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HAUSDORFF DIMENSION OF THE DOUBLE POINT SET OF THE WESTWATER PROCESS**

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Abstract

Let $X=\{X_t\}_{t\in[0,1]}$ be the Westwater process which is the coordinate process under 3-dimensional polymer measure $\nu(g)$ constructed by J. Westwater. In this paper, the Hausdorff dimension problem for the double point set of X is investigated. As a result, it is proved that

$$\dim D_2=1, \quad \nu(g)-\text{a.e.},$$

where $D_2=\{x\in R^3: X_t=X_s=x \text{ for some } s<t\in[0,1]\}$ is the double point set of X

§ 1. Introduction

Let $X=\{X_t\}_{t\in[0,1]}$ denote the Westwater process which is the coordinate process under 3-dimensional polymer measure $\nu(g)$ constructed by M. J. Westwater. In [12], we have proved that X has intersection local time and its path has double points. In the same paper, it is also obtained that the Hausdorff dimension of the set of double times for X is $1/2$.

Let μ be the Wiener measure in 3-dimensions, and $B=\{B_t\}_{t>0}$ be the Wiener process in 3-dimensions. It is well-known that the path of B has double points, and the Hausdorff dimension of the set of double points denoted by $\bar{D}_2$ is 1 (see [2], [10] or [8]). Recently, Le Gall[7] greatly improved their results. He investigated the Hausdorff measure of $\bar{D}_2$ and proved that the correct measure function of $\bar{D}_2$ is

$$k_2(x)=x(\log|\log x|)^2, \quad x>0.$$

We now denote the set of double points of X by D_2. Then a natural problem is to compute the Hausdorff dimension of D_2 and, further, to give the correct measure function of D_2. In the present paper, we discuss the above problem. The main result is as follows.

Theorem 1.1 *With probability* 1 *to* $\nu(g)$

$$\dim D_2=1$$

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where dim D_2 *denotes the Hausdorff dimension of* $D_2=\{x\in R^3 \mid \exists s<t\in[0,1],\ x=X_t=X_s\}$.

We prove this theorem in two steps. We estimate first the lower bound for dim D_2, and then estimate the upper bound. Using Kanfman's idea[4] and imitating an argument in [3, § 5], we can get the lower bound in Section 3. To get the upper bound, we apply the method used in [10] in Section 4. In Section2, we make some preparations for the proof in the last two sections.

§ 2. Preliminary

In the present section, we state and prove several lemmas which are used in the next sections. For Brownian motion B, [10, Lemma 3] plays an important role in estimating the upper bound for dim $\bar{D}_2$. To estimate the upper bound for dim D_2, we need a result similar to [10, Lemma 3] for Westwater process X. In fact, Lemma 2.3 can help us to get the similar result. In [3, § 5], Geman, Horowitz and Rosen apply a Kaufman's lemma in [4] to get the lower bound for dim $\bar{D}_2$. To conclude the desirable result, we need also a similar lemma for Westwater process X. And Lemma 2.4, although the result is perhaps known, helps us to realize the goal. Since [9, Corollary 2.1.4] is used for many times, for convenience, we state it as follows.

Lemma 2.1. *Given a probability space* $(\Omega, \mathscr{F}, P)$, *let* $X(\cdot, \omega)$ *be continuous for all* $w\in\Omega$, *and* X: $[0,1]\times\Omega\to R^3$ *be a* $\mathscr{B}_{[0,1]}\times\mathscr{F}$*-measurable function. If there exist numbers* $\alpha>0$, $r>0$ *and* $C<\infty$ *such that*

$$E_P|X(t)-X(s)|^r\leqslant C|t-s|^{1+\alpha},\quad 0\leqslant s, t\leqslant 1,$$

then for any $\nu\in(2, 2+\alpha)$, *and* $\lambda>0$,

$$P\Big(\sup_{0\leqslant s<t\leqslant 1}\frac{|X(t)-X(s)|}{|t-s|^{\beta}}\geqslant\frac{8\nu}{\nu-2}(4\lambda)^{1/r}\Big)\leqslant CA/\lambda$$

where $\beta=(\nu-2)/r$ *and*

$$A=\int_0^1\int_0^1|t-s|^{1+\alpha-\nu}\,dsdt.$$

For Westwater process X, we have the following property[11].

Lemma 2.2. *There exists a number* $C(n, \varepsilon)<\infty$ *for each* $n\geqslant 1$ *and* $\varepsilon\in(0,1)$ *s such that*

$$E_{\nu(g)}|X_t-X_s|^n\leqslant C(n,\varepsilon)|t-s|^{n/2-\varepsilon},\quad \forall\, t,s\in[0,1].$$

Now we prove the next two lemmas.

Lemma 2.3. *There exists a constant* $C(\varepsilon)<\infty$ *for each* $\varepsilon\in(0,1/2)$ *such that*

$$\nu(g)\Big(\bigcup_{i=2^{m-1}}^{2^m}\{|X(i2^{-m})|\leqslant 2^{-(1/2-\varepsilon)m}\}\Big)\leqslant C(\varepsilon)2^{-(1/2-2\varepsilon)m}$$

for any $m\geqslant 1$.

Proof Note [9, Lemma 3], then

$$\mu\Big(\bigcup_{i=2^{m-1}}^{2^m}\{|B(i2^{-m})|\leqslant 2^{-(1/2-\varepsilon)m}\}\Big)\leqslant C_3 2^{-(1/2-\varepsilon)m}.$$

According to [6, Theorem 2.1] (or see [11]), we have

$$\begin{aligned}\nu(g)&\Big(\bigcup_{i=2^{m-1}}^{2^m}\{|X(i2^{-m})|\leqslant 2^{-(1/2-\varepsilon)m}\}\Big)\\ &=E_\mu\{f_m I_{(\bigcup_{i=2^{m-1}}^{2^m}\{|B(i2^{-m})|\leqslant 2^{-(1/2-\varepsilon)m}\})}\}\\ &\leqslant E_\mu^\delta\{f_m^{1/\delta}\}E_\mu^{1-\delta}(I_{(\bigcup_{i=2^{m-1}}^{2^m}\{|B(i2^{-m})|\leqslant 2^{-(1/2-\varepsilon)m}\})})\\ &\leqslant C(\delta)\rho^m 2^{-(1/2-\varepsilon)m(1-\delta)}\\ &\leqslant C(\varepsilon)2^{-(1/2-2\varepsilon)m}\end{aligned}$$

where $\rho=2^{1/2\varepsilon+\delta\varepsilon}>1$, and δ is chosen to be ε. Moreover, the definition of $\{f_m\}_{m\geqslant 1}$ is given in [6], and f_m satisfies that there exists a $C(\delta)<\infty$ for each $\rho>1$ such that

$$E_\mu f_m^{1/\delta}\leqslant C^{1/\delta}(\delta)(\rho^{1/\delta})^m,\quad \forall m\geqslant 1.$$

This completes our proof.

Lemma 2.4. *For 3-dimensional Wiener process $B=(B_t)_{0\leqslant t<\infty}$, if F denotes the set of $B(\cdot)$ for which there exists an N such that for any interval I of length 2^{-m}, $m\geqslant N$, $B^{-1}(I)\cap[0,1]$ is contained in the union of less than m^2 intervals of the form $[k4^{-m},(k+1)4^{-m}]$, $0\leqslant k\leqslant 4^m$, then*

$$\mu(F)=1. \tag{2.1}$$

Proof The following argument is quite similar to that in [4]. In fact, for one dimensional Wiener process $W=(W_t)_{0\leqslant t<\infty}$, we have

$$\limsup_{s-t\to 0}|W_s-W_t|/\sqrt{2|s-t|\log 1/|s-t|}=1,\quad \text{a.e.}$$

Hence, for three dimensional Wiener process $B=(B_t)_{0\leqslant t<\infty}$,

$$\limsup_{t-s\to 0}|B_t-B_s|/\sqrt{2|t-s|\,|\log|t-s||}\leqslant\sqrt{3}.$$

It follows that for any $k=0,1,\cdots,4^m-1$,

$$\sup_{t\in[k4^{-m},(k+1)4^{-m}]}|B_t-B_{k4^{-m}}|\leqslant\sqrt{6\,4^{-m}\,m\log 4}\leqslant\sqrt{12\sqrt{2}\,m}\,2^{-m},\quad \text{a.e.}$$

Therefore, it is sufficient to prove that

$$\mu(G)=1 \tag{2.2}$$

where G denotes the set of $B(\cdot)$ for which there exists an N such that for any interval I' of length $5\sqrt{m}\,2^{-m}$, $m\geqslant N$, $B^{-1}(I')\cap[0,1]$ contains less than m^2 points of the form $k4^{-m}$ for $k=0,1,\cdots,4^m$.

To prove (2.2), we now concern the number of m-tuples $0\leqslant k_1<\cdots<k_m\leqslant 4^m$ for which

$$|B(k_{i+1}4^{-m})-B(k_i4^{-m})|\leqslant 10\sqrt{m}\,2^{-m},\quad 1\leqslant i<m. \tag{2.3}$$

Obviously, the probability which appears such m-tuple is that

$$\leqslant(C_1m^{1/2})^{m-1}(k_2-k_1)^{-3/2}\cdots(k_m-k_{m-1})^{-3/2}$$

where $C_1 \in (0, \infty)$ is a constant. The sum of this probability for all m-tuple is

$$\leqslant C_2(C_1 m^{1/2})^{m-1} 4^m \leqslant (C_3 m^{1/2})^m$$

where $C_2,\ C_3 \in (0,\infty)$ are constants.

Now, we denote by $c_{k_i, k_{i+1}}$ the event that (2.3) occurs. Then, the number of m-tuples is

$$A_m = \sum_{0<k_1<\cdots<k_m<4^m} I_{\bigcap_{i=1}^{m-1} c_{k_i, k_{i+1}}}.$$

By Chebyshev inequality, we have

$$\mu(A_m \geqslant m^2 (C_3 m^{1/2})^m) \leqslant m^{-2}.$$

Thus, Borel-Cantelli lemma implies that almost surely there exists an N such that

$$A_m < m^2 (C_3 m^{1/2})^m, \quad \forall m \geqslant N.$$

However, if there exists an interval I' of length $5\sqrt{m}\, 2^{-m}$ such that $B(k4^{-m}) \in I'$ for s values of k, then

$$\binom{s}{m} \leqslant m^2 (C_3 m^{1/2})^m.$$

It is not difficult to see that $s \leqslant m^2$. Clearly, s is exactly what we wanted.

The proof is finished.

§ 3. Lower Bound for dim D_2

In the present section, we begin with a lemma which is used again in the next section.

Lemma 3.1. *There exist finite positive constants C_1 and C_2 for any given $M \geqslant 1$ and $s \in (0, 1/2)$ such that*

$$\nu(g)\Big(\sup_{|t-s| \leqslant n^{-1}} |X_t - X_s| \geqslant C_2 n^{-1/2+s}\Big) \leqslant C_1 n^{-M}, \quad \forall n \geqslant 1. \qquad (3.1)$$

Proof. Actually, (3.1) is obtained immediately from Lemma 2.1 and Lemma 2.2. More precisely, in Lemma 2.1, we choose $r = 2M/s$, $\lambda = n^M$. By Lemma 2.2, we can choose $\alpha = 2M/(2s) - 1 - s$. Thus, $\nu = M/s - M + 2$, $\beta = (1-s)/2$, and

the left hand side of (3.1)

$$\leqslant \nu(g)\Big(\sup_{|t-s| \leqslant n^{-1}} \frac{|X_t - X_s|}{|t-s|^\beta} \geqslant C_2 n^{s/2}\Big)$$

$$\leqslant \nu(g)\Big(\sup_{0 \leqslant t < s \leqslant 1} \frac{|X_t - X_s|}{|t-s|^\beta} \geqslant \frac{8\nu}{\nu-2} (4n^M)^{s/2M}\Big)$$

$$\leqslant CAn^{-M}$$

where we pick $C_2 = \dfrac{8\nu}{\nu-2}{}^{s/2M}$, and $C_1 = CA$.

Remark 2. 1. Since $s \in (0, 1/2)$ and $M \geqslant 1$ are arbitrary, the constant C_2 in (3.1) may be chosen to be 1. Moreover, from the above proof we easily see that

$$\nu(g)\Big(\sup_{|t-s|\leqslant n^{-1}}|X(t)-X(s)|\geqslant Ln^{-1/2+s}\Big)$$
$$\leqslant C_1L^{-M}n^{-M},\quad \forall n\geqslant 1$$

for any $L\geqslant 1$ and $M\geqslant 1$, where $C_1\in(0,\infty)$ does not depend on L and n.

Having Lemma 2.4 and Lemma 3.1, we can prove the next lemma.

Lemma 3.2. *For the Westwater process $X=(X_t)_{0<t<1}$ and any given $s\in(0,1/2)$, if $\bar{F}$ denotes the set of $X(\cdot)$ for which there exists an N such that for any interval of length 2^{-m+s}, $m\geqslant N$, $X^{-1}(I)\cap[0,1]$ is contained in the union of less than 2^{2sm} intervals of the form $[k4^{-m},(k+1)4^{-m}]$, $0\leqslant k\leqslant 4^m$, then*

$$\nu(g)(\bar{F})=1. \tag{3.2}$$

Proof Actually, Lemma 3.1 tells us that for any given $M\geqslant 1$

$$\nu(g)\Big(\sup_{|t-s|\leqslant 4^{-m}}|X(t)-X(s)|\geqslant 2^{-(1-s)m}\Big)\leqslant C_M2^{-Mm},\quad \forall m\geqslant 1$$

where the constant $C_M\in(0,\infty)$. According to this property, we easily see that for proving (3.2) it is sufficient to prove that

$$\nu(g)(\bar{G})=1, \tag{3.3}$$

where $\bar{G}$ denotes the set of $X(.)$ for which there exists an N such that for any interval I' of length $2.2^{-(1-s)m}$, $m\geqslant N$, $X^{-1}(I')\cap[0,1]$ contains less than 2^{2sm} points of the form $k4^{-m}$, for $k=0,1,\cdots,4^m$.

To show (3.3), we also concern the number of m-tuples $0\leqslant k_1<\cdots<k_m\leqslant 4^m$ for which

$$|X(k_{i+1}4^{-m})-X(k_i4^{-m})|\leqslant 4\cdot 2^{-(1-s)m}. \tag{3.4}$$

Meanwhile, let $\bar{C}_{k_i,k_{i+1}}$ denote the event that (3.4) occours, and

$$\bar{A}_m=\sum_{0\leqslant k_1<\cdots<k_m\leqslant 4^m}I_U,$$

where

$$U=\bigcap_{i=1}^{m-1}\bar{C}_{k_i,k_{i+1}}.$$

Thus [6, Theorem 2.1] tells us that

$$\begin{aligned}\nu(g)&(\bar{A}_m\geqslant 2^{2sm}(C_12^{sm})^m)\\&=E_\mu\{f_{2m}I_{(A'_m\geqslant 2^{2sm}(C_12^{sm})^m)}\}\\&\leqslant E^{\delta}_{\mu}f^{1/\delta}_{2m}\cdot E^{1-\delta}_{\mu}I_{(A'_m\geqslant 2^{2sm}\cdot(C_12^{sm})^m)}\\&\leqslant C_\delta\rho^m[\mu(A'_m\geqslant 2^{2sm}(C_12^{sm})^m]^{1-\delta},\end{aligned} \tag{3.5}$$

where $\rho>1$ and A'_m is got from $\bar{A}_m$ only when $\bar{C}_{k_i,k_{i+1}}$ is replaced by $C'_{k_i,k_{i+1}}$, which denotes the event (3.4) as soon as X is changed into B. Since

$$\mu\Big\{\bigcap_{i=1}^{m-1}C'_{k_i,k_{i+1}}\Big\}\leqslant(C_12^{sm})^m(k_2-k_1)^{-3/2}\cdots(k_m-k_{m-1})^{-3/2},$$

by a similar argument in Lemma 2.4, we have

$$\mu(A'_m\geqslant 2^{2sm}(C_12^{sm})^m)\leqslant 2^{-2sm}.$$

Thus, choosing suitable $\delta\in(0,1/2)$ and $\rho>1$, we get that

the right hand side of $(3.5)\leqslant C2^{-sm}$.

From the discussion in Lemma 2.4, we know that the above estimation leads us to the desirable result (3.3).

The proof is completed.

Now we prove that

$$\dim D_2 \geqslant 1, \ \nu(g)\text{-a.e.} \tag{3.6}$$

Otherwise, we can assume that

$$\dim D_2 = \alpha < 1 - \varepsilon_0. \tag{3.7}$$

Then for any $\delta > 0$ and $\varepsilon > 0$, we can find arbitrarily large n, and a sequence of discs B_i of raddi $\leqslant 2^{-(1-\varepsilon)n}$ such that $D_2 \subset \bigcup_i B_i$ and

$$\sum_i (\text{radius } B_i)^{1-\varepsilon_0} \leqslant \delta.$$

Define n_i by $2^{-(1-\varepsilon)n_i} \leqslant \text{radius } B_i \leqslant 2^{-(1-\varepsilon)(n_i-1)}$, and let $\bar{B}_i$ be the disc centered at B_i with radius $2^{-(1-\varepsilon)(n_i-1)}$. Meanwhile, we let

$$M_0 = \{(t, s) \mid X_t = X_s\},$$

and

$$\bar{X}(t_1, t_2) = (X_{t_1}, X_{t_2}),$$

then

$$M_0 \cap [0, 1]^2 \subset \bigcup_i \bar{X}^{-1}(\bar{B}_i^2) \cap [0, 1]^2$$

$$\subset \bigcup_i \{\text{the union of } 2^{4\varepsilon n_i} \text{ cubes of the form: } \prod_{i=1}^{2} [k_1 4^{-n_i+1}, (k_1+1) 4^{-n_i+1}]\}.$$

Let I_{ij}, $1 \leqslant j \leqslant 2^{4\varepsilon n_i}$, denote these cubes. By (3.7) we have for large n_i

$$\begin{aligned}\sum_{i,j} [e(I_{it})]^{1/2-\varepsilon_1/2} &\leqslant \sum_i 2^{4\varepsilon n_i} 4^{-(1/2-\varepsilon_1/2)(n_i-1)} \\ &= 4^{1/2-\varepsilon_1/2} \sum_i 2^{-(1-\varepsilon_1-4\varepsilon)n_i} \\ &\leqslant 4^{1/2-\varepsilon_1/2} \sum_i 2^{-(1-\varepsilon_0)n_i} \leqslant \sum_i (\text{radius } B_i)^{1-\varepsilon_0} \\ &\leqslant \delta\end{aligned}$$

where we choose ε, $\varepsilon_1 \in (0, 1/2)$ such that

$$\varepsilon_1 - 4\varepsilon \leqslant \varepsilon_0.$$

This implies $\dim (M_0 \cap [0, 1]^2) \leqslant 1 - \varepsilon_1$.

However, [12, Theorem 5.2] tells us that

$$\dim (M_0 \cap [0, 1]^2) \geqslant 1, \ \nu(g)\text{-a.e.}$$

This contradiction shows that we must have (3.6).

§ 4. Upper Bound for dim D_2

We begin with a lemma.

Lemma 4.1. *There exists a constant $C(\varepsilon) \in (0, \infty)$ for any $\varepsilon \in (0, 1/2)$, such that*

$$\nu(g)\{|X(t)| \leqslant 2^{-(1/2-\varepsilon)m} \text{ for some } t \in [1/2, 1]\} \leqslant C(\varepsilon) 2^{-(1/2\varepsilon)m} \tag{4.1}$$

for any $m \geqslant 1$.

Proof Actually, the left hand side of (4.1)

$$=\nu(g)\Big(\bigcup_{i=2^{m-1}}^{2^m-1}\{|X(t)|\leqslant 2^{-(1/2-\varepsilon)m}\text{for some } t\in[i2^{-m},(i+1)2^{-m}].\Big)$$

However, Lemma 3.1 tells us that it is sufficient to prove that

$$\nu(g)\Big\{\bigcup_{i=2^{m-1}}^{2^m-1}|X(i2^{-m})|\leqslant 2.2^{-(1/2-\varepsilon)m}\Big\}\leqslant C(\varepsilon)2^{-(1/2-2\varepsilon)m}$$

for any $m\geqslant 1$. This is just the conclusion of Lemma 2.3.

The proof is finished.

Remark 4. 1. Similarly, we have for any fixed $s\in(0,1/2)$,

$$\nu(g)\{|X(t)-X(s)|\leqslant 2^{-(1/2-\varepsilon)m}\text{ for some } t\in[1/2,1]\}$$
$$\leqslant C(s,\varepsilon)2^{-(1/2-2\varepsilon)m},\ \forall m\geqslant 1.$$

Proof Without loss of the generality, we assume that (this is reasonable because of Lemma 3.1) $s=i\cdot 2^{-m}$ for some $i=0,1,\cdots,2^{m-1}$. Observing the proof of Lemma 2.3 and Lemma 4.1, we see that it is sufficient to show

$$\mu\Big(\bigcup_{j=2^{m-1}}^{2^{m-1}}\{|B(j2^{-m})-B(i2^{-m})|\leqslant 2.2^{-(1/2-\varepsilon)m}\}\Big)$$
$$\leqslant C(\delta,i)2^{-(1/2-\varepsilon)m(1-\delta)}. \tag{4.2}$$

Since

$$|s-i2^{-m}|<2^{-m},$$

we easily know that

$$|j2^{-m}-i2^{-m}|\geqslant C'(s)>0$$

for any $j=2^{m-1},\cdots,2^{m-1}$. Noting this property, we get (4.2) immediately from [10, Lemma 3].

Actually, from the above proof we also see that for any fixed $s\in(0,1)$ and $\varepsilon_0\in(0,1-s)$,

$$\nu(g)\{|X(t)-X(s)|\leqslant 2^{-(1/2-\varepsilon)m}\text{ for some } t\in(s+\varepsilon_0,1]\}$$
$$\leqslant C(s,\varepsilon_0,\varepsilon)2^{-(1/2-2\varepsilon)m},\ \forall m\geqslant 1. \tag{4.3}$$

Remark 4.2. It is not difficult to get the following from (4.3)

$$\nu(g)\{|X(t)-X(s)|\leqslant m^{-(1/2-\varepsilon)},\text{ for some } t\in(s+\varepsilon_0,1]\}$$
$$\leqslant C(s,\varepsilon_0,\varepsilon)m^{-(1/2-2\varepsilon)},\ \forall m\geqslant 1,$$

where $s\in(0,1)$ is any fixed point, and $\varepsilon_0\in(0,1-s)$.

We are now in a position to estimate the upper bound of dim.D_2 Let

$$\mathscr{L}(s,t,w)=\{X(u,w):s\leqslant u\leqslant t\}$$

for any $s,t\in[0,1]$ and $s<t$, and

$$Q_2(\omega)=\mathscr{L}(0,1/4,\omega)\cap\mathscr{L}(1/2,1,\omega).$$

We first prove that for almost surely $\omega\in\Omega$

$$\dim Q_2(\omega)\leqslant 1. \tag{4.4}$$

Split up the interval [0, 1/4] into m equal pieces by the time points

$$t_i=1+i/(4m)\quad(i=0,1,\cdots,m).$$

The maximum displacement of $X(t)-X(t_i)$ in $t_i \leqslant t \leqslant t_{i+1}$ will be of the order less than $m^{-(1/2+\varepsilon)}$ for any $\varepsilon \in (0, 1/2)$. More precisely, if

$$Y_{i,m}=m^{-1/2+\varepsilon} \sup_{t_i<t<t_{i+1}} |X(t)-X(t_i)|,$$

then $Y_{i,m}$, $i=0, 1, \cdots, m-1$, are random variables and by Remark 2.1

$$\nu(g)\{Y_{i,m}\geqslant L\}\leqslant CL^{-M}m^{-M}$$

for sufficient large $M\geqslant 1$. Just as in [10, § 5], we define a discrete random variable $\rho_{i,m}$ as follows:

if $Y_{i,m}\leqslant 1$, put $\rho_{i,m}=m^{-(1/2-\varepsilon)}$;

if $2^s<Y_{i,m}\leqslant 2^{s+1}$, put $\rho_{i,m}=(1+2^{s+1})m^{-(1/2-\varepsilon)}$, $s=0, 1, \cdots$.

Now we consider the points of $\mathscr{L}$ (0, 1/4, ω) which are approached within $m^{-(1/-\varepsilon)}$ by the piece $\mathscr{L}(1/2, 1, \omega)$. Clearly, if we cover this set of near$-m^{-(1/2-\varepsilon)}$ returns then we will certainly have covered $Q_2(\omega)$. If any point of $\mathscr{L}(t_i, t_{i+1}, \omega)$ is to be a point of near$-m^{-(1/2-\varepsilon)}$ return, then the piece $\mathscr{L}(1/2, 1, \omega)$ must at least enter the sphere $S_{i,m}$ with center $X(t_i)$ and radius $\rho_{i,m}$. By Remark 4.2 we know that the probability $p_{i,m}$ of a return $S_{i,m}$ in the interval [1/2, 1] satisfies

$$p_{i,m}\leqslant C\,\rho_{i,m}\cdot m^{\varepsilon}$$

Set

$$d_{i,m}(\omega)=0 \text{ if in } [1/2, 1] \text{ no return occurs;}$$

$$d_{i,m}(\omega)=2\rho_{i,m}(\omega) \text{ if the return occurs.}$$

$$l_m(\omega)=\sum_{i=0}^{m-1}[d_{i,m}(\omega)]^{1+\delta}$$

for any fixed $\delta\in(0, 1/2)$. Then the random variable $l_m(\omega)$ is the sum of the $1+\delta-$th powers of the diameters of the spheres $S_{i,m}$ which are re-entered at least one times in the intervals considered.

Obviously, by Remark 2.1

$$E_{\nu(g)}(d_{i,m}^{1+\delta})\leqslant E_{\nu(g)}[2\rho_{i,m}]^{1+\delta}$$

$$\leqslant C_2\{[m^{-1/2+\varepsilon}]^{1+\delta}[m^{-1/2+\varepsilon}]\cdot m^{\varepsilon}+\sum_{s=0}^{\infty}\nu(g)(\rho_{i,m}=(1+2^s)m^{-(1/2-\varepsilon)})$$

$$\cdot[(1+2^s)m^{-(1/2-\varepsilon)}]^{1+\delta}\cdot(1+2^s)m^{-(1/2-\varepsilon)}\cdot m^{\varepsilon}$$

$$\leqslant C_2\{m^{-(1/2-\varepsilon)}[2+\delta]\cdot m^{\varepsilon}+\sum_{s=0}^{\infty}C_1(1+2^s)^{-M}m^{-M}\cdot[(1+2^s)m^{-(1/2-\varepsilon)}]^{2+\delta}\}\cdot m^{\varepsilon}$$

$$\leqslant C_3 m^{-1-(\delta/6-\delta^2/9)},\ \forall m\geqslant 1$$

where we pick $\varepsilon=\delta/9$. Finally,

$$E_{\nu(g)}[l_m(\omega)]=\sum_{i=0}^{m-1}E_{\nu(g)}[d_{i,m}(\omega)]^{1+\delta}\leqslant C_3 m^{-(\delta/6-\delta^2/9)}.$$

It is easy to see that there is almost surely a finite real number $M(\omega)$ such that for a subsequence $m_1, m_2, \cdots$

$$l_{m_i}(\omega)\leqslant M(\omega),\ i=1, 2, \cdots, \tag{4.5}$$

On the other hand, if

$$q_m = \max_{0 \leqslant i \leqslant m-1} \rho_{i,m},$$

noting the definition of $\rho_{i,m}$ we have from Lemma 3.1

$$\nu(g)(q_m \geqslant \delta_1/2) \leqslant m \cdot C_4 m^{-M}, \ \forall m \geqslant 1$$

for some constant $C_4 \in (0, \infty)$, where $M \geqslant 1$ is a large enough constant and $\delta_2 > 0$ is any fixed constant. Then the Borel-Cantelli lemma implies that only finitely many of the events $\{q_m \geqslant \delta_1/2\}$ occur, so we may assume that all the covering spheres have diameter less than δ_2. Therefore (4.5) leads to

$$\dim Q_2(\omega) \leqslant 1 - \delta.$$

Since $\delta \in (0, 1/2)$ is arbitrary, (4.4) is correct.

By the same argument as before, we can show

$$\dim Q_2^{(s,t)}(\omega) \leqslant 1, \ \nu(g)-\text{a.e.},$$

where $s < t$, and $s, t \in (0, 1)$, and

$$Q_2^{(s,t)}(\omega) = \mathscr{L}(0, s, \omega) \cap \mathscr{L}(t, 1, \omega).$$

Thus, it is not difficult to get

$$\dim Q_2^{(\delta)}(\omega) \leqslant 1, \ \nu(g)\text{-a.e.}$$

where

$$Q_2^{(\delta)}(\omega) = \{x = X(s, \omega) = X(t, \omega) \text{ for some } s, t \in [0, 1] \text{ with} |s-t| \geqslant \delta\}.$$

Obviously, $Q_2^{(1/n)}(\omega) \uparrow D_2$ as $n \uparrow \infty$. Since $X(t, \omega)$ is continuous with respect to $t \in [0, 1]$, we easily see

$$\text{diam}\ (Q_2^{(1/n)}(\omega), D_2 \backslash Q_2^{(1/n+1)}(\omega)) > 0, \ \forall n \geqslant 1.$$

[1, Lemma 1.4] helps us to obtain

$$\dim D_2 \leqslant 1, \ \nu(g)\text{-a.e.}$$

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