Chin. Ann. of Math.
13B:1(1992), 68—79.

ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE DIRAC OPERATOR IN EUCLIDEAN SPACE

J. CNOPS*

Abstract

The author considers the possibility of generalizing the theory of classical polynomials to the higher dimensional case. The starting point is the splitting up of the second order differential operator of these polynomials into the derivation operator, considered as an operator between Hilbert spaces and its adjoint. In the case of several dimensions the derivation operator is replaced by the Dirac operator. As however the set of polynomials in the vector variable x is not dense in the Hilbert modules considered, first a decomposition of these modules in terms of spherical monogenic functions is proved. Then by applying the theory to each of the constituents, generalizations of the Gegenbauer and the Hermite polynomials are obtained.

§1. Introduction

Spectral theory. We say a selfadjoint operator A has pure point spectrum on the Hilbert space H if H is the direct orthogonal sum of the eigenspaces of A. In [3] we studied the Hilbert space $L_2(X, \mu, H)$, where μ is a measure on the set X and H is a Hilbert space, of L_2-functions on X with values in H and the extension of an operator on H, which was defined pointwise.

The Clifford algebra. We consider the 2^m-dimensional real vector space $\mathscr{A}(R^m)$ (or $\mathscr{A}$, shortly) given by the basis vectors $\{e_A = A \subset \{1, 2, \cdots m\}\}$ with the notation $e_0 = e_\phi$ and $e_A = e_{h_1 \cdots h_k}$ for $A = \{h_1, \cdots, h_k\}$ and $1 \le h_1 < \cdots < h_k \le m$.

On this vector space an associative product is defined by

$$e_{h_1} e_{h_2} \cdots e_{h_k} = e_{h_1 \cdots h_k} \text{ for } 1 \le h_1 < \cdots < h_k \le m,$$

which is governed by the rules $e_i^2 = -1$ and $e_i e_j = -e_i e_j$ for $i \ne j$. Hence $\mathscr{A}$ is the linear associative algebra generated by the elements $e_1, e_2, \cdots, e_m$. It is clear however that $\mathscr{A}$ is not commutative. Moreover for $m \geqslant 3$, $\mathscr{A}$ has zero divisors and henc is not a field. Since e_0 is the unit element for multiplication we can identify $\lambda \in R$ with λe_0. An involution on $\mathscr{A}$ is defined by $\bar{e}_i = -e_i$ and $\overline{ab} = \bar{b}\bar{a}$. The Euclidean

Manuscript received March 2, 1990.
* Seminar of Algebra and Functional Analysis, State University, Ghent.

norm in $\mathscr{A}$ will be denoted by $|.|_0$. A vector $x(x_1, \cdots, x_m) \in R$ can be identified with the Clifford number

$$x = \sum_{i=1}^{m} e_i x_i.$$

For the product of a vector with itself we have $x^2 = -|x|_0^2$. A unit vector will be indicated in the sequel by a Greek letter, e. g. ξ, θ, etc. The decomposition of x will be denoted as $x = r\xi$ where $r = |x|_0$. Analogically we use the decomposition $b = \rho\beta$.

Hilbert-modules over $\mathscr{A}$. Let H be an $\mathscr{A}$-module (i.e. a vector space over $\mathscr{A}$). A function$(.,\ .)$: $H \times H \to \mathscr{A}$ is called an inner product on H if for all f, g, $h \in H$ and $\lambda \in \mathscr{A}$,

(i) $(f,\ g\lambda + h) = (f,\ g)\lambda + (f,\ h)$;

(ii) $(f,\ g) = \overline{(g,\ f)}$;

(iii) $\mathrm{Re}(f, f) \geqslant 0$ and $\mathrm{Re}(f, f) = 0 \Leftrightarrow f = 0$.

From this $\mathscr{A}$-valued inner product a real valued inner product over H can be derived by

$$(f,\ g)_R = \mathrm{Re}(f,\ g).$$

H looked upon as a vector space over $\mathbf{R}$ with this real valued inner product will be denoted as H_R. The norm on H is given by $\|f\|^2 = (f, f)_R$. A module H with an $\mathscr{A}$-valued inner product is called a Hilbert module if H_R is a Hilbert space. The definitions of closed, densely defined, adjoint and selfadjoint operators thus can be generalized immediately. Notice however that since $\mathscr{A}$ in general contains non-invertible elements, Gram-Schmidt procedures cannot be applied without caution. So, in contrast with the case of Hilbert spaces, it is not neccessarily true that each operator with pure point spectrum gives rise to an orthogonal basis of eigenvectors. The Riesz representation theorem however remains valid. For more details on Hilbert $\mathscr{A}$-modules we refer to [5].

An important example of Hilbert modules are the modules $L_2(\Omega,\ \omega)$ of $\mathscr{A}$-valued measurable functions over $\Omega \subset \mathbf{R}^m$ with the inner product

$$(f,\ g) = \int_\Omega \bar{f} g \omega.$$

Also we shall use the Hilbert module $L_2(S^{m-1})$ with the inner product

$$\langle f,\ g \rangle = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \bar{f} g dS$$

in which ω_{m-1} is the surface of the unit sphere S^{m-1}.

The Dirac operator and the Laplacian. The Dirac operator is given by

$$D = \sum_{i=1}^{m} e_i \partial_{x_i}.$$

If f is a C^1-function in a domain Ω, then f is called left (right) monogenic if

$Df=0$ (resp. $fD=\sum_{i=1}^{m}\partial_{x_i}fe_i=0$) in Ω.

Since $D^2=-\Delta$, where Δ is the Laplacian, each monogenic function is harmonic and hence analytic. The operator D can be separated in a radial and a spherical part

$$D=\xi\left(\partial_r+\frac{1}{r}\Gamma\right).$$

In [4] we proved that the Dirac operator on the sphere, Γ, is selfadjoint on the Hilbert module $L_2(S^{m-1})$. The eigenmodules are given by

$$\mathscr{P}_k=\{f\in C^1(S^{m-1}):\Gamma f=-kf\},$$

the module of inner spherial monogenic functions of degree k, and

$$Q_k=\{f\in C^1(S^{m-1}):\Gamma f=(k+m-1)f\},$$

the module of outer spherical monogenic functions of degree k. The orthogonal sum of $\mathscr{P}_k$ and Q_{k-1} (where we put $Q_{-1}=\{0\}$) is $\mathscr{H}_k$, the module of spherical harmonics of degree k. With P_k, (Q_k) we shall denote an arbitrary element of $\mathscr{P}_k(Q_k)$. The modules P_k are mutually orthogonal, as are the modules Q_k. Moreover each P_k is orthogonal to each Q_l. The mapping which maps P_k on ξP_k is an isometry between $\mathscr{P}_k$ and Q_k.

For details on this paragraph we refer to [2].

1.1 Γ as operator on L_2-modules over a radially symmetric domain

Let Ω be a radially symmetric domain and ω a radially symmetric weight function. With the notation

$$\Omega_R=\{r\in\mathbf{R}:\exists\boldsymbol{x}\in\Omega:|\boldsymbol{x}|=r\}$$

the module $L_2(\Omega,\omega)$ can be identified with $L_2(\Omega_R, r^{m-2}\omega l, L_2(S^{m-1}))$ where $r^{m-1}\omega l$ is the measure arising when writing the integral of the inner product in spherical coordinates. Hence the extension of the operator Γ, as defined in [3], is selfadjoint on the module $L_2(\Omega,\omega)$, which shall be denoted by L_2 for short in the rest of this paragraph. For each k take an orthogonal basis $\{P_k^{(i)}: i=1,\cdots,K(m,k)\}$ of $\mathscr{P}_k$. Then we define

$$R_k^{(i)+}=\{f:f(r\xi)=P_k^{(i)}(r\xi)a(r),\ \text{for } r \text{ a. e.}, a(r)\ \text{Clifford valued}\},$$
$$R_k^{(i)-}=\{f:f(r\xi)=\xi P_k^{(i)}(r\xi)b(r),\ \text{for } r \text{ a.e.}, b(r)\ \text{Clifford valued}\},$$
$$R_k^{(i)}=R_k^{(i)+}\oplus^{\perp}R_k^{(i)-}$$

and

$$L_{2,k}^{(i)+}=L_2\cap R_k^{(i)+},$$
$$L_{2,k}^{(i)-}=L_2\cap R_k^{(i)-},$$
$$L_{2,k}^{(i)}=L_2\cap R_k^{(i)}.$$

Then we have the decomposition

$$L_2=\bigoplus_{k,i}^{\perp}L_{2,k}^{(i)}.$$

1.2. The Dirac operator

The importance of the modules $R_k^{(i)}$ stems from the fact that they are invariant for the Dirac operator. Indeed we have, for a real valued, derivable function f that

$$\begin{aligned} D(P_k^{(i)}(x)f(r)) &= D(f(r)P_k^{(i)}(x)) \\ &= (Df(r))P_k^{(i)}(x) + f(r)(DP_k^{(i)}(x)) \\ &= \xi f'(r)P_k^{(i)}(x) \\ &= \xi P_k^{(i)}(x)f'(r), \end{aligned} \tag{1}$$

since

$$\begin{aligned} Df(r) &= \xi\left(\partial r + \frac{1}{r}\Gamma\right)f(r) \\ &= \xi f'(r). \end{aligned}$$

This is also valid after right multiplication with a Clifford constant and hence, by superposition, for Clifford valued f since we can write f as $\sum_{A\subset\{1,\dots,n\}} f_A e_A$. In a similar way one gets

$$\begin{aligned} &D(\xi P_k^{(i)}(x)f(r)) \\ &= D(f(r)r^{m+2k-1})\frac{\xi P_k^{(i)}(x)}{r^{m+2k-1}} \\ &= \xi\partial r(f(r)r^{m+2k-1})\frac{\xi P_k^{(i)}(x)}{r^{m+2k-1}} + (f(r)r^{m+2k-1})D\left(\frac{\xi P_k^{(i)}(x)}{r^{m+2k-1}}\right) \\ &= P_k^{(i)}(x)\left(f'(r) + \frac{m+2k-1}{r}f(r)\right), \end{aligned} \tag{2}$$

since $\xi P_k^{(i)}(\xi) \in Q_k$ and hence the second term is zero.

These explicit expressions show that the action of the Dirac operator on the module $R_k^{(i)}$ can be reduced to that of the Dirac operator in $2k$ dimensions higher on the module of radially symmetric functions. We shall formalize it as follows:

Take $P_k^{(i)}$ a fixed inner spherical monogenic function of degree k in m dimensions. With the notations $R_{0,m+2k}$ for the module of radially symmetric functions in $m+2k$ dimensions, $R_{k,m}^{(i)}$ for the module $R_k^{(i)}$ in m dimenssions, $x = r\xi$ as variable in $\mathbf{R}^m$, $b = \rho\beta$, $|\beta| = 1$ as variable in $\mathbf{R}^{m+2k}$ and D_x and D_b the respective Dirac operators we can define the following bijection of $R_{k,m}^{(i)}$ on $R_{0,m+2k}$: Let $F(x) = P_k^{(i)}(x)f(r) + \xi P_k^{(i)}(x)g(r)$, then $F^*(b) = f(\rho) + \beta g(\rho)$ is the radially symmetric extension of F. Conversely, if $G(b) = h(\rho) + \beta k(\rho)$ is a radially symmetric function in $\mathbf{R}^{m+2k}$, then we use the notation $G^+(x) = P_k^{(i)}(x)h(r) + \xi P_k^{(i)}(x)k(r)$. Clearly $G^{+*} = G$, and $F^{*+} = F$. Moreover, from the formulae for the Dirac operator it is immediately clear that

$$D_x F - (D_b F^*)^+ \tag{3}$$

and by reiteration

$$D_x^n F = (D_b^n F^*)^+. \tag{4}$$

So we have e. g. for the powers of the variables x and b that for n even $n = 2l$,

$$\begin{aligned} D_b \boldsymbol{b}^n &= D_b((-1)^l \rho^n) \\ &= (-1)^l \beta n \rho^{n-1} \\ &= -n \boldsymbol{b}^{n-1}, \end{aligned}$$

for n odd, $n=2l+1$,

$$\begin{aligned} D_b \boldsymbol{b}^n &= D_b((-1)^l r^n \beta \\ &= (-1)^{l+1}(n+m+2k-1)\rho^{n-1} \\ &= -(n+m+2k-1)\boldsymbol{b}^{n-1}. \end{aligned}$$

Hence

$$D_x \boldsymbol{x}^n P_k(\boldsymbol{x}) = b(n, k) \boldsymbol{x}^{n-1} P_k(\boldsymbol{x}) \tag{5}$$

with

$$b(n, k) = \begin{cases} -n, & n \text{ even}, \\ -(n+m+2k-1), & n \text{ odd}. \end{cases} \tag{6}$$

§ 2. Orthogonal Polynomials Associated with the Dirac Operator

In the theory of the classical orthogonal polynomials one starts from two Hilbert spaces H_i, $i=0, 1$,

$$H_i = L_2(\Omega, \omega_i), \quad i=0, 1$$

where Ω is an open domain in $\mathbf{R}$, $\omega_i \in C^\infty(\Omega)$ and the operator $L=d/dx$ is a closed densely defined operator from $H_0 \to H_1$. If L^* is the adjoint operator of L, which also has the form of a differential operator, then L^*L is a selfadjoint operator with pure point spectrum. The orthogonal polynomials considered are eigenfunctions of this operator L^*L. The only solutions in the real case are the Jacobi polynomials (with the special cases Gegenbauer, Tchebycheff and Legendre polynomials), Laguerre and Hermite polynomials. It is clear however that for each operator L solutions of this problem can be sought. In the sequel we shall take $L=D$.

2.1. The radially symmetric case

We again take R_0 to be the module of radially symmetric functions, i.e. functions of the form $f(\boldsymbol{x}) = A(r) + \xi B(r)$. Let now Ω be an open radially symmetric domain in $\mathbf{R}^m$ with two radially symmetric C^∞ weight functions w_0, w_1. We define the Hilbert modules H_0 and H_1 by

$$H_i = L_2(\Omega, w_i) \cap R_0 = L_{2,0}(\Omega, w_i)$$

and denote the inner products by $(,)_i$. With polynomials we mean polynomials in the variable $\boldsymbol{x}$ with right, Clifford valued, coefficients, i.e. functions of the form

$$p_n(\boldsymbol{x}) = \sum_{i=0}^{n} \boldsymbol{x}^i a_i, \quad a_i \in \mathscr{A}.$$

Clearly each polynomial is a radially symmetric function

$$p_n(\boldsymbol{x}) = \sum_{2i \le n} (-1)^i r^{2i} a_{2i} + \xi \sum_{2j+1 \le n} (-1)^j r^{2j+1} a_{2j+1}.$$

The right module of polynomials is indicated as $\mathscr{A}[x]$. Let $(P_n)_{n=0}^{\infty}$ be a sequence of polynomials, each P_n of degree n. This sequence is an algebraic basis of $\mathscr{A}[x]$ if and only if the coefficient for the highest degree of each P_n is an invertible Clifford number. By multiplication with a suitable Clifford constant we get polynomials with a real valued highest degree coefficient. In the sequel we shall always mean with (P_n) such a sequence. We now look for domains Ω with weight functions w_0, w_1 such that

* D can be extended to a closed densely defined operator from $H_0 \to H_1$.
* D^*D is a selfadjoint differential operator with pure point spectrum

$$\sigma=\{\lambda_0, \lambda_1, \cdots\} \tag{7}$$

such that there exists a sequence of eigenpolynomials (P_n), forming a basis for H_1.

This investigation will contain two steps: first we look for Ω, w_0, w_1 and L where L is a differential operator and the following weak conditions hold:

* for each pair of polynomials we have

$$(Df, g)_1=(f, Lg)_0,$$

* there exist P_0, P_1, P_2 such that

$$LDP_n=P_n\lambda_n, \quad n=0, 1, 2$$

with λ_n a Clifford constant.

In the first step we show that the only solutions meeting the weak conditions are those with the weight functions $(1+x^2)^\alpha$ (for certain values of α) and $\exp(-x^2/2)$. In the second step we show that in these cases also the strong conditions are met.

Suppose that Ω, w_0, w_1 and L meet the weak conditions. Then we have

$$\begin{aligned}(Df, g)_1&=\int_\Omega \overline{Df}\, w_1 dx\\ &=-\int_\Omega (\bar{f}D) g w_1 dx\\ &=\int_\Omega \bar{f} D(gw_1)dx-\int_{\partial\Omega} \bar{f}\, d\sigma . g w_1,\end{aligned}$$

where, if Ω is not a bounded domain, the integral over $\partial\Omega$ has to be looked upon as

$$\int_{\partial\Omega} \bar{f} d\sigma\, g w_1=\lim_{k\to\infty}\int_{\partial\Omega_k} \bar{f} d\sigma\, g w_1$$

with $\Omega_k=\Omega\cap B(k)$. For L to be a differential operator we must have that $w_1=0$ on $\partial\Omega$, and respectively for Ω not bounded

$$\lim_{k\uparrow\infty}\int_{\partial\Omega_k} \bar{f}\, d\sigma\, g w_1=0$$

for all polynomials f and g. In this case we have that

$$\begin{aligned}(Df, g)_1&=\int_\Omega \bar{f} D(w_1 g)dx\\ &=(f, w_0^{-1}D(w_1 g))_0.\end{aligned}$$

Hence L has the form

$$L=w_0^{-1}Dw_1.$$

We now put $X=w_1w_0^{-1}$. Then

$$LD=w_0^{-1}Dw_1D$$
$$=w_0^{-1}DXw_0D$$
$$=XD^2+\left[(DX)+X\frac{(Dw_0)}{w_0}\right]D.$$

Clearly for each polynomial P_0 of degree zero

$$LDP_0=L0=0.$$

Let P_1 be the polynomial of degree 1 we look for,

$$P_1(x)=xa_1+b_1.$$

Then there has to be a λ_1 such that

$$XD^2P_1+\left[(DX)+X\frac{(Dw_0)}{w_0}\right]DP_1=P_1\lambda_1.$$

But

$$D^2P_1=0,$$
$$DP_1=-ma_1,$$

where a_1 is a real constant, and hence

$$\left[(DX)+X\frac{(Dw_0)}{w_0}\right]=-P_1\frac{\lambda_1}{a_1m}.$$

Since w_0 and w_1 are radially symmetric, $w_i(x)=w_i(r)$, X is radially symmetric and we have

$$DX(x)=\xi X'(r),$$
$$Dw_0(x)=\xi w_0'(r).$$

Hence λ_1 is real, $b_1=0$ and

$$\left[(DX)+X\frac{(Dw_0)}{w_0}\right]=xA, \tag{8}$$

where A is real. For $P_2=x^2a_2+xb_2+c_2$ we have

$$XD^2P_2+xA.DP_2=P_2\lambda_2,$$

where

$$D^2P_2=2ma_2,$$
$$DP_2=-2xa_2-mb_2.$$

Since A is real, we have

$$xADP_2=x(DP_2)A.$$

Hence the second member at left is a polynomial of degree not more than two, i.e. X is a polynomial of degree not more than two and has the form (since X is real)

$$X=x^2B+C,$$

B, C real constants, and w_0 satisfies (8)

$$-2rB+(-r^2B+C)\frac{w_0'(r)}{w_0(r)}=rA.$$

This leads to the following possibilities (we put $A+2B=2\gamma$):

(1) $B=0$: we put $C=1$ and get

$$w_0'(r)=2\gamma r w_0(r).$$

From this we have

$$w_0=\beta\exp(r^2\gamma),$$
$$w_1=w_0.$$

The condition that w_1 has to be zero on the boundary of Ω is only met if $\gamma<0$. We normalize $\gamma=-1/2$, $D=1$ and get the solution

$$w_0=w_1=\exp(x^2/2),$$
$$\Omega=\mathbf{R}^m.$$

(2) $C=0$: with the normalization $B=1$ the equation for w_0 becomes

$$w_0'(r)=\frac{2\gamma}{r}w_0(r)$$

from which

$$w_0(r)=\beta r^{2\gamma}.$$

This does not give a solution to the weak problem.

(3) $B\neq 0$, $C\neq 0$:

$$(-r^2B+C)w_0'(r)=2r\gamma w_0(r),$$
$$w_0(r)=(-r^2B+C)^{\gamma/-B}.$$

For $BC<0$ this does not lead to a solution of the weak problem; for $BC>0$ we normalize $-B=C=1$ to get

$$w_0(r)=(1+x^2)^\alpha,$$
$$w_1(r)=(1+x^2)^{\alpha+1},$$

where $\alpha+1>0$ so that w_1 satisfies the boundary condition, and $\Omega=B(r)$.

If the strong conditions are met and hence for each n there exists a polynomial P_n and a constant λ_n such that

$$XD^2P_n+xADP_n=P_n\lambda_n,$$

then, from a comparison of the coefficients for the highest degree at both sides we see that λ_n is real. Moreover the equation gives a recursive relation between the coefficients of P_n, and hence all the coefficients are real. As a consequence we can write

$$P_n(x)=\sum_{i=0}^{n}x^ia_i=\sum_{i=0}^{n}a_ix^i.$$

2.2. Rodrigues' formula

To show that the solutions of the weak problem satisfy the strong conditions we use an indirect method. First we show that the orthogonal polynomials can be defined by a generalized formula of Rodrigues. Using this formula it is straightforward to prove that the polynomials indeed are eigenfunctions of the differential formula.

For $n\in N$ we define the Hilbert module

$$H_n=L_{2,0}(\Omega,\ w_n),$$

where ω_n is given by

$$w_n=X^n w_n.$$

It is clear that H_n for $n\geqslant 0$ contains all polynomials since

$$f\in H_n\leftrightarrow X^{n/2}f\in H_0.$$

We now define the operators L_n by

$$L_n f=w_n^{-1}(Dw_{n+1}f).$$

These operators are defined for f if f is a C^1-function. Moreover using the explicit expression for w_n we have that

$$\begin{aligned}L_n f&=\frac{1}{w_0X^n}D(X^{n+1}w_0f)\\&=\frac{1}{w_0X^n}((n+1)\ (DX)X^nw_0+X^{n+1}Dw_0)f+\frac{w_0X^{n+1}}{w_0X^n}Df\\&=\left[(n+1)(DX)+X\frac{Dw_0}{w_0}\right]f+XDf,\end{aligned}$$

where the term between square brackets is a polynomial of degree 1 with real coefficients. Hence, if P is a polynomial of degree l, then L_nP is a polynomial of degree $l+1$(for the weak solutions given the highest order terms cannot cancel each other). If we introduce the notations

$$g_{tn}=L_t\dots L_{n-1}1,$$

it is clear that for $n\geqslant t\geqslant 0$, g_{tn} is a polynomial of degree $n-t$.

Theorem 2.1. *For t fixed the sequence of polynomials g_{tn}, $n=t,\ t+1,\ \cdots$ forms an orthogonal set in H_t. Moreover the functions g_{tn} satisfy the differential equation*

$$L_tDg_{tn}=\lambda_{tn}g_{tn}.$$

Proof For two arbitrary polynomials f and g we have $(Df,\ g)_{k+1}=(f,\ L_kg)_k$. Indeed

$$\begin{aligned}(Df,\ g)_{k+1}&=\int_\Omega\overline{Df}w_{k+1}g\,dx\\&=\int_\Omega\bar{f}D(w_{k+1}g)dx\\&=\int_\Omega\bar{f}w_k(L_kg)dx\end{aligned}$$

since $w_{k+1}=X^kw_1$ vanishes on the boundary of Ω. Hence we have for $k\in N$

$$\begin{aligned}(g_{tn},\ x^k)_t&=(L_t\cdots L_{n-1}1,\ x^k).\\&=(1,\ D^{n-t}x^k)_n.\end{aligned}$$

Hence, if $k<n-t$ then $(g_{tn},\ x^k)_t=0$. Since each g_{tn} is a polynomial of degree $n-t$, for each n, $(g_{tn})_{n\in N}$ is a sequence of orthogonal polynomials.

It is clear that L_tDg_{tn} is a polynomial of degree $n-t$ which is orthogonal to all x^k for $k<n-t$ since

$$(L_tDg_{tn},\ x^k)_t=(g_{tn},\ L_tDx^k)_t$$

and L_tDx^k is a polynomial of degree k. Hence $L_t\,Dg_{tn}$ is a multiple of g_{tn}.

Remark We have

$$L_t\cdots L_{n-1}1=w_t^{-1}Dw_{t+1}w_{t+1}^{-1}Dw_{t+2}\cdots w_{n-1}^{-1}Dw_n1$$
$$=w_t^{-1}D^{n-t}w_tX^{n-t},$$

which gives the classical Rodrigues formula.

2.3. Not radially symmetric functions

In the paragraph above the module $L_{2,0}(\Omega,\ w)$ was studied. For the solution of the problem for the module $L_2(\Omega,\ w)$ we shall make use of the decomposition of functions in terms of spherical monogenic functions. Using the decomposition

$$L_2(\Omega,\ w)=\oplus^\perp L_{2,k}^{(i)}(\Omega,\ w)$$

it is sufficient to find an orthogonal basis for each $L_{2,k}^{(i)}$ to obtain a basis for the complete L_2-module. We now introduce the notation

$$\Omega_p=\{x\in\mathbf{R}^p;\ |x|\in\Omega_R\}.$$

Hence $\Omega=\Omega_m$. We denote by $R_{k,p}^{(i)}$ the module of k-symmetric functions in p dimensions associated with $P_k^{(i)}$ and

$$L_{2,k,p}^{(i)}=L_2(\Omega_p,\ w)\cap R_{k,p}^{(i)}.$$

Then there is an isometry between $L_{2,k,m}^{(i)}$ and $L_{2,0,m+2k}$, given up to a constant by

$$\tau;\ L_{2,k,m}^{(i)}\rightarrow L_{2,0,m+2k},$$
$$\tau(f)=f^*.$$

If $\{f_n;f_n(b)=a_n(\rho)+\beta c_n(\rho),\ a_n,\ c_n$ real valued, $n=0,\ 1,\ \cdots\}$ is a basis for $L_{2,0,m+2k}$ then

$$\{\tau^{-1}f_n=f_n^+\}$$

is a basis of $L_{2,k,m}^{(i)}$. With the notation D_p for the Dirac operator in p dimensions and $L_{n,p}$ for its adjoint in p dimensions,

$$L_{n,p}f=w_n^{-1}(D_pw_{n+1}f),$$

we now have also

Theorem 2.2. *If $\{P_n(b)\}$ is an orthogonal basis of $L_{2,0,2k+m}$ consisting of polynomials with real valued coefficients, then $\{P_n^+(x)\}$ is an orthogonal basis of $L_{2,k,m}^{(i)}$.*

If there exist λ_n such that

$$L_{0,2k+m}D_{2k+m}P_n=P_n\lambda_n,$$

then

$$L_{0,m}D_m(P_n^+)=P_n^+\lambda_n.$$

Proof The orthogonality follows from the isometry between $L_{2,0,2k+m}$ and $L_{2,k,m}^{(i)}$. Also we have

$$
\begin{aligned}
P_n^+\lambda_n &= (L_{0,m+2k}D_{m+2k}P_n)^+\\
&= XD_{m+2k}^2P_n+\beta A_1D_{m+2k}P_n\\
&= XD_m^2(P_n^+)+xA_1D(P_n^+)\\
&= L_{0,m}D_m(P_n^+).
\end{aligned}
$$

Here it makes sense to pose the generalized problem: which domains Ω with weight functions w_0, w_1 satisfy the following conditions

* it is possible to extend D to a closed, densely defined operator from $L_2(\Omega, w_0)$ $\to L_2(\Omega, w_1)$;

* D^*D is a selfadjoint differential operator with pure point spectrum $\sigma=\{\lambda_0, \lambda_1, \cdots\}$ such that each eigenmodule has an orthogonal basis of functions of the form $p_{n,k}P_k^{(i)}$, where $p_{n,k}$ is a polynomial of degree n.

In analogy with the radially symmetric case we get

Theorem 2.3. *Let Ω, w_0, X satisfy the strong conditions. With the notations for $t\leq n$,*

$$g_{t,n,k}P_k=L_t\cdots L_{n-1}P_k$$

(Notice that the definition is independent of the chosen P_k) there exist constants such that

$$p_{n,k}P_k=g_{0,n,k}P_kc_k=(w_0^{-1}(D^n)w_n)P_kc_k.$$

As a conclusion we have that there are two cases in which the classical orthogonal polynomials allow an extension to the more dimensional case in such a way that a second order differential operator and a generalized Rodrigues formula, both in terms of the Dirac operator, exist. They are:

* the generalized Hermite polynomials, defined as

$$H_{n,m,k}(x)P_k^{(i)}=e^{r^2/2}(-D)^n\exp(-r^2/2)P_k^{(i)}$$

which are orthogonal on the complete space for the weight function $\exp(-r^2/2)$, and

* the generalized Gegenbauer polynomials, defined as

$$C_{n,m,k}^{\alpha}(x)P_k^{(i)}=(1-r^2)^{-\alpha}D^n(1-r^2)^{\alpha+n}P_k^{(i)}$$

which are orthogonal on the unit ball for the weight function $(1-r^2)^{\alpha}$ if $\alpha>-1$.

References

[1] Berthier, A. M., Spectral theory and wave operators for the Schrödinger equation, Pitman, London, 1982.
[2] Brackx, F., Delanghe, R. & Sommen, F., Clifford Analysis, Pitman, London, 1982.
[3] Cnops, J., Extension of operators to L_2-spaces over a Hilbert space (to appear).
[4] Cnops, J., The spectrum of the Dirac operator on the sphere (to appear in Simon Stevin).
[5] Delanghe, R. & Brackx, F., Hypercomplex function theory and Hilbert modules with reproducing kernel, *Proc. London Math. Soc., third series*, **37** (1978), 545—576.
[6] Dunford, N. & Schwartz, J., Linear Operators II, Interscience publishers, New York, 1963.
[7] Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi., F., Higher transcendental functions, Vol. I-II, Mc Graw-Hill, New York, 1953.

[8] Hochstadt, H., The functions of mathematical physics, Pure and applied mathematics, vol XXIII, Wiley-Interscience, New York, 1971.

[9] Koornwinder, T., The addition formula for Jacobi polynomials and spherical harmonics, *SIAM J. Appl. Math.*, **25**(1973), 236—246.

[10] Sommen, F., Plane elliptic systems and monogenic functions in symmetric domains, *Rend. Circ. Mat. Palermo*, **6**(1984), 259—269.

[11] Szegö, G., Orthogonal polynomials, AMS Colloquium Publications vol. XXIII, AMS, Providence, Rhode Island, 1939.

[12] Triebel, H., Höhere Analysis, Hochschulbücher für Mathematik 76, VEB Deutcher Verlag der Wissenschaften, Berlin, 1972.