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$\varDelta$-CONTEXT FOR NONSINGULARLY RELATIVE PRIMITIVE RINGS

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Abstract

In this paper the concept $\varDelta$-context is introduced, as a generalization of Morita Context, to give a characterization of nonsingularly relative primitive rings.

§ 0. Introduction

Jacobson[3] gave a characterization for primitive rings with a nonzero socle by proving the existence and uniqueness of the corresponding dual space. Among many generalizations of this thought the most famous one might be Morita Context. Armitsur[2] used MC to study the structure of a class of rings. Zelmanowitz[8] introduced the concept Partial Context generalizing MC. In order to give a stricter characterization for nonsingularly relative primitive rings, we introduce here a special kind of Partial Context, called $\varDelta$-context, and then give the existence and uniqueness of $\varDelta$-context for nonsingularly relative primitive rings.

§ 1. Preliminaries and Definitions

Through out this paper, $\varDelta$ will denote a division ring, R will denote a ring with identity, F will denote a right Gabriel topology over R. A ring R is called a nonsingularly relative primitive ring if it is nonsingularly F-primitive for some Gabriel topology F, i. e., R admits a faithful nonsingular F-cocritical module[7]. For localization concepts and their properties, the reader is referred to [1] and [5]. Now we recall some results given in [7].

Proposition 1.1.[[7], 1.5] *Suppose M is a faithful F-cocritical R-module and $\varDelta = End_R\ \overline{M}$ is the corresponding division ring, where $\overline{M}$ is the quasi-injective hull of M. Then*

$$M \setminus Z_R(M) = \{x \in M : \exists r \in R,\ r \neq 0,\ \varDelta M r = \varDelta x r\}.$$

Proposition1.2.[[7], 2.11, 2.12] *Suppose R is a nonsingularly F-primitive ring, and*

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M_1 and M_2 are two faithful F-cocritical R-modules. Then M_1 is subisomorphic to M_2 (i. e., they have isomorphic nonzero submodules), and the corresponding division rings $\Delta_1 = End_R\, \overline{M}_1$ and $\Delta_2 = End_R \overline{M}_2$ are isomorphic to each other.

Now we give the basic definition. For a right R-module V_R and a left R-module ${}_RW$, $V \otimes_R W$ will denote the tensor product of V_R and ${}_RW$.

Definition 1.3. *Let Δ be a division ring. A Δ-context is the sextuple $(R, V, W, X, (\cdot,\cdot), [\cdot,\cdot])$, where R is a ring, ${}_\Delta V_R$ and ${}_RW_\Delta$ are two-sided modules which are vector spaces over Δ, X is a nonzero left-R-right-R-submodule of $W \otimes_\Delta V$, $(\cdot, \cdot)$ and $[\cdot,\cdot]$ are two two-sided module homomorphisms, $(\cdot,\cdot)$: ${}_RX_R \to {}_RR_R$, $[\cdot, \cdot]$: ${}_\Delta(V \otimes_R W)_\Delta \to {}_\Delta\Delta_\Delta$, satisfying the following conditions:*

(i) *If (w, v) is defined (i. e., $w \otimes v \in X$), then for every w' in W, $(w, v)w' = w[v, w]$.*

(ii) *If (w, v) is defined, then for every v' in V, $v'(w, v) = [v', w]v$.*

(iii) *For every finite subset $\{w_1, \cdots, w_n\}$ in W, there is a nonzero element v in V such that all (w_i, v) are defined, $i = 1, \cdots, n$.*

Such a Δ-context is called faithful, if it satisfies two more conditions:

(iv) *V is a faithful right R-module.*

(v) *For every nonzero element v in V, there is an element w in W such that $[v, w] = 0$.*

A Δ-context $(R, V, W, X, (\cdot, \cdot), [\cdot, \cdot])$ is called full if $X = W \otimes_\Delta V$. Obviously a full Δ-context is a Morita Context.

For $w \in W$, D_w will often denote $\{v \in V: (w, v) \text{ is defined}\}$, which is a right R-submodule of V.

Lemma 1.4. *The following is true for a faithful Δ-context $(R, {}_\Delta V_R, {}_RW_\Delta, X, (\cdot, \cdot), [\cdot,\cdot])$.*

(i) *If $(w, v) = 0$ then either $w = 0$ or $v = 0$.*

(ii) *If for all v in V, $[v,w] = 0$, then $w = 0$.*

(iii) *Let $v \in V$. If $v(X) = 0$, then $v = 0$, where (X) denotes the homomorphic image of X in R under $(\cdot,\cdot)$.*

(iv) *W is a faithful left R-module.*

Proof (i) Suppose $(w, v) = 0$ and $w = 0$. For every w' in W, $w[v, w'] = (w,v)w' = 0$, thus $[v,w'] = 0$ since $[v,w']$ is an element in the division ring Δ. Now we know $v = 0$ by 1.3.(v).

(ii) Let v' be a nonzero element in D chosen by 1.3. (iii). For every v in V, $v(w,v') = [v,w]v' = 0$, hence $(w,v') = 0$ by 1.3.(iv). Then $w = 0$ by (i).

(iii) Suppose $v(X) = 0$. For every $w \in W$, choose a nonzero element v' in D_w. Then $[v,w]v' = v(w,v') = 0$. Hence $[v,w] = 0$ since Δ is a division ring. Now $v = 0$ by (iii).

(iv) Let $r\in R$. If $rW=0$, then for every v in V, $[vr, W]=[v, rW]=0$. Hence $vr=0$ by 1.3.(v), and $r=0$ by 1.3.(iv).

§ 2. Existence and Uniqueness Theorems

Theorem 2.1. *R is a nonsingularly relative primitive ring if and only if R admits a faithful Δ-context for some division ring Δ.*

Proof Necessity. Suppose R is a nonsingularly F-primitive ring, where F is a Gabriel topology, and V_R is a faithful nonsingular F-cocritical module. We may assume that V_R is quasi-injective by [7], 1.2. Let $\Delta=\mathrm{End}_R V$, which is a division ring. Let $W=P\mathrm{Hom}_R(V,R)$, the set of all partial linear functionals of V_R. Then $W\neq 0$ by [9], Theorem 3.2. For $w\in W$, $w\neq 0$, let D_w be the domain of w, which is a nonzero submodule of V_R. Let X be the two-sided submodule of ${}_R(W\otimes_\Delta V)_R$ generated by all of the elements $w\otimes_\Delta v$, $w\in W$, $v\in D_w$. For $v\in D_w$, define $(w, v)=w(v)$. Obviously this definition can be extended to a two-sided module homomorphism from X to ${}_RR_R$. For $v\in V$, $w\in W$, let $[v, w]$ be the homomorphism from D_w to V, $[v, w](v')=vw(v')=v(w, v')$, $v'\in D_w$. Since V_R is quasiinjective, this homomorphism can be extended to an endomorphism of V_R, i.e., we may consider $[v, w]\in\Delta$. It is not difficult to verify that $[\cdot, \cdot]$ defines a two-sided module homomorphism from ${}_\Delta(V\otimes_R W)_\Delta$ to ${}_\Delta\Delta_\Delta$. Now we claim that we have constructed a faithful Δ-context. 1.3. (i) follows by definition. Let $w\in W$, $w'\in W$, $v\in D_w$, $r=(w, v)\in R$, $d=[v, w']\in\Delta$. Then for every $z\in d^{-1}D_w\cap D_w\neq 0$ (noting that V is uniform), $wd(z)=w(dz)=w([v, w']z)=w(v(w', z))=w(v)(w', z)=rw'(z)$. Therefore $wd=rw'$, and this is 1.3.(ii). Since V_R is uniform, $\bigcap_{i=1}^{n} D_{w_i}\neq 0$ for $\{w_i\}_{i=1}^n\in W$, i. e., 1.3. (iii) holds. V_R is faithful by definition. Now suppose v is a nonzero element in V. By 1.1, there exists an element r in R, such that $Vr=\Delta vr\neq 0$. Let f be the partial linear functional defined on vrR, $f(vrs)=rs$, $s\in R$. Then $f\in W$, and $(f, vr)=r\in(X)$. Hence $v(X)\neq 0$ since $0\neq vr\in v(X)$. It is easy to see that this implies that $[v, w]=0$ is impossible, i.e., 1.3. (v) follows.

Sufficiency. Suppose R admits a faithful Δ-context $(R, {}_\Delta V_R, {}_RW_\Delta, X, (\cdot, \cdot), [\cdot, \cdot])$ for some division ring Δ. We will prove that V_R is nonsingular and uniform. Then V_R is a faithful nonsingular G-cocritical module, where G denotes the Goldie topology, and hence R is a nonsigularly G-primitive ring. For two nonzero elements v_1, v_2 in V, choose an element w_1 in W such that $[v_1, w_1]\neq 0$. We may assume that $[v_1, w_1]=1$, for we may substitute w_1 by $w_1[v_1, w_1]^{-1}$ if necessary. Similarly choose w_2 in W such that $[v_2, w_2]=1$. By 1.3. (iii) there is a nonzero element v in V such that both (w_1, v) and (w_2, v) are defined. Then

$v_1(w_1, v)=v=v_2(w_2, v)$. Therefore $v_1R\cap v_2R\neq 0$, and V_R is uniform. Finally we prove that V_R is nonsingular. Let v be a nonzero element in V. Choose a nonzero element w in W such that $[v, w]\neq 0$, and a nonzero element v' in D_w. Then $I=(w, v')R$ is a nonzero right ideal of R. If $va=0$ for $a=(w, v')a'=(w, v'a')\in I$, then $[v, w]v'a'=v(w, v'a')=0$. Hence $v'a'=0$, and $a=0$. Thus $\mathrm{ann}_R(v)\cap I=0$ and $\mathrm{ann}_R(v)$ is not essential in R, i. e., $Z_R(V)=0$.

Corollary 2.2. *R is a primitive ring with a nonzero socle if and only if R admits a faithful full Δ-context for some division ring Δ.*

Proof If R is a primitive ring with a nonzero socle, then in the necessity part of the proof of Theorem 2.1, V_R is simple, $D_w=V$ for every w in W, and hence $X=W\otimes_\Delta V$. Conversely, assume the faithful Δ-context is also full. For two arbitrary nonzero elements v_1, v in V, choose w_1 in W such that $[v_1, w_1]=1$. Then $v=v_1(w_1, v)\in v_1R$, i. e., $v_1R=V$. Hence V_R is simple. Since V_R is also nonsingular, we know R is primitive with a nonzero socle.

Noting that a faithful full Δ-context is two-sided faithful Morita Context, we see that the sufficiency of 2.2 is a direct conclusion from the fact that the class of all primitive rings with nonzero socles is a normal class[4]. The necessity of 2.2 could also be constructed directly as in the form (R, eR, Re) over the division ring eRe. Now we give two lemmas before we discuss the uniqueness of the Δ-context.

Lemma 2.3. *Suppose $(R, {}_\Delta V_R, {}_RW_\Delta, X, (\cdot,\cdot), [\cdot,\cdot])$ is a faithful Δ-context. Then V_R is quasi-injective and $\Delta=\mathrm{End}_RV$.*

Proof We know V_R is nonsingular uniform by the proof of the sufficiency of Theorem 2.1. Hence V_R is monoform. Assume f is a partial endomorphism of V_R defined on a nonzero submodule U of V_R. Choose $u\in U$, $w\in W$, such that $[u, w]=1$. Since V_R is uniform, we may choose a nonzero element v in $U\cap D_w$. Let $d=[f(u), w]$. Noting $v=[u, w]v=u(w, v)$, we have $f(v)=f(u(w, v))=f(u)(w, v)=[f(u), w]v=dv$. Since V_R is monoform, f and d are identical on U. Thus f could be extended to V, i. e., V_R is quasi-injective and $\mathrm{End}_R V=\Delta$.

Lemma 2.4. *Suppose $(R, V, W, X, (\cdot,\cdot), [\cdot,\cdot])$ is a faithful Δ-context and I is a nonzero right ideal of R. Then there exist a nonzero element w in W and a nonzero submodule U of V_R such that $(w, u)\in I$ for every element u in U.*

Proof Since V is faithful, $VI\neq 0$. Thus $VI(X)\neq 0$ by 1.4. (iii), and $I(X)\neq 0$, i.e., there are $w'\in W$, $v'\in V$, such that $I(w',v')=(Iw', v')=0$. Now let $w=aw'\in Iw'$, $a\in I$, $w\neq 0$, $U=D_w\neq 0$. Then for every $u\in U$, $(w, u)=a(w', u)\in I$.

Theorem 2.5. *Suppose R is a ring, Δ_1 and Δ_2 are two division rings, $(R, {}_{\Delta_1}V_{1R}, {}_RW_{1\Delta_1})$ and $(R, {}_{\Delta_2}V_{2R}, {}_RW_{2\Delta_2})$ are faithful Δ_1- and Δ_2-context respectively. Then $\Delta_1\cong\Delta_2$, $V_{1R}\cong V_{2R}$ (subisomorphism), ${}_RW_{1R}\cong W_2$.*

Proof We know that V_{1R} and V_{2R} are nonsingular uniform quasi-injective modules, and $\Delta_1=\mathrm{End}_R V_1, \Delta_2=\mathrm{End}_R V_2$ by 2.3 and the sufficiency proof of Theorem 2.1. Then by Theorem 1.2 Δ_1 and Δ_2 are isomorphic and V_{1R} and V_{2R} are subisomorphic to each other. Let $V^*=P\mathrm{Hom}_R(V_1,R)\cong P\mathrm{Hom}_R(V_2, R)$ (noting that V_1 and V_2 are uniform modules and subisomorphic to each other). In order to prove ${}_RW_1\cong{}_RW_2$, we only need to prove ${}_RW_1\cong{}_RV^*\cong{}_RW_2$. For $w\in W_1$, $(w,\cdot)$ defines a homomorphism from D_w to R. By 1.4. (i), this gives a natural imbedding of ${}_RW_1$ in ${}_RV^*$. Assume f is a nonzero element of V^*, from a nonzero submodule U of V_R onto a nonzero right ideal I of R. Since V_R is monoform, f is an injective map and is an isomorphism from U to I. According to 2.4, choose a nonzero submodule U' of V_R, $w\in W$, such that $(w, u')\in I$ for every $u'\in U$. We may consider w as an isomorphism from U' onto some nonzero right ideal I' contained in I. Let d be the homomorphism: $U'\xrightarrow{w}I'\xrightarrow{f^{-1}}U$, $d=f^{-1}w$. Then $d\in\Delta$. Now it is easy to see under the embedding ${}_RW_1\to{}_RV^*$, $f=wd^{-1}\in W_1$, i. e., ${}_RW_1\cong{}_RV^*$. Similarly ${}_RW_2\cong{}_RV^*$. This ends the proof.

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