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THE FOURIER TRANSFORM AND THE PARSEVAL FORMULA IN HARMONIC ANALYSIS FOR OPERATORS

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Abstract

In this paper, integrable operators with respect to right tranlation are defined and their elementary properties are discussed. The main resuits of Fourier transform for opertors and the Parseval formula for operators are also proved.

§ 1. Introduction

U. B. Tewari and Shobha Madan studied a harmonic analysis for operators on the homogeneous Banach spaces defined over a compact Abelian group in [1]. We attempt to develop the discussion of a harmonic analysis for operators on the homogeneous Banach spaces defined over a locally compact Abelian group. We observe the following situation: when we replace the compact group by a locally compact noncompact group G, a change of case will happen. For example, at the moment it is impossible to define the Fourier transform for the invariant operators on homogeneous Banach space B on G. This is an analogue of the case: the Fourier transform for constant function on G does not exist when G is a locally compact noncompact.

When we focus our attention on "integrable operator with respect to right translation on B" (see Definition 1.1), however, it is possible to initiate the Fourier transform for operator. At the moment the Fourier transform for operator possesses almost all elementary properties which are similar to the Fourier transform of usual complex valued function on G.

Let B be a set of some complex valued functions on G and be a Banach space under the norm $\|\cdot\|_B$. The space B has following properties:

(1) If $f\in B$, then $R_tf\in B$ and $\|R_tf\|_B=\|f\|_B$, where R_t denotes the translation

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operator, $t\in G$.

(2) If $f\in B$, then $\lim_{t\to\theta}\|R_tf-f\|_B=0$, where θ is the identity in G. Above B is said to be a homogeneous Banach space on G. We assume in addition one more condition: Let Γ denote the dual group of G, $\gamma(t)\equiv(t,\gamma)$ denotes a character of G. If $\gamma\in\Gamma$, then $\gamma f\in B$ and $\|\gamma f\|_B=\|f\|_B$. B is said to be a characteristic homogeneous Banach space. We know easily that the mapping $f\mapsto\gamma f$ is continuous for $f\in B$, $\gamma\in\Gamma$. If the set of all p-th integrable functions on G is denoted by $L^p(G)(1\leqslant p<\infty)$, $C_0(G)$ is the set of all continuous functions vanishing at the infinite, then both $C_0(G)$ and one of the $L^p(G)(1\leqslant p<\infty)$ are the characteristic homogeneous Banach spaces.

Definition 1.1. *Let $L^1(G, B)$ be the Banach space of all B valued integrable functions on G, T be in $\mathscr{L}(B)$, the set of all bounded linear operators from B to B. If TR_tf is an element in $L^1(G, B)$ for each $f\in B$, then T is said to be an "integrable operator on B with respect to right translation". We denote these operators by the notation $\mathscr{L}_B^1$.*

Theorem 1.2. *The following are equivalent:*

(1) $T\in\mathscr{L}_B^1$.

(2) $\|TR_tf\|_B\in L^1(G)$, *for each* $f\in B$.

(3) $\|R_{-t}TR_tf\|_B\in L^2(G)$, *for each* $f\in B$.

(4) $R_{-t}TR_tf\in L^1(G, B)$, *for each* $f\in B$.

Proof First, since $\|TR_tf-Tf\|_B\leqslant\|T\|_{\mathscr{L}}\|R_tf-f\|_B\to0(t\to\theta)$ is valid for each $f\in B$, TR_tf is continuous with respect to variable t.

Next, from estimation of the following inequalities for each $f\in B$,

$$\begin{aligned}\|R_{-t}TR_tf-Tf\|_B&\leqslant\|R_{-t}TR_tf-R_{-t}Tf\|_B+\|R_{-t}Tf-Tf\|_B\\&\leqslant\|T\|_{\mathscr{L}}\|R_tf-f\|_B+\|R_{-t}(Tf)-Tf\|_B\to0(t\to\theta),\end{aligned}$$

we get the continuity of $R_{-t}TR_tf$ with respect to variable t.

Finally, $\|R_{-t}TR_tf\|_B=\|TR_tf\|_B$ holds, because the following inequalities are obvious

$$\|R_{-t}TR_tf\|_B\leqslant\|TR_tf\|_B=\|R_tR_{-t}TR_tf\|_B\leqslant\|R_{-t}TR_tf\|_B,\ f\in B,\ t\in G.$$

Hence, we come to the conclusion of Theorem 1.2 immediately.

Theorem 1. 3. *If $T\in\mathscr{L}_B^1$, then both $(t,\gamma)TR_tf$ and $(t,\gamma)R_{-t}TR_tf$ are elements in $L^1(G, B)$, where $f\in B$, $\gamma\in\Gamma$.*

Proof Since both (t,γ) and $R_{-t}TR_tf$ are continuous with respect to variable t, $(t,\gamma)R_{-t}TR_tf$ is also continuous. Further, since

$$\|(t,\gamma)R_{-t}TR_tf\|_B=\|R_{-t}TR_tf\|_B,$$

using Theorem 1.2, we see $(t,\gamma)R_{-t}TR_tf\in L^1(G, B)$. The proof of the conclusion that $(t,\gamma)TR_tf\in L^1(G, B)$ is an analogue of above procedure.

Example 1.4. Let ϕ be a bounded measurable function on G. If the operator T_ϕ is defined by

$$T_\phi : L^1(G) \to L^1(G)$$
$$f(\cdot) \mapsto \phi(\cdot) f(\cdot),$$

then $T_\phi \in \mathscr{L}^1_{L^1(G)}$ is equivalent to $\phi \in L^1(G)$.

The invariant operator T in $L^1(G)$ (nontrivial), i. e., the multiplier in $L^1(G)$ (see [2]), however, does not belong to $\mathscr{L}^1_{L^1(G)}$. Since there is a unique element μ in $M(G)$, the set of all bounded measures on G, such that $Tf = \mu * f$ for all $f \in L^1(G)$, we see that $\|TR_t f\|_{L^1(G)} = \|R_t Tf\|_{L^1(G)} = \|Tf\|_{L^1(G)} = \|\mu * f\|_{L^1(G)}$ is a constant independent of t. Therefore $TR_t f \notin L^1(G, L^1(G))$.

§ 2. The Fourier Transform for Operators in $\mathscr{L}^1_B$

Definition 2.1. *Let $T \in \mathscr{L}^1_B$. Using Theorem 1.3 we know that $(t, \gamma) R_{-t} T R_t f \in L^1(G, B)$ for each $f \in B$, $\gamma \in \Gamma$, namely, Banach valued integration $\int_G (t, \gamma) R_{-t} T R_t f d\lambda(t)$ is significant (see [3]), where λ is a Haar measure on G. It is easy to verify that the mapping*

$$\pi_\gamma(T) : B \to B$$
$$f \mapsto \int_G (t, \gamma) R_{-t} T R_t f d\lambda(t)$$

is linear. For a fixed $T \in \mathscr{L}^1_B$, $\pi_\gamma(T)$ is an element depending on γ in $L(B)$, the set of all linear operators from B to B. We easily see that the mapping

$$\pi_\gamma : \mathscr{L}^1_B \to L(B)$$
$$T \mapsto \pi_\gamma(T)$$

is also linear. For $\gamma \in \Gamma$, $T \in \mathscr{L}_B$, the mapping

$$\hat{T} : \Gamma \to L(B)$$
$$\gamma \mapsto \pi_\gamma(T)$$

is said to be the Fourier transform for an operator T. When we do not need to avoid confusion, $\hat{T}(\gamma) = \pi_\gamma(T)$ will be also said to be the Fourier transform as $f(x)$ is said to be a complex valued function f. Indeed, $\hat{T}(\gamma)$ is the value of the Fourier transform for an operator T in $\mathscr{L}^1_B$ at $\gamma \in \Gamma$.

Example 2.2. Via computing we see that the Fourier transform of the operator T_ϕ in Example 1.4 belongs to $L(L^1(G))$ and $(\hat{T}_\phi(\gamma) f)(s) = (s, -\gamma) \hat{\phi}(\gamma) f(s)$ holds for each $f \in L^1(G)$. Hence $\hat{T}_\phi(\gamma) = (\cdot, -\gamma) \hat{\phi}(\gamma)$. Omittig the factor $(\cdot, -\gamma)$ we can identify $\hat{T}_\phi(\gamma)$ with $\hat{\phi}(\gamma)$. Therefore, the Fourier transform for an operator has generalized the usual notion of the Fourier transform for an integrable function on G.

Theorem 2.3. *Let $T \in \mathscr{L}^1_B$, $f \in B$. Then*

(1) $\hat{T}(\gamma)f$ *is continuous in the norm topology of* B.

(2) $\hat{T}(\gamma)f\in C_0(\Gamma, B)$.

Proof (1) Let $\gamma_1, \gamma_2\in\Gamma$, $\varepsilon>0$. Then there sxists a compact set K in G, such that $\int_{G-K}\|R_{-t}TR_tf\|_B d\lambda(t)<\varepsilon$. Since

$$\|\hat{T}(\gamma_1)f-\hat{T}(\gamma_2)f\|_B=\left\|\left(\int_{G-K}+\int_K\right)((t, \gamma_1)-(t, \gamma_2))R_{-t}TR_tfd\lambda(t)\right\|_B$$

$$\leqslant 2\varepsilon+\int_K|(t, \gamma_1-\gamma_2)-1|\,\|R_{-t}TR_tf\|_B d\lambda(t),$$

we have $\lim\limits_{\gamma_1\to\gamma_2}\|\hat{T}(\gamma_1)f-\hat{T}(\gamma_2)f\|_B=0$ when γ_1 and ε tend to γ_2 and 0 respectively.

(2) Write $F(t)\equiv R_{-t}TR_tf$. Then $F\in L^1(G, B)$. Because B valued simple functions are dense in $L^1(G, B)$, it follows that for arbitrary small positive number ε, there is a B valued function of form $\sum\limits_{j=1}^n b_j\chi_{E_j}(t)$ such that $\|F(t)-\sum\limits_{j=1}^n b_j\chi_{E_j}(t)\|_{L^1(B)}<\varepsilon$, where $\{x_j\}_1^n\subset B$, $\{E_j\}_1^n$ is a collection of disjoint Borel measurable sets in G and χ_{E_j} is the characteristic function. $\|\cdot\|_{L^1(B)}$ denotes the norm in the space $L^1(G, B)$. Furthermore, for each $j(1\leqslant j\leqslant n)$ we have

$$\int_G(t, \gamma)b_j\chi_{E_j}(t)d\lambda(t)=b_j\hat{\chi}_{E_j}(\gamma)\in C_0(\Gamma, B).$$

In the following inequalities

$$\|\hat{T}(\gamma)f\|_B=\left\|\int_G(t, \gamma)F(t)d\lambda(t)\right\|_B$$

$$\leqslant\left\|\int_G(t, \gamma)\Big(F(t)-\sum_{j=1}^n b_j\chi_{E_j}(t)\Big)d\lambda(t)\right\|_B+\left\|\int_G(t, \gamma)\sum_{j=1}^n b_j\chi_{E_j}(t)d\lambda(t)\right\|_B$$

$$\leqslant\varepsilon+\sum_{j=1}^n\left\|\int_G(t, \gamma)b_j\chi_{E_j}(t)d\lambda(t)\right\|_B,$$

letting γ and ε tend to ∞ and 0 respectively we get

$$\lim_{\gamma\to\infty}\|\hat{T}(\gamma)f\|_B=0.$$

Theorem 2.4. *Let* $T\in\mathscr{L}_B^1$. *Then* $\hat{T}$ *is* $\{u_\alpha\}$ *summable to* T *in the strong operator topology of* $\mathscr{L}(B)$, *namely*,

$$\lim_\alpha\left\|\int_\Gamma\hat{u}_\alpha(\gamma)\hat{T}(\gamma)d\eta(\gamma)f-Tf\right\|_B=0$$

holds for each $f\in B$. *When* $\{u_\alpha\}$ *is an approximate identity possessing compact support and norm of one in* $L^1(G)$, η *is a Haar measure on* Γ.

Proof Since $\{u_\alpha\}\subset L^1(G)\cap L^2(G)$, $\{\hat{u}_\alpha(\gamma)\}$ have inversions of the Fourier transform (see [4]). We have the following equalities

$$\int_\Gamma\hat{u}_\alpha(\gamma)\hat{T}(\gamma)d\eta(\gamma)f$$

$$=\int_\Gamma\Big(\int_G(s, \gamma)u_\alpha(s)d\lambda(s)\Big(\int_G(t, \gamma)R_{-t}TR_tfd\lambda(t)\Big)d\eta(\gamma)$$

$$=\int_G R_{-t}TR_tfd\lambda(t)\int_\Gamma (t,\gamma)d\eta(\gamma)\int_G (s,\gamma)u_\alpha(s)d\lambda(s)$$

$$=\int_G R_{-t}TR_tfd\lambda(t)\int_\Gamma (t,\gamma)\hat{u}_\alpha(\gamma)d\eta(\gamma)$$

$$=\int_G u_\alpha(-t)R_{-t}TR_tfd\lambda(t).$$

Using Theorem 1.2 we see that for an arbitrary small positive number ε there is a symmetrical neighborhood $O(\theta)$ containing the identity θ of G such that $\|R_{-t}TR_tf-Tf\|_B<\varepsilon$. We can choose $\{u_\alpha\}$such that their supports are contained by $0(\theta)$. From $Tf=\int_G u_\alpha(t)\ Tfd\lambda(t)$ we get following equalities

$$\int_G u_\alpha(-t)R_{-t}TR_tfd\lambda(t)-Tf$$

$$=\int_G u_\alpha(-t)(R_{-t}TR_tf-Tf)d\lambda(t)$$

$$=\left(\int_{0(\theta)}+\int_{G-0(\theta)}\right)u_\alpha(-t)(R_{-t}TR_tf-Tf)d\lambda(t).$$

In the first term, the norm is not greater than $\varepsilon\int_{0(\theta)}|u_\alpha(-t)|d\lambda(t)=\varepsilon$; the second term is zero. Therefore,

$$\lim_\alpha\left\|\int_\Gamma \hat{u}_\alpha(\gamma)\hat{T}(\gamma)d\eta(\gamma)f-Tf\right\|_B$$

$$=\lim_\alpha\left\|\int_G u_\alpha(-t)R_{-t}TR_tfd\lambda(t)-Tf\right\|_B<\varepsilon.$$

Letting ε tend to zero, we get the equality in Theorem 2.4.

Corollary 2.5. *Let T_1, $T_2\in\mathscr{L}^1_B$. If $\hat{T}_1(\gamma)=\hat{T}_2(\gamma)$ for all $\gamma\in\Gamma$, then $T_1=T_2$.*

Proof Apply Theorem 2.4.

We suppose that S, $T\in\mathscr{L}^1_B$. It is easy to verify ST, $TS\in\mathscr{L}^1_B$. The following theorem will discuss the Fourier transform for compound operator ST.

Theorem 2.6. *Let S, $T\in\mathscr{L}^1_B$, $\gamma\in\Gamma$, $f\in B$. Then $\hat{S}(\gamma-\cdot)\hat{T}(\cdot)f$ is $\{u_\alpha\}$ summable to $\widehat{ST}(\gamma)f$ in the norm of B, namely, the following relation holds*

$$\lim_\alpha\left\|\int_\Gamma \hat{u}_\alpha(\delta)\hat{S}(\gamma-\delta)\hat{T}(\delta)d\eta(\delta)f-\widehat{ST}(\gamma)f\right\|_B=0.$$

Proof Write $F_\alpha=\int_\Gamma \hat{u}_\alpha(\delta)\hat{T}(\delta)d\eta(\delta)$. Let $t\in G$. Using Theorem 2.4 we know $\lim\limits_\alpha\|F_\alpha(R_tf)-TR_tf\|_B=0$, i. e., $\{F_\alpha(R_tf)\}$ converges to TR_tf pointwise. Thus

$$\widehat{st}(\gamma)f=\int_G(t,\gamma)(R_{-t}STR_t)fd\lambda(t)$$

$$=\int_G(t,\gamma)R_{-t}S(\lim_\alpha F_\alpha(R_tf))d\lambda(t).$$

We have noticed that $(t,\gamma)R_{-t}S(\lim F_\alpha(R_tf))$ converges to $(t,\gamma)(R_{-t}STR_tf)$ on $G\times\Gamma$ pointwise and $\int_G\|(t,\gamma)R_{-t}STR_tf\|_Bd\lambda(t)<+\infty$. Using Lebesgue's Dominated

Convergence Theorem for the Bochner integral (see [3]) we get

$$\begin{aligned}\widehat{st}(\gamma)f&=\lim_{\alpha}\int_{G}(t,\gamma)R_{-t}SF_{\alpha}(R_{t}f)d\lambda(t)\\&=\lim_{\alpha}\int_{\Gamma}\hat{u}_{\alpha}(\delta)\Big(\int_{G}(t,\gamma-\delta)R_{-t}SR_{t}\hat{T}(\delta)fd\lambda(t)\Big)d\eta(\delta)\\&=\lim_{\alpha}\int_{\Gamma}\hat{u}_{\alpha}(\delta)\hat{S}(\gamma-\delta)\hat{T}(\delta)fd\eta(\delta).\end{aligned}$$

Therefore, the following holds:

$$\lim_{\alpha}\left\|\int_{\Gamma}\hat{u}_{\alpha}(\delta)\hat{S}(\gamma-\delta)\hat{T}(\delta)d\eta(\delta)f-ST(\gamma)f\right\|_{B}=0.$$

Definition 2.7. *Let $T\in\mathscr{L}_{B}^{1}$, $\mu\in M(G)$. Define the following*

$$\mu*T:B\to B$$

$$f\mapsto\int_{G}R_{t}TR_{-t}fd\mu(t).$$

$\mu*T$ is said to be the convolution of μ and T.

Theorem 2.8. *Let $T\in\mathscr{L}_{B}^{1}$, $\mu\in M(G)$. Then $(\mu*T)=\hat{\mu}\hat{T}$.*

Proof Straightforward.

§ 3. The Parseval Formula in the Harmonic Analysis for Operators

Lemma 3.1. *Suppose $T\in\mathscr{L}_{B}^{1}$, $\gamma\in\Gamma$. Then there exists an invariant operator H_{γ} which satisfies the following equality for all $f\in B$,*

$$(\pi_{\gamma}(T)f)(s)=(-s,\gamma)(H_{\gamma}f)(s).$$

Proof For each $u,v\in G$, we have

$$\begin{aligned}R_{v}((u,\gamma)\pi_{\gamma}(T)f)(u)&=R_{v}(u,\gamma)\int_{G}(t,\gamma)R_{-t}TR_{t}f(u)d\lambda(t)\\&=(u-v,\gamma)R_{v}\int_{G}(t,\gamma)R_{-t}TR_{t}f(u)d\lambda(t)\\&=(u,\gamma)\int_{G}((t-v),\gamma)R_{v-t}TR_{t-v}R_{v}f(u)d\lambda(t)\\&=((u,\gamma)\pi_{\gamma}(T))R_{v}f(u).\end{aligned}$$

Write$(\cdot,\gamma)\pi_{\gamma}(T)=H_{\gamma}$, then $R_{v}H_{\gamma}=H_{\gamma}R_{v}$; i. e., H_{γ} is an invariant operator and $\pi_{\gamma}(T)=(-\cdot,\gamma)H_{\gamma}$.

Theorem 3.2. *Suppose that B is a Banach algebra without the order (see [2]). Then $\pi_{\gamma}(T)$is a bounded linear operator, and there exists a constant $M>0$ such that $\|\pi_{\gamma}(T)\|\leqslant M$ for all $\gamma\in\Gamma$.*

Proof By [2] we know H_{γ} is a bounded linear operator. Thus $\pi_{\gamma}(T)$ is also bounded linear. From Theorem 2.3 we get $\pi_{\gamma}(T)f\in C_{0}(\Gamma,B)$ for any $f\in B$. Then the number set $\{\|\pi_{\gamma}(T)f\|_{B}\}_{\gamma\in\Gamma}$ is bounded. Using Uniform Boundedness Theorem for operators on Banach space we conclude that the number set $\{\|\pi_\gamma(T)\|\}_{\gamma\in\Gamma}$ is bounded, i. e., there exists a constant $M>0$, such that $\|\pi_\gamma(T)\|\leqslant M$ for all $\gamma\in\Gamma$.

We now start to discuss the Parseval formula in harmonic analysis for operators. From [2] we know that $A_2(G)=\{f\,|\,f\in L^1(G),\ \hat{f}\in L^2(\Gamma)\}$ is a homogeneous Banach algebra without the order under the norm $\|\cdot\|_{A_2(G)}=\|\cdot\|_{L^1(G)}+\|\cdot\|_{L^2(\Gamma)}=\|\cdot\|_{L^1(G)}+\|\cdot\|_{L^2(G)}$. Indeed, $A_2(G)=\{f\,|\,f\in L^1(G)\cap L^2(G)\}$ is a Seg al algebra (see [5]). Let $f\in L^2(G)$. We have

$$\left|\int_G fg\mathrm{d}\lambda\right|\leqslant\|f\|_{L^2(G)}\|g\|_{L^2(G)}\leqslant\|f\|_{L^2(G)}\|g\|_{A_2(G)}$$

for all $g\in A_2(G)$. Therefore, $A_2(G)\subset L^2(G)\subset A_2(G)^*$ and $\langle f, g\rangle=(f, g)$ where $(\cdot, \odot)$ denotes the inner product of a Hilbert space.

Theorem 3.3. *If* $T\in\mathscr{L}^1_{A(G)}$, *then*

(1) $T^*\in\mathscr{L}^1_{A_2(G)^*}$.

(2) $\pi_\gamma(T^*)=\pi_{-\gamma}(T)^*$.

(3) $\pi_\gamma(T^*)$ *is bounded linear.*

Proof From Theorem 3.2 we see that $\pi_\gamma(T)^*$ is a bounded linear operator on $A_2(G)^*$. For any $f\in A_2(G)^*$, $g\in A_2(G)$, the following formula holds:

$$\begin{aligned}\langle\pi_\gamma(T)^*f, g\rangle&=\langle f, \pi_\gamma(T)g\rangle\\&=\left\langle f, \int_G(t, \gamma)R_{-t}TR_tgd\lambda(t)\right\rangle\\&=\int_G(t, -\gamma)\langle R_{-t}T^*R_tf, g\rangle d\lambda(t)\\&=\left\langle\int_G(t, -\gamma)R_{-t}T^*R_tfd\lambda(t), g\right\rangle\\&=\langle\pi_{-\gamma}(T^*)f, g\rangle.\end{aligned}$$

Therefore, (1) $T^*\in\mathscr{L}^1_{A_2(G)^*}$. (2) $\pi_{-\gamma}(T^*)=\pi_\gamma(T)^*$. (3) $\pi_\gamma(T^*)$ is bounded linear.

Corollary 3.4. $T\in\mathscr{L}^1_{A_2(G)}$ *is self conjugate if and only if the equality* $\pi_{-\gamma}(T)=\pi_\gamma(T)^*$ *holds for all* $\gamma\in\Gamma$.

Proof Apply Theorem 3.3 and Corollary 2.5 to complete the proof.

Theorem 3.5. *Let* $T\in\mathscr{L}^1_{A_2(G)}$. *If* $\hat{T}(\gamma)f\in L^2(\Gamma, L^2(G))$, *then the following formula holds*

$$\langle\pi_{\theta_\gamma}(T^*T)f, f\rangle=\left\langle\int_\Gamma\pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma)f, f\right\rangle,$$

where $f\in A_2(G)$, θ_γ *is the identity in* Γ. *Above formula can be said to be the Parseval formula for operators.*

Proof By Theorem 3.3, Theorem 2.4 and Theorem 2.6 we see that

$$\lim_\alpha\left\langle\int_\Gamma\hat{u}_\alpha(\gamma)(T^*)^\wedge(-\gamma)\hat{T}(\gamma)fd\eta(\gamma)-(T^*T)^\wedge(\theta_\gamma)f\right\rangle_{A_2(G)^*}=0$$

holds for all $f\in A_2(G)$. By Theorem 3.3 we have $(T^*)^\wedge(-\gamma)=\hat{T}(\gamma)^*$. Thus the following formula holds:

(a) $$\lim_\alpha \left\| \int_\Gamma \hat{u}_\alpha(\gamma)\hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f - (T^*T)^\wedge(\theta_\gamma)f \right\|_{A(G)^*} = 0.$$

Since $\hat{T}(\gamma)f \in L^2(\Gamma, L^2(G))$, the integral

$$\left\langle \int_\Gamma \hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle = \int_\Gamma \|\hat{T}(\gamma)f\|^2_{L^2(G)} d\eta(\gamma)$$

is significant. Suppose that ε is a small positive number. Then there exists a compact set $K \subset \Gamma$ such that

(b) $$\left| \left\langle \int_\Gamma \hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle - \left\langle \int_K \hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle \right| < \varepsilon.$$

By [6] we know that $\{u_\alpha\}$ have a property: $\lim\limits_\alpha \hat{u}_\alpha(\gamma) = 1$ uniformly holds for $\gamma \in K$. Hence

(c) $$\left\langle \int_K \hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle - \left\langle \lim_\alpha \int_K \hat{u}_\alpha(\gamma)\hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle = 0.$$

(d)
$$\begin{aligned}
&\left| \left\langle \lim_\alpha \int_K \hat{u}_\alpha(\gamma)\hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle - \left\langle \lim_\alpha \int_\Gamma \hat{u}_\alpha((\gamma)\hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle \right| \\
&= \left| \left\langle \lim_\alpha \int_{\Gamma-K} \hat{u}_\alpha(\gamma)\hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle \right| \\
&\leqslant \lim_\alpha \int_{\Gamma-K} |\hat{u}_\alpha(\gamma)| \, \|\hat{T}(\gamma)f\|_{L^2(G)} d\eta(\gamma) < \varepsilon.
\end{aligned}$$

Using (a), (b), (c) and (d) to do an estimation of amplifying we get

$$\left| \left\langle \int_\Gamma \hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle - \langle (T^*T)^\wedge(\theta_\gamma)f, f \rangle \right| \leqslant \varepsilon + 0 + \varepsilon + 0 = 2\varepsilon.$$

Letting ε tend to zero, we have

$$\left\langle \int_\Gamma \hat{T}(\gamma)^*\hat{T}(\gamma)d\eta(\gamma)f, f \right\rangle = \langle (T^*T)^\wedge(\theta_\gamma)f, f \rangle,$$

namely,

$$\langle \pi_{\theta_\gamma}(T^*T)f, f \rangle = \left\langle \int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma)f, f \right\rangle.$$

Theorem 3.6. *Suppose $T \in \mathscr{L}^1_{A^2(G)}$. If $\hat{T}(\gamma)f \in L^2(\Gamma, L^2(G))$ for all $f \in A_2(G)$, then*

$$\pi_{\theta_\gamma}(T^*T) = \int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma).$$

Proof From Theorem 3.5 we know that the following equalities

$$\langle \pi_{\theta_\gamma}(T^*T)f, f \rangle = \left\langle \int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma)f, f \right\rangle = \int_\Gamma \|\pi_\gamma(T)f\|^2_{L^2(G)} d\eta(\gamma)$$

hold for all $f \in A_2(G)$ when the conditions of Theorem 3.6 is satisfied. Using the extremely identical relationship in the Hilbert space we see that the equality

$$\langle \pi_{\theta_\gamma}(T^*T)f, g \rangle = \left\langle \int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma)f, g \right\rangle$$

holds for all $f, g \in A_2(G)$. Thus

$$\pi_{\theta_\gamma}(T^*T)f = \int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma)f$$

holds for all $f\in A_2(G)$. Therefore

$$\pi_{\theta_r}(T^*T)=\int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta(\gamma).$$

Example 3.7. Suppose that T is an operator as follows:

$$T_\phi: A_2(G)\to A_2(G)$$
$$f\mapsto \phi f,$$

where ϕ is a bounded function in $L^2(G)$. By Example 2.2 we know

$$\pi_\gamma(T_\phi)=(\cdot,\ -\gamma)\hat{\phi}(\gamma),\quad \pi_\gamma(T_\phi)^*=(\cdot,\ \gamma)\hat{\phi}(\gamma)^*,$$

$$T_\phi^*T_\phi=|\phi|^2,\qquad \pi_{\phi_r}(T_\phi^*T_\phi)=\int_G |\phi|^2 d\lambda,$$

$$\int_\Gamma \pi_\gamma(T)^*\pi_\gamma(T)d\eta=\int_\Gamma |\hat{\phi}|^2 d\eta.$$

Therefore, $\int_G |\phi|^2 d\lambda(t)=\int_\Gamma |\hat{\phi}|^2 d\eta(\gamma)$. This is just the well known Parseval formula in $L^2(G)$.

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