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THE MINIMAX DIRECTION FOR THE DIRECT PRODUCT OF A CONVEX CONE WITH ITS APPLICATION TO TESTING PROBLEMS**

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Abstract

Let C be a closed convex cone in R^k and let C^p be the p-th direct product of C. This paper gives some results of the minimax direction with respect to C^p and an inner product based on $\Gamma\otimes\Lambda$, where Γ is a $k\times k$ diagonal matrix with positive diagonal elements, Λ is a $p\times p$ positive definite matrix and $\Gamma\otimes\Lambda$ is the Kronecker product of Γ and Λ. It is also shown that the results may be applied to test the homogeneity of k normal mean vectors where the mean vectors are restricted by a given partial order.

§ 1. Introduction

Let C be a convex cone in R^k and let C^p be the p-th direct product of C. In R^{pk}, a vector is denoted by a $p\times k$ matrix $a=(a_1, \cdots, a_k)$, say, and an inner product and a norm are defined respectively by

$$(a, b)_{\Gamma\otimes\Lambda}=\sum_{j=1}^{k} a_j'\Lambda^{-1}b_j/r_j \quad \text{and} \quad \|a\|_{\Gamma\otimes\Lambda}=\sqrt{(a, a)_{\Gamma\otimes\Lambda}},$$

where Γ is a $k\times k$ diagonal matrix with positive diagonal elements $r_1, \cdots, r_k$, Λ is a $p\times p$ positive definite matrix and $\Gamma\otimes\Lambda$ is the Kronecker product of Γ and Λ.

A $p\times k$ matrix a_0 is said to be of the minimax direction with respect to (wrt) $(C^p, \Gamma\otimes\Lambda)$ if it satisfies

$$\inf_{b\in C^p} \Delta_{\Gamma\otimes\Lambda}(a_0, b)=\sup_{a\in R^{pk}} \inf_{b\in C^p} \Delta_{\Gamma\otimes\Lambda}(a, b), \tag{1}$$

where

$$\Delta_{\Gamma\otimes\Lambda}(a, b)=\frac{(a, b)_{\Gamma\otimes\Lambda}}{\|a\|_{\Gamma\otimes\Lambda}\|b\|_{\Gamma\otimes\Lambda}}$$

for non-zero vectors a and b. The minimax direction is said to be unique if there is another matrix a_{00} satisfying (1) which implies $\Delta_{\Gamma\otimes\Lambda}(a_0, a_{00})=1$.

When $p=1$, some properties of the minimax direction were given in [7] and its applications can be found in [6], [2], [7] and [4]. This paper considers the general case. A main theorem is given in section 2 and its applications to statistical testing problems are discussed in section 3.

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§ 2. Main Result

According to $p=1$ and $k=1$, we also define inner products and other related terminologies for Γ and Λ similarly as in Section 1.

Theorem 1. *Assume that C is closed. If there exists a k-dimensional vector z such that $\Delta_\Gamma(c, z)>0$ for all $c\in C$, then the minimax direction wrt $(C^p, \Gamma\otimes\Lambda)$ uniquely exists. Let α and β be of minimax directions wrt (Q, Λ) and (C, Γ) respectively, where Q is the positive orthant in R^p. Then $a_0=\alpha\beta'$ is of the minimax direction wrt $(C^p, \Gamma\otimes\Lambda)$.*

In order to prove the above theorem, we need some lemmas.

Let D be a polyhedral convex cone in R^s. Then D can be denoted as

$$D=\{d\in R^s;\ d=\lambda_1 d_1+\cdots+\lambda_m d_m,\ \lambda_i\geqslant 0;\ i=1, \cdots, m\}$$

for some d_i's, i. e., D is the smallest convex cone containing $d_1, \cdots, d_m$, and the d_i's are said to be edge vectors of D.

For a given $s\times s$ positive definite matrix Σ, we define $(.\,,.)_\Sigma$ and Δ_Σ similarly as in Section 1. Since $\Delta_\Sigma(d, h)$ may be considered as the cosine of the angle between d and h in R^s wrt the inner product $(.\,,.)_\Sigma$, it is not difficult to show that a vector d_0 is of minimax direction wrt (D, Σ) if and only if it satisfies

$$\min_{1\leqslant i\leqslant m} \Delta_\Sigma(d_0, d_i)=\sup_{d\in R^s}\min_{1\leqslant i\leqslant m}\Delta_\Sigma(d, d_i). \tag{2}$$

From discussions of section 8 in [1], we can prove the following lemma.

Lemma 1. *If there is a vector h such that $\Delta_\Sigma(h, d_i)>0$, $i=1, \cdots, m$, then the minimax direction wrt(D, Σ) uniquely exists. A vector d_0 is of the minimax direction if and only if there exists a subset M' of $M=\{1, \cdots, m\}$ such that $d_0=\sum_{j\in M'}\lambda_j d_j$ with $\lambda_j>0$ and for $j\in M'$*

$$\Delta_\Sigma(d_0, d_j)=\min_{1\leqslant i\leqslant m}\Delta_\Sigma(d_0, d_i).$$

If D is a closed and convex cone, there exists an increasing sequence $\{D_n\}$ of polyhedral convex cones, convergent to D with $D_n\subset D$ for all n.

Lemma 2. *Assume that there is a vector h such that $\Delta_\Sigma(h, d)>0$ for all $d\in D$. Let β_n be of minimax direction wrt (D_n, Σ). Then the limit β of $\{\beta_n\}$ exists and β is of minimax direction wrt(D, Σ).*

Proof Without loss of generality, we assume that norms of β_n are all one. Since the surface of unit ball is compact, for any subsequence of $\{\beta_n\}$, there is a sub-subsequence $\{\beta_t\}$ and a vector β such that $\beta_t\to\beta$ as $t\to\infty$. Note that, by Lemma 1, $\beta_t\in D_t$ and hence $\beta_t\in D$ and $\beta\in D$.

Let $f_t(x)=\min_{d\in D_t}\Delta_\Sigma(x, d)$ and $f(x)=\min_{d\in D}\Delta_\Sigma(x, d)$. The closedness of D_t implies that the intersection of D_t and the surface of unit ball is compact and hence $f_t(x)$

is continuous. Similarly, $f(x)$ is also a continuous function. As $\{D_t\}$ converges to D, $f_t(x) \to f(x)$ as $t \to \infty$ for any $x \in R^s$ and $\{f_t\}$ is a uniformly convergent sequence on D.

If β is not of minimax direction wrt (D, Σ), there is a vector β^* satisfying $f(\beta^*) > f(\beta)$. Let $\varepsilon = f(\beta^*) - f(\beta)$. Since $\beta_t \in D$ and $\beta \in D$ there is large enough t such that

$$f_t(\beta^*) - f(\beta^*) > -\frac{\varepsilon}{3} \quad \text{and} \quad f(\beta) - f_t(\beta_t) > -\frac{\varepsilon}{3}.$$

Then we have

$$f_t(\beta^*) - f_t(\beta_t) > \frac{\varepsilon}{3} > 0$$

and this is contrary to the definition of β_t.

To complete the proof, we need to show the uniqueness of the minimax direction wrt (D, Σ). In the fact, if β_1 and β_2 both are of minimax direction wrt (D, Σ) and $\Delta_\Sigma(\beta_1, \beta_2) < 1$, i. e., β_1 and β_2 are not on the same half-line, the condition implies $f(\beta_1) = f(\beta_2) > 0$. Let $\beta = \lambda_1\beta_1 + \lambda_2\beta_2$ with $\lambda_i > 0$. By the inequation

$$\|\beta\|_\Sigma < \lambda_1\|\beta_1\|_\Sigma + \lambda_2\|\beta_2\|_\Sigma,$$

we have $f(\beta) > f(\beta_1)$. This is contrary to the definition of β_1.

Proof of Theorem 1 At first, we prove the case that C is a polyhedral convex cone. Let the edge vectors of C be $g_1, \cdots, g_m$. Note that the positive orthant Q is a polyhedral convex cone in R^p and has p edge vectors $q_1, \cdots, q_p$, say, where q_i is the vector whose i-th element is one and all others are zero. It is easy to check that a $p \times k$ matrix e is an edge vector of C^p if and only if exactly one row vector of e is an edge vector of C and all others are k-dimensional zero vectors. Thus C^p has $p \times m$ edge vectors. Denote them by $e_{\mu\nu}$, $\mu = 1, \cdots, p$ and $\nu = 1, \cdots, m$, then which may be expressed by

$$e_{\mu\nu} = q_\mu g'_\nu, \quad \mu = 1, \cdots, p; \ \nu = 1, \cdots, m.$$

Since α is of minimax direction wrt(Q, Λ), from Lemma 1, there is a subset P' of $P = \{1, \cdots, p\}$ such that $\alpha = \Sigma\, \lambda_i q_i$ with $\lambda_i > 0$ for $i \in P'$. Similarly, there is a subset M' of $M = \{1, \cdots, m\}$ and $\beta = \Sigma\, \tau_j g_j$ with $\tau_j > 0$ for $j \in M'$. Then

$$\begin{aligned} a_0 = \alpha\beta' &= \sum_{i \in P'} \sum_{j \in M'} (\lambda_i\tau_j) q_i g'_j \\ &= \sum_{i \in P'} \sum_{j \in M'} (\lambda_i\tau_j) e_{ij}, \end{aligned}$$

where $\lambda_i\tau_j > 0$. As Q is the positive orthant, $\Delta_\Lambda(\alpha, q_i) > 0$ for $i \in P$. The conditions imply $\Delta_\Gamma(\beta, g_j) > 0$ for $j \in M$. Then we have

$$\begin{aligned} \Delta_{\Gamma\otimes\Lambda}(a_0, e_{ij}) &= \Delta_\Gamma(\alpha, q_i)\,\Delta_\Lambda(\beta, g_j) \\ &= \min_{1 \leq \mu \leq p} \Delta_\Gamma(\alpha, q_\mu) \min_{1 \leq \nu \leq m} \Delta_\Lambda(\beta, g_\nu) \\ &= \min_{(\mu,\nu)} \Delta_\Gamma(\alpha, q_\mu)\,\Delta_\Lambda(\beta, g_\nu) \\ &= \min_{(\mu,\nu)} \Delta_{\Gamma\otimes\Lambda}(a_0, e_{\mu\nu}) \end{aligned}$$

for $i\in P'$ and $j\in M'$, in which the second equation follows Lemma 1, and, by Lemma 1 also, a_0 is of minimax direction wrt$(C^p, \Gamma\otimes\Lambda)$.

Now let C be a closed and convex cone and let $\{C_n\}$ be an increasing sequence of polyhedral convex cones, convergent to C with $C_n\subset C$ for all n. Then $\{C_n^p\}$ is an increasing sequence and converges to C^p. Let β_n and β be of minimax directions wrt (C_n, Γ) and (C, Γ) respectively. Then $a_{0n}=\alpha\beta_n'$ is of minimax direction wrt $(C_n^p, \Gamma\otimes\Lambda)$. Following Lemma 2, we see that $a_0=\alpha\beta'$ is of minimax direction wrt $(C^p, \Gamma\otimes\Lambda)$.

§ 3. Applications

Let $\overline{X}_j=(\bar{x}_{1j}, \cdots, \bar{x}_{pj})$ be the mean of a sample with size n_j from a p-variant normal population $N_p(\theta_j, \Lambda)$, $j=1, \cdots, k$, where Λ is known. We consider testing the homogeneity of the normal means H_0: $\theta_1=\cdots=\theta_k$ against H_1-H_0, where H_1 denotes the hypothesis that the normal means are restricted by a partial order.

At first, we consider that for an umbrella order, that is,

$$H_1: \theta_1\leqslant\cdots\leqslant\theta_h\geqslant\cdots\geqslant\theta_k, \tag{3}$$

where h is known and $\theta_\mu\leqslant\theta_\nu$ means that all the elements of $\theta_\nu-\theta_\mu$ are non-negative. If $h=k$, the ordering is said to be simple, the likelihood ratio (LR) test was described by Barlow et al. (1972) for $p=1$ and generalized by Sasabuchi et al. (1983) to $p\geqslant2$. But, in the case $p\geqslant2$, they only gave an iterative algorithm computing the maximum likelihood estimate and we need simulation to obtain the upper α point (cf. Nomakuchi and Shi, 1988). When $h\in\{1, \cdots, k\}$, the LR test was considered by Shi (1988) for $p=1$ and it is very difficult to generalize his results to $p\geqslant2$.

In this section, by using results of Section 2, we will give a simple test statistic which was said to be the optimal contrast test by Robertson et al. (1988) and the most stringent somewhere most powerful test by Schaafsma and Smid (1966).

A contrast test, corresponding to a vector a, would reject H_0 for large values of

$$U_a=(a, \overline{X})_{\Gamma\otimes\Lambda}=\sum_{j=1}^{k} n_j a_j'\Lambda^{-1}\overline{X}_j, \tag{4}$$

where $\overline{X}=(\overline{X}_1, \cdots, \overline{X}_k)'$, Γ is a $k\times k$ diagonal matrix with diagonal elements $1/n_1, \cdots, 1/n_k$ and $a=(a_1, \cdots, a_k)'$, with $\sum_j n_j a_j=0$ is said to be the contrast coefficient. An optimal contrast test is, chosen an optimal contrast coefficient a_0, say, to maximize its minimum power over all $a\in H_1-H_0$. Let

$$C=\{c\in R^k; c_1\leqslant\cdots\leqslant c_h\geqslant\cdots\geqslant c_k, \sum_j n_j c_j=0\}.$$

Following the discussions in Chapter 4 of [4], we can prove that U_{a_0} is an optimal contrast test if and only if a_0 is of minimax direction wrt$(C^p, \Gamma\otimes\Lambda)$.

It is clear that the convex cone C satisfies the conditions of Theorem 1 and we need to derive α and β, the minimax directions wrt (Q, Λ) and (C, Γ), respectively.

Remark. Let $\delta=(\delta_1, \cdots, \delta_p)'$, where δ_i is the square root of the i-th diagonal element of Λ^{-1}. For a number p', $1\leqslant p'\leqslant p$, we partition δ into two parts: $\delta=(\delta'_{(1)}, \delta'_{(2)})'$, where $\delta_{(1)}$ consists of the first p' components of δ, and partition the matrix Λ correspondingly:

$$\Lambda=\begin{pmatrix}\Lambda_{11} & \Lambda_{12}\\ \Lambda_{21} & \Lambda_{22}\end{pmatrix}.$$

From Lemma 1, there is a subset of P' of P, without loss of generality, $P'=\{1, \cdots, p'\}$, such that

$$(\Lambda_{11}-\Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21})\delta_{(1)}>0, \tag{5}$$

$$\Lambda_{22}^{-1}\Lambda_{21}\delta_{(1)}+\delta_{(2)}\leqslant 0, \tag{6}$$

and the minimax direction wrt (Q, Λ) is

$$\alpha=(\alpha'_{(1)}, 0')',$$

where 0 is the $p-p'$ dimensional zero vector and

$$\alpha_{(1)}=(\Lambda_{11}-\Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21})\delta_{(1)}.$$

If $\Lambda\delta>0$, condition (6) is vacuous and $\alpha=\Lambda\delta$. For details see [8].

Now we derive β, the minimax direction wrt (C, Γ). The convex cone C has $k-1$ edge vectors $g_i=(g_{i1}, \cdots, g_{ik})'$, $i=1, \cdots, k-1$, in which g_{ij}, $j=1, \cdots, k$, are given as follows:

$$\begin{aligned} \text{when } i<h, \quad & g_{ij}=-1/s_i && \text{for } j\leqslant i,\\ & \quad\;\; =1/(s_k-s_i) && \text{for } j>i;\\ \text{when } i\geqslant h, \quad & g_{ij}=1/s_i && \text{for } j\leqslant i,\\ & \quad\;\; =-1/(s_k-s_i) && \text{for } j>i, \end{aligned}$$

where $s_i=n_1+\cdots+n_i$. If $\beta=(\beta_1, \cdots, \beta_i)'$ can be written as a linear combination of the edge vectors of the form

$$\beta=\lambda_1 g_1+\cdots+\lambda_{k-1}g_{k-1}$$

with $\lambda_i>0$, $i=1, \cdots, k-1$, and satisfies

$$\sum_j n_j\beta_j g_{1j}=\sum_j n_j\beta_j g_{2j}=\cdots=\sum_j n_j\beta_j g_{k-1j}, \tag{7}$$

then, by Lemma 1, the vector β is of the minimax direction wrt (C, Γ). We solve the above equation and obtain the following

Lemma 3. *The elements of the minimax direction β are*

$$\begin{aligned} \beta_j&=\{\sqrt{s_{j-1}(s_k-s_{j-1})}-\sqrt{s_j(s_k-s_j)}\}/n_j, && j<h,\\ &=\{\sqrt{s_{h-1}(s_k-s_{h-1})}+\sqrt{s_h(s_k-s_h)}\}/n_h, && j=h,\\ &=\{\sqrt{s_j(s_k-s_j)}-\sqrt{s_{j-1}(s_k-s_{j-1})}\}/n_j, && j>h. \end{aligned}$$

Letting $a_0=\alpha\beta'$, we see that the optimal contrast test rejects H_0 for large values of

$$U_{a_0}=(a_0,\ \overline{X})_{\Gamma\otimes\Lambda}=\sum_{j=1}^{k}\beta_j n_j \alpha'\Lambda^{-1}\overline{X}_j. \tag{8}$$

Theorem 2. *The optimal contrast test for testing H_0 against the umbrella order (3) is of the form (8), where α is of minimax direction wrt (Q, Λ) given in Remark and β is of minimax direction wrt (C, Γ) given in Lemma 3. Under H_0, the test is normally distributed with mean zero and variance $\|\alpha\beta'\|_{\Gamma\otimes\Lambda}$.*

Another useful partial ordering is the simple tree order

$$H_1: \theta_1 \leqslant \theta_j,\ j=2,\ \cdots,\ k. \tag{9}$$

When $p=1$, the LR test for testing H_0 against H_1-H_0 was studied by Barlow et al. (1972) and Robertson et al. (1988) and it is also difficult to generalize the results for $p\geqslant 2$. We, now, derive the optimal contrast test for this testing problem.

Let $C=\{c\in R^k;\ c_1\leqslant c_j,\ j=2,\ \cdots,\ k,\ \sum_j n_j c_j=0\}$. Then C is a closed and convex cone having $k-1$ edge vectors

$$g_i=\left(-\frac{1}{n_1},\ \cdots,\ -\frac{1}{n_i},\ \frac{k-1}{n_{i+1}},\ -\frac{1}{n_{i+2}},\ \cdots,\ -\frac{1}{n_{i+k}}\right),$$

for $i=1,\ \cdots,\ k-1$. Because of symmetry, it is easy to check that vector

$$\beta=\left(-\frac{1}{n_1},\ \frac{1}{s},\ \cdots,\ \frac{1}{s}\right)', \tag{10}$$

where $s=n_2+\cdots,\ +n_k$, satisfies equation (7) and hence β is of the minimax direction wrt (C, Γ).

Theorem 3. *The optimal contrast test for testing H_0 against the simple order (9) is of the form (8), where α is given in Remark and β is given in (10). Under H_0, the test is normally distributed with mean zero and variance $\|\alpha\beta'\|_{\Gamma\otimes\Lambda}$.*

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