). \tag{10}$$

Finally, putting $A=p$, $B=0$, $C=4$ and $N=p^k$, $k\geqslant 2$, in Theorem 2.4, we have $\Delta=16p$ and $\alpha=\frac{2\sqrt{p}}{p}$.

For two integers m and n such that $m+n=k$ and $m,\ n\geqslant 0$, we have

$$g.\ c.\ d.\ (p^{2m+1},\ 2p^{m+1}w,\ pw^2-4p^{2n})$$
$$=\begin{cases}1, & \text{if } n=0 \text{ and } m=k;\\ p, & \text{if } n=k \text{ and } m=0, \text{ or } n,\ m\geqslant 1 \text{ and } p\nmid w;\\ p^2 g.\ c.\ d.\ (p^{2(m-1)+1},\ 2p^{(m-1)+1}w_1,\ pw_1^2-4p^{2(n-1)}), & \text{if } m,\ n\geqslant 1,\ p\mid w \text{ and } w=pw_1.\end{cases}$$

Hence we get

$$\sigma(p^k)J(16p^{2k+1};\ 16p)\Psi\left(\frac{2\sqrt{p}}{p}\right)$$
$$=\sum_{w(\bmod p^k)}\Psi\left(\frac{w+\frac{2\sqrt{p}}{p}}{p^k}\right)+J(16p^{2k+1};\ 16p^{2k-1})\Psi(p^{k-1}2\sqrt{p})$$
$$+J(16p^{2k+1};\ 16p^{2k-1})\sum_{\substack{n+m=k\\ n,m\geqslant 1\\ w(\bmod p^m)}}\Psi\left(\frac{p^{n-1}2\sqrt{p}+w}{p^m}\right)$$
$$+\sum_{\substack{n+m=k\\ n,m\geqslant 1\\ w(\bmod p^{m-1})}}(J(16p^{2k+1};\ 16p^{2k-3}\ g.\ c.\ d.\ (p^{2(m-1)+1},\ 2p^{(m-1)+1}\ w,\ pw^2-4p^{2(n-1)})^{-2})$$
$$-J(16p^{2k+1};\ 16p^{2k-1}))\Psi\left(\frac{p^{n-1}\frac{2\sqrt{p}}{p}+w}{p^{m-1}}\right)$$
$$=T\left(p^k,\ \frac{2\sqrt{p}}{p}\right)+J(16p^{2k+1};\ 16p^{2k-1})\sum_{\substack{n+m=k-1\\ n,m\geqslant 0\\ (\bmod p^m)}}\Psi\left(\frac{w+p^n2\sqrt{p}}{p^m}\right)$$

$$+J(16p^{2k+1};16p^{2k-3}),\sum_{\substack{n+m=k-2\\ n,m\geq 0\\ w(\mathrm{mod}\,p^m)}} J(16p^{2k-3};16p^{2k-3}\, g.c.d.(p^{2m+1},2p^{m+1}w,pw^2-4p^{2n})^{-2})$$

$$\times\Psi\left(\frac{w+p^n\dfrac{2\sqrt{p}}{p}}{p^m}\right)$$

$$-J(16p^{2k+1};16p^{2k-1})\sum_{\substack{n+m=k-3\\ n,m\geq 0\\ w(\mathrm{mod}\,p^m)}}\Psi\left(\frac{w+p^n2\sqrt{p}}{p^m}\right)\quad (\text{if } k=2, \text{ this term}=0)$$

$$-J(16p^{2k+1};16p^{2k-1})\sum_{w(\mathrm{mod}\,p^{k-2})}\Psi\left(\frac{w+\dfrac{2\sqrt{p}}{p}}{p^{k-2}}\right)$$

$$=T\left(p^k,\frac{2\sqrt{p}}{p}\right)+J(16p^{2k+1};16p^{2k-1})(T_{p^{k-1}}\Psi)(2\sqrt{p})$$

$$+J(16p^{2k+1};16p^{2k-3})\sigma(p^{k-2})J(16p^{2k-3};16p)\Psi\left(\frac{2\sqrt{p}}{p}\right)$$

$$-J(16p^{2k+1};16^{2k-1})(T_{p^{k-3}}\Psi)(2\sqrt{p})\quad(\text{if } k=2, \text{ this term}=0)$$

$$-J(16p^{2k+1};16p^{2k-1})T\left(p^{k-2},\frac{2\sqrt{p}}{p}\right),$$

which combining with (5), (6) and (8) implies

$$T\left(p^k,\frac{2\sqrt{p}}{p}\right)=\frac{g_p^{(k)}}{g_p^{(k-1)}}T\left(p^{k-2},\frac{2\sqrt{p}}{p}\right)+(p^k+p^{k-1})g_p^{(k)}\Psi(2\sqrt{p})$$

$$-\frac{g_p^{(k)}}{g_p^{(k-1)}}(T_{p^{k-1}}\Psi)(2\sqrt{p})+\frac{g_p^{(k)}}{g_p^{(k-1)}}(T_{p^{k-3}}\Psi)(2\sqrt{p})$$

$$(\text{if } k=2, \text{ this term}=0),\ k\geq 2. \tag{11}$$

We have already seen that Proposition 3.1 holds if $k=0$, 1 (i. e. $\lambda_0=1$, $\lambda_1=pg_p^{(1)}+1$, $\mu_0=1$ and $\mu_1=pg_p^{(1)}$ by (10)). For $k\geq 2$, we will prove it using the induction. According to (9) and (11), and the assumptions of the induction, for $k\geq 2$, we have

$$\begin{aligned}\mu_k&=\frac{g_p^{(k)}}{g_p^{(k-1)}}(\mu_{k-2}-\lambda_{k-1}+\lambda_{k-3})+(p^k+p^{k-1})g_p^{(k)}\\&=\frac{g_{(p)}^{(k)}}{g_p^{(k-1)}}(p^{k-2}g_p^{(k-2)}-p^{k-1}g_p^{(k-1)}-p^{k-2}g_p^{(k-2)})+(p^k+p^{k-1})g_p^{(k)}\\&=p^kg_p^{(k)},\end{aligned}$$

and

$$\begin{aligned}\lambda_k&=\lambda_{k-2}+\left(1-\frac{g_p^{(k)}}{g_p^{(k-1)}}\right)\mu_{k-1}+(p^k+p^{k-1})g_p^{(k)}\\&=\sum_{n=0}^{k-2}p^ng_p^{(n)}+\left(1-\frac{g_p^{(k)}}{g_p^{(k-1)}}\right)p^{k-1}g_p^{(k-1)}+(p^k+p^{k-1})g_p^{(k)}\\&=\sum_{n=0}^{k}p^ng_p^{(n)},\end{aligned}$$

which finishes the proof of Proposition 3.1.

Proof of Theorem 1.2. From Proposition 3.1, we get the claim (1) of Theorem

1.2. According to this, we have

$$
\begin{aligned}
J_g(s) &= \sum_{k=0}^{\infty} \frac{\lambda_k}{p^{ks}} = \sum_{k=0}^{\infty} p^{-ks} \sum_{n=0}^{k} p^n g_p^{(n)} \\
&= \sum_{n=0}^{\infty} p^n g_p^{(n)} \sum_{k=n}^{\infty} p^{-ks} = (1-p^{-s})^{-1} \sum_{n=0}^{\infty} g_p^{(n)} p^{-n(s-1)} \\
&= (1-p^{-s})^{-1} \Big(\sum_{0\leq n<\omega} p^{-n(s-1)} + \sum_{n\geq\omega}^{\infty} p^{n-\omega} p^{-n(s-1)} \Big) \\
&= (1-p^{-s})^{-1} ((1-p^{-\omega(s-1)})(1-p^{-(s-1)})^{-1} + p^{-\omega(s-1)}(1-p^{-(s-2)})^{-1}) \\
&= (1-p^{-s})^{-1}(1-p^{-(s-1)})^{-1}(1-p^{-(s-2)})^{-1}(1-p^2p^{-s}+p^{\omega}(p^2-p)p^{-(\omega+1)s}) \\
&= (1-\lambda)^{-1}(1-p^2\lambda)^{-1} \cdot \begin{cases} 1, & \text{if } \omega=0, \\ 1-(p-1)\sum_{n=1}^{\omega}(p\lambda)^n, & \text{if } \omega\geq 1, \end{cases}
\end{aligned}
$$

with $\lambda=p^{-s}$. So we have finished the proof of Theorem 1.2.

§ 4. The Proof of Proposition 1.4 and The Concluding Remark

It is clear that Proposition 1.4 is true.

As a concluding remark, we point out that, similar to the proof of the Ramanujan–Petersson conjecture given by P. Delinge (see [3]), if we can prove that Dirichlet series $J_g(s)$ is a zeta function for an absolutely nonsingnlar projective variety over finite field F_p. then according to A. Weil–P. Delinge theorem [2], wewill get a proof of the Ankeny Artin–Chowla–Conjecture.

Referlnces

[1] Ankeny, N. C., Artin, E. & Chowla. S., The class number of real quadratic number fields, *Ann. of Math.*, **56**:2(1952), 479—493.

[2] Delinge, P., La Conjecture de Weil, *I. Publ. Math. IHES*, **43**(1974), 273—307.

[3] Katz, N., An overview of Delinge's proof of the Riemann hypothesis for varieties over finite fields, *Amer. Math. Soc. Proc. Symp. Pure Math.*, **28**(1976), 275—305.

[4] Lu, H. Hirzebruch sums and Hecke operators, 1989 *Journal of Number Theory*, **38**: 2(1991), 185—195.

[5] Lu, H., Hecke operator and Pellian equation conjecture (I), 1988 (unpuplished).