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A NECESSARY AND SUFFICIENT CONDITION THAT BIHOLOMORPHIC MAPPINGS ARE STARLIKE ON A CLASS OF REINHARDT DOMAINS***

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Abstract

This paper studies the Reinhardt domains B defined as

$$B \equiv \left\{ Z=(z_1, z_2, \cdots, z_n \in C^n \,\middle|\, \sum_{i=1}^{n} |z_i|^{p_i} < 1,\ 2p_n > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 1. \right\}$$

The Schwartz lemma for B is established. Using it the authors give a necessary and sufficient condition that a local biholomorphic mapping from B to C^n is starlike. It is reduced to the Suffridge's theorem in the case $p_1 = p_2 = \cdots = p_n > 1$.

§ 1. Introduction

Some problems and topics related to biholomorphic starlike mappings have been considered in [1-5]. In particular, using the principle of subordination T. J. Suffridge has established the necessary and sufficient condition that a mapping be local biholomorphic and map the bounded domains in C^n

$$D_p \equiv \left\{ Z=(z_1, z_2, \cdots, z_n) \in C^n \,\middle|\, \sum_{i=1}^{n} |z_i|^p < 1,\ p > 1 \right\}$$

onto starlike domains in C^n.

In this paper we will deal with the following Reinhardt domains

$$B \equiv \left\{ Z=(z_1, z_2, \cdots z_n) \in C^n \,\middle|\, \sum_{i=1}^{n} |z_i|^{p_i} < 1,\ 2p_n > p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n > 1 \right\}.$$

First of all, we establish the Schwartz lemma for B which extent the Schwartz lemma for D_p in C^n[6]. The Schwartz lemma can be applied to study the biholomorphic starlike mappings instead of the principle of subordination in references [1, 2, 5]. By the way, we have a counterexample to show that if the condition $2p_n > p_1$ was dropped, then the Schwartz lemma would be not true. The remaining part of

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this paper is devoted to generalizing the Suffridge's theorem for D_p to B by using Schwartz lemma for B.

We introduce some notations in this paper.

Let $u(Z)=\sum_{i=1}^{n}|z_i|^{p_i}$. We denote the distance function from the origin in C^n by $\rho(W)$, the segment rW, $0\leqslant r\leqslant 1$, joining the origin and the point W in C^n by $\sigma(W)$. The unit disk in C will be denoted by D, the disk central at the origin with radius r by $D(r)$, if central at t by $D(t, r)$.

§ 2. A Schwartz Type Lemma

Theorem 1. *Suppose $\phi: B\to B$ is a holomorphic mapping with $\phi(0)=0$ and $J_\phi(0)=\nu I$, $0<\nu\leqslant 1$, where I is unit matrix. Then*

$$u(Z)\geqslant u(\phi(Z))$$

holds for all $Z\in B$.

At frist, we prove a lemma.

For a fixed $Z\in B\setminus\{0\}$, $u(Z)<1$, we can choose two positive real numbers r_0, r and a system of rational numbers $\frac{l_i}{m_i}\leqslant p_i$, l_i and m_i are mutually primitive integers for $i=1, 2, \cdots, n$. They satisfy

$$\frac{l_1}{m_1}\geqslant\frac{l_2}{m_2}\geqslant\cdots\geqslant\frac{l_n}{m_n}>1,\quad 2\frac{l_n}{m_n}>\frac{l_1}{m_1}, \tag{1}$$

$$r_0|z_i|^{\frac{l_i}{m_i}}\leqslant r|z_i|^{p_i},\ u(Z)<r_0<r<1,$$

if $\frac{l_i}{m_i}$ are sufficiently close to p_i for all $i=1, 2, \cdots, n$. Fix $\frac{l_i}{m_i}$ and take

$$Y(t)=ZTU,\quad Y^*(t)=Z\bar{T}U \tag{2}$$

where

$$T=\mathrm{diag}((r_0t^L)^{\frac{m_1}{l_1}}, (r_0t^L)^{\frac{m_2}{l_2}}, \cdots, (r_0t^L)^{\frac{m_n}{l_n}}),$$

$$U=\mathrm{diag}(u^{-\frac{m_1}{l_1}}(Z), u^{-\frac{m_2}{l_2}}(Z), \cdots, u^{-\frac{m_n}{l_n}}(Z)),$$

$$L=l_1l_2\cdots l_n,\ t\in D(r_1),\ r_1=\left(\frac{1}{r}\right)^{\frac{1}{L}}.$$

Since

$$\sum_{i=1}^{n}|y_i|^{\frac{l_i}{m_i}}=r_0|t|^L\frac{\sum_{i=1}^{n}|z_i|^{\frac{l_i}{m_i}}}{u(Z)}=r|t|^L<1,$$

$$\sum_{i=1}^{n}|y_i^*|^{\frac{l_i}{m_i}}=r_0|t|^L\frac{\sum_{i=1}^{n}|z_i|^{\frac{l_i}{m_i}}}{u(Z)}=r|t|^L<1.$$

Further, because of $\frac{l_i}{m_i}\leqslant q_i$

$$\sum_{i=1}^{n} |y_i|^{p_i} \leqslant \sum_{i=1}^{n} |y_i|^{\frac{l_i}{m_i}} \leqslant 1,$$
$$\sum_{i=1}^{n} |y_i^*|^{p_i} \leqslant \sum_{i=1}^{n} |y_i^*|^{\frac{l_i}{m_i}} \leqslant 1. \tag{3}$$

In light of (3), (2) defines a holomorphic mapping and an antiholomorphic mapping from $D(r_1)$ to B. We have

Lemma 1. *Suppose $\phi=(\phi_1, \phi_2, \cdots, \phi_n)$ is a holomorphic mapping from B to B with $\phi(0)=0$. If $1+\mu<r_1$ then the sets*

$$N_i=\{t\in D(1+\mu) \mid \phi_i\circ Y(t)=0 \text{ or } \overline{\phi}_i\circ Y^*(t)=0\},$$
$$N=\bigcup_{i=1}^{n} N_i,$$

are all finite. Or $\phi\circ Y(t)\equiv 0$, $\overline{\phi}\circ Y^*(t)\equiv 0$.

Proof Otherwise, there is at least a zero of $\phi_i(ZTU)$, or $\overline{\phi}_i(Z\overline{T}U)$, which is not isolated in $D(r_1)$. It is not possible because of the fact that the combinational mappings $\phi_i\circ Y(t)$ and $\overline{\phi}_i\circ Y^*(t)$ are both holomorphic functions in $D(r_1)$.

Call the finite zeros in N_i by $t_j^{(i)}$, $j=1, 2, \cdots, n_i$.

Consider the following n complex functions of one complex variable t

$$\mathscr{A}^i(t)=\phi_i^{\frac{l_i}{2m_i}}(ZTU)\overline{\phi}_i^{\frac{l_i}{2m_i}}(Z\overline{T}U),$$
$$0\leqslant\arg\phi_i(ZTU),\ \arg\overline{\phi}_i(Z\overline{T}U)<2\pi,\ i=1, 2, \cdots, n.$$

Obviously, $\mathscr{A}^i(t)$, $i=1, 2, \cdots, n$, are holomorphic function of t in $D(1+\mu)\setminus\{N\}$.

Now let us prove Theorem 1.

For a fixed $Z\in B\setminus\{0\}$, suppose $\mathscr{A}^i(t)$ is not identically vanishing. Take a sufficient small $\delta>0$ satisfying

$$\delta<\frac{1}{2}\min\{|t_1-t_2|,\ \frac{1}{2}\operatorname{dist}(t_3, \partial D(1+\mu)), \forall t_1, t_2, t_3\in N\cup\{u^{1/L}(Z)\}\}. \tag{4}$$

Now construct a system of C^∞ functions $f^{(j)}$ in $D(r_1)$ satisfying the following conditions:

1. $f^{(i)}(\eta)\in[0, 1]$, if $\eta\in D(r_1)$,

2. $$f^{(i)}(\eta)=\begin{cases}0, & \text{if } \eta\in\bigcup_{t\in N_i} D\left(t, \frac{1}{2}\delta\right),\\ 1, & \text{if } \eta\in D(r_1)\setminus\{\bigcup_{t\in N_i}\overline{D(t, \delta)}\}.\end{cases} \tag{5}$$

We know the gradients of the functions obey[7]

$$|\nabla f^{(i)}(\eta)|\leqslant\frac{C_i}{\delta},\quad i=1, 2, \cdots, n, \tag{6}$$

where C_i's are some constants only depending on the given Z. Let

$$\mathscr{B}^{(i)}(t)=\frac{1}{2\pi i}\int_{D(1+\mu)}\frac{1}{\eta-t}\cdot\frac{\partial(f^{(i)}(\eta)\mathscr{A}^i(\eta))}{\partial\overline{\eta}}\,d\eta\wedge d\overline{\eta},\ t\in D(1+\mu). \tag{7}$$

Recalling the definition of $f^{(i)}(\eta)$ and the analytic property of $\mathscr{A}^i(t)$, we have

$$2\pi|\mathscr{B}^{(i)}(t)|=\left|\int_{D(1+\mu)}\frac{1}{\eta-t}\circ\frac{\partial(f^{(i)}(\eta)\mathscr{A}^{i}(\eta))}{\partial\bar{\eta}}\,d\eta\wedge d\bar{\eta}\right|$$
$$=\left|\sum_{j=1}^{u_i}\int_{E_j}\frac{\mathscr{A}^{i}(\eta)}{\eta-t}\circ\frac{\partial(f^{(i)}(\eta))}{\partial\bar{\eta}}\,d\eta\wedge d\bar{\eta}\right|$$
$$\leqslant\sum_{j=1}^{u_i}\int_{E_j}\frac{|\mathscr{A}^{i}(\eta)|}{|\eta-t|}\circ\left|\frac{\partial(f^{(i)}(\eta))}{\partial\bar{\eta}}\right|d\eta\wedge d\bar{\eta}, \tag{8}$$

where

$$E_j=\{D(t_j^{(i)},\delta)\}\setminus\left\{D\left(t_j^{(i)},\frac{1}{2}\delta\right)\right\}.$$

Substituting (6) into (8), we have by mean value theorem

$$2\pi|\mathscr{B}^{(i)}(t)|\leqslant\sum_{j=1}^{u_i}|\mathscr{A}^{i}(\eta_j^{(i)})|\int_{E_j}\frac{1}{|\eta-t|}\cdot\left|\frac{\partial(f^{(i)}(\eta))}{\partial\bar{\eta}}\right|d\eta\wedge d\bar{\eta}$$
$$\leqslant\sum_{j=1}^{u_i}\frac{C_i}{\delta}|\mathscr{A}^{i}(\eta_j^{(i)})|\left(\int_{E_j\cap|\eta-t|>\delta}\frac{1}{|\eta-t|}d\eta\wedge d\bar{\eta}+\int_{E_j\cap D(t,\delta)}\frac{1}{|\eta-t|}d\eta\wedge d\bar{\eta}\right)$$
$$\leqslant\sum_{j=1}^{u_i}\frac{C_i}{\delta}|\mathscr{A}^{i}(\eta_j^{(i)})|\left(\frac{1}{\delta}\int_{\frac{1}{2}\delta<|\eta-t_j^{(i)}|<\delta}d\eta\wedge d\bar{\eta}+\int_{|\eta-t|<\delta}\frac{1}{|\eta-t|}d\eta\wedge d\bar{\eta}\right)$$
$$=\sum_{j=1}^{u_i}\frac{C_i}{\delta}|\mathscr{A}^{i}(\eta_j^{(i)})|\left(\frac{3}{4}\delta\pi+2\pi\delta\right)$$
$$=\sum_{j=1}^{u_i}\frac{11C_i\pi}{4}|\mathscr{A}^{i}(\eta_j^{(i)})|,$$

where $\eta_j^{(i)}$ satisfy $\frac{1}{2}\delta<|\eta_j^{(i)}-t_j^{(i)}|<\delta$. Thus we obtain

$$|\mathscr{B}^{(i)}(t)|\leqslant\mathscr{K}\sum_{j=1}^{u_i}|\mathscr{A}^{i}(\eta_j^{(i)})|,$$
$$\sum_{j=1}^{n}|\mathscr{B}^{(i)}(t)|\leqslant\mathscr{K}\mathscr{M}, \tag{9}$$
$$\mathscr{M}=\sum_{i=1}^{u}\sum_{j=1}^{u_i}|\mathscr{A}^{i}(\eta_j^{(i)})|,$$

where $\mathscr{K}=\max\{11/8\cdot C_i,\ i=1,2,\cdots,n\}$ only depends on the given Z. From (7) and by using the Theorem 1.1.3 in [8]

$$\bar{\partial}\mathscr{B}^{(i)}(t)=\bar{\partial}(f^{(i)}(t)\mathscr{A}^{i}(t))$$

in $t\in D(1+\mu)$. This implies for a given $s>0$,

$$C(t)=\sum_{i=1}^{n}\frac{\mathscr{B}^{(i)}(t)-f^{(i)}(t)\mathscr{A}^{i}(t)}{t^J}$$

is a holomorphic function of t in $D(1+\mu)\setminus D(s)$, $u^{1/L}>s>0$. Now let us estimate the maximum absolute value on the boundaries $\partial D(1+\mu)$ and $\partial D(s)$. Since

$$|C(t)|\leqslant\sum_{i=1}^{n}(|\mathscr{B}^{(i)}(t)|+|f^{(i)}(t)\mathscr{A}^{i}(t)|)\cdot\frac{1}{|t|^L}, \tag{10}$$

we obtain

$$s^L|C(t)|_{t\in\partial D(s)}\leqslant\mathscr{K}\mathscr{M}+\sum_{i=1}^{n}|\mathscr{A}^{i}(t)|_{t\in\partial D(s)} \tag{11}$$

by (9) and $f^{(i)}\leqslant1$ when $t\in\partial D(s)$, for any $i=1,2,\cdots,n$. Similarly, when $t\in\partial D(1+\mu)$, $f^{(i)}(t)=1$, we have

$$\sum_{i=1}^{n}|f^{(i)}(t)\mathscr{A}^{i}(t)|=\sum_{i=1}^{n}|\mathscr{A}^{i}(t)|$$
$$=\sum_{i=1}^{n}|\phi_i^{\frac{l_i}{2m_i}}(ZTU)\overline{\phi}_i^{\frac{l_i}{2m_i}}(Z\overline{T}U)|$$
$$\leqslant\sqrt{\sum_{i=1}^{n}|\phi_i(ZTU)|^{\frac{l_i}{m_i}}\sum_{i=1}^{n}|\overline{\phi_i}(Z\overline{T}G)|^{\frac{l_i}{m_i}}}.$$

Therefore, when $\frac{l_i}{m_i}$ is sufficiently close to p_i, for all $i=1, 2, \cdots, n$,

$$|\mathscr{C}(t)|_{t\in\partial D(1+\mu)}\leqslant 1+\mathscr{K}\mathscr{M}. \tag{12}$$

Based upon the maximal principle of analytic functions, we can get

$$\left|\sum_{i=1}^{n}[\mathscr{B}^{(i)}(t)-f^{(i)}(t)\mathscr{A}^{i}(t)]\right|\leqslant|t|^{L}\max\left(1+\mathscr{K}\mathscr{M}, \frac{\left(\mathscr{K}\mathscr{M}+\sum_{i=1}^{n}|\mathscr{A}^{i}(t)|_{t\in\partial D(s)}\right)}{s^{L}}\right). \tag{13}$$

In particular, we take $t=u^{1/L}(Z)$. Then (13) becomes

$$\left|\sum_{i=1}^{n}[\mathscr{B}^{(i)}(u^{1/L}(Z))-f^{(i)}(u^{1/L}(Z))\mathscr{A}^{i}(u^{1/L}(Z))]\right|$$
$$\leqslant u(Z)\max\left(1+\mathscr{K}\mathscr{M}, \frac{\left(\mathscr{K}\mathscr{M}+\sum_{i=1}^{n}|\mathscr{A}^{i}(t)|_{t\in\partial D(s)}\right)}{s^{L}}\right). \tag{14}$$

Recalling the definitions of $f^{(i)}(t)$ and $\mathscr{A}^{i}(t)$, we have

$$f^{(i)}(u^{1/L}(Z))=1,\ |\mathscr{A}^{i}(u^{1/L}(Z))|=|\phi_i(ZR)|^{\frac{l_i}{m_i}}, \tag{15}$$

where $R=\operatorname{diag}\{r_0^{\frac{m_1}{l_1}}, r_0^{\frac{m_2}{l_2}}, \cdots, r_0^{\frac{m_n}{l_n}}\}$. Therefore

$$\sum_{i=1}^{n}|f^{(i)}(u^{1/L}(Z))\mathscr{A}^{i}(u^{1/L}(Z))|=\sum_{j=1}^{n}|\phi_i(ZR)|^{\frac{l_i}{m_i}}. \tag{16}$$

Thus we obtain

$$\sum_{i=1}^{n}|\phi_i(ZR)|^{\frac{l_i}{m_i}}\leqslant\sum_{i=1}^{n}|\mathscr{B}^{(i)}(u^{\frac{1}{L}}(Z))|+u(Z)\max\left(1+\mathscr{K}\mathscr{M}, \frac{\left(\mathscr{K}\mathscr{M}+\sum_{j=1}^{n}|\mathscr{A}^{i}(t)|_{t=|s|}\right)}{s^{L}}.\right) \tag{17}$$

Here we suppose $u^{1/L}(Z)$ is not a zero point of some $\phi_i(ZTU)$. If $u^{1/L}(Z)$ is a zero point of some $\phi_i(ZTU)$, (14) still holds. At the beginning of the proof we suppose $\mathscr{A}^{i}(t)$ does not identically vanish. If $\mathscr{A}^{i}(t)\equiv 0$ it is not necessary to construct the function $\mathscr{B}^{(i)}(t)$, and (17) still holds.

Now we let $\delta\to 0$. Then $\eta_j^{(i)}\to t_j^{(i)}$. Since $\mathscr{A}^{i}(t)$ is continuous with respect to t, it follows that

$$\mathscr{A}^{i}(\eta_j^{(i)})\to\mathscr{A}^{i}(t_j^{(i)})=0.$$

From (9) $\mathscr{M}\to 0$ and $\mathscr{B}^{(i)}(u^{1/L}(Z))\to 0$. Therefore (17) becomes

$$\sum_{i=1}^{n}|\phi_i(ZR)|^{\frac{l_i}{m_i}}\leqslant u(Z)\max\left(1, \frac{\sum_{i=1}^{n}|\mathscr{A}^{i}(t)|_{t\in\partial D(s)}}{s^{L}}\right). \tag{18}$$

The relation holds for any fixed s, $u^{1/L}(Z)>s>0$. It is easily seen that

$$|\phi_i(ZTU)|_{t\in\partial D(s)}$$
$$=s^{L\frac{m_i}{l_i}}\left|\left(\nu z_i\left(\frac{r_0}{u(Z)}\right)^{\frac{m_i}{l_i}}+\sum_{j,k=1}^{n}s^{L\left(\frac{m_j}{l_j}+\frac{m_k}{l_k}-\frac{m_i}{l_i}\right)}a_{jk}z_jz_k\left(\frac{r_0}{u(Z)}\right)^{\frac{m_j}{l_j}+\frac{m_k}{l_k}+\cdots}\right)\right|$$
$$\leqslant s^{L\frac{m_i}{l_i}}\left(\left(\frac{r_0}{u(Z)}\right)^{\frac{m_i}{l_i}}\nu|z_i|+O(s^{L(\frac{m_j}{l_j}+\frac{m_k}{l_k}-\frac{m_i}{l_i})}\right).$$

In condition (1) we have $\frac{m_j}{l_j}+\frac{m_k}{l_k}-\frac{m_i}{l_i}>0$. Thus

$$\lim_{s\to 0}\sum_{i=1}^{n}\frac{|\mathscr{A}^i(t)|_{t\in\partial D(s)}}{s^L}\leqslant 1.$$

We conclude

$$\sum_{i=1}^{n}|\phi_i(ZR)^{\frac{l_i}{m_i}}|\leqslant u(Z).$$

Let $\frac{l_i}{m_i}\to p_i$ for all $i=1, 2, \cdots, n$. Then $r_0\to 1$. This completes the proof of the theorem.

Remark. If the condition $2p_n>p_1$ is dropped then Theorem 1 is not true. To explain it we give an example.

We take $n=2$, $B=\{(z_1, z_2)\in C^n \mid |z_1|^5+|z_1|^2<1,\}$ and $\phi(z_1, z_2)=(sz_1, sz_2+(1-2s)z_0^2)$, where s is a sufficiently small positive number. Then $\phi(0)=0$, $J_\phi(0)=sI$, $s>0$, and $u(\phi(Z))=|\phi_1|^5+|\phi_2|^2=s^5|z_1|^5+|sz_2+(1-2s)z_1^2|^2\leqslant s^5|z_1|^5+(s|z_2|+(1-2s)|z_1|^2)^2$. On $|z_1|=1$, $|z_2|=1$, the right hand side of the previous inequality is not greater than $s^5+(1-2s)^2=1-2s+s^2+s^5<1$. Thus $u(\phi(Z))<1$ when $|z_1|=1$, $|z_2|=1$. But $B\subset\{(z_1, z_2)\in C^2, \mid |z_1|\leqslant 1,\ |z_2|\leqslant 1\}$. Therefore ϕ: $B\to B$ is an into mapping. On the other hand, if we take $z_2=0$ then

$$|\phi_1(z_1, 0)|^5+|\phi_2(z_1, 0)|^2=s^5|z_1|^5+(1-2s)^2|z_1|^4\leqslant|z_1|^5=u(z_1, 0)$$

is not true when we take $|z_1|=1-5s$ and s is sufficiently small. So the Schwartz lemma fails if the condition $2p_n>p_1$ is dropped.

This counterexample was given by Professor Carl H. FitzGerald.

§ 3. The Necessary Conditon for Biholomorphic Mapping to be Starlike in B

Now we can use the Schwartz lemma for B in the above section to give the necessary conditon for a biholomorphic mapping to be starlike in B.

Theorem 2. *Suppose f: $B\to C^n$ is a starlike biholomorphic mapping. Then*

$$\langle du\cdot f^{-1}, d\rho\rangle|_{W=f(Z)}\geqslant 0$$

holds for any $Z\in B^p\setminus\{0\}$, where $\langle\ ,\ \rangle$ is the inner product in C^n.

Proof Let $f(B)$ denote the image of B under f. For a fixed $Z\in B$, a subset of B is defined as

$$\varepsilon_a = \{Y \in B \mid u(Y) < u(Z) = a\}.$$

Obviously, ε_a or $\varepsilon_{u(Z)}$ is an open set. So is $f(\varepsilon_a)$ because of the open mapping theorem and $\{f(\overline{\varepsilon_a})\} = \overline{\{f(\varepsilon_a)\}}$. By the starlike hypothesis for the mapping f, the segment $\sigma(W) = rW$, $0 < r < 1$, joining the origin and the point $W = f(Z)$, is in $f(B)$, i. e., $rW \in \{f(B)\}$ and $f^{-1}(rW) \in B$, for all $0 \leqslant r \leqslant 1$. We claim that if

$$rW \in f(\bar{\varepsilon}_a), \tag{19}$$

then the directional derivative of u along the direction $d\rho$

$$\langle du \cdot f^{-1}, d\rho \rangle \big|_{W=f(Z)} = \frac{\partial u}{\partial \rho} \geqslant 0.$$

The theorem holds. To show that (19) holds we suppose there is such an $r_0 < 1$ that $r_0 W \notin \{f(\overline{\varepsilon_a})\}$, i. e., $u(f^{-1}(r_0 W)) > a$. Define a new mapping $K(Z)$ from B to B by $K(Z) = f^{-1}(r_0 f(Z))$. By the hypothesis for f, $K(Z)$ is holomorphic with $K(0) = 0$ and $J_K(0) = r_0 I$. Using the previous Theorem 1 we have

$$u(Z) \geqslant u(K(Z)) = u(f^{-1}(r_0 W)) > a. \tag{20}$$

Then (20) contradicts $u(f^{-1}(W)) = a$. Thus (19) is true.

§ 4. The Sufficient Condition for Biholomorphic Mapping to be Starlike in B

In this section we give the sufficient condition for a biholomorphic mapping to be starlike in B

Theorem 3. *Suppose f: $B \to C^n$ is a holomorphic immersion with $f(0) = 0$ and*

$$\langle du \cdot f^{-1}, d\rho \rangle \big|_{W=f(Z)} \geqslant 0$$

holds for any $Z \in B$. The f is biholomorphic and starlike with respect to the origin in B.

To prove this theorem we need two lemmas. We denote

$$\varepsilon_a = \{Z \in B \mid u(Z) < a,\ a > 0\}.$$

Lemma 2. *Suppose f: $B \to C^n$ is a holomorphic immersion with $f(0) = 0$. Let*

$$\langle du \cdot f^{-1}, d\rho \rangle \big|_{W=f(Z)} \geqslant 0$$

hold for any $Z \in B$. If f is biholomorphic on ε_a then $f(\bar{\varepsilon}_a)$ is starlike with respect to the origin in C^n.

Proof We observe $\bar{\varepsilon}_a$ is a close set. Obviously, the image set $f(\bar{\varepsilon}_a)$ is also closed under the holomorphic mapping. Thus, for a given $Z \in \bar{\varepsilon}_a$, the intersection of the two closed sets, the segment $\sigma(f(Z))$ and $f(\bar{\varepsilon}_a)$, is also closed. Call it $r(Z)$.

Secondly we show that for the given $Z \in \bar{\varepsilon}_a$, there exists a $\delta_1 > 0$ such that

$$(1-t) f(Z) \in f(\bar{\varepsilon}_a) \tag{21}$$

holds for $\delta_1 > t > 0$. If it is not true, we can find such a real sequence $\{t_i\}_{i=1}^{\infty}$ which satisfies $\lim\limits_{i \to \infty} \{t_i\} = 0$ and $(1 - t_i) f(Z) \notin f(\bar{\varepsilon}_a)$. Let $B(f(Z), \delta)$ be the open ball centred at $f(Z)$ with radius δ and $B(f(Z), \delta)\subset\{f(U_z)\}$. When i is sufficiently large.

$$(1-t_i)f(Z)\in B(f(Z), \delta).$$

Hence a point Z_i exists in the neighbourhood U_z such that $f(Z_i)=(1-t_i)f(Z)$ due to the biholomorphic condition of f in U_Z. But $f(Z_i)\notin f(\bar{s}_a)$, hence $u(Z_i)>a\geqslant u(Z)$. However, by the assumption of $\langle du\cdot f^{-1}, d\rho\rangle|_{W=f(Z)}\geqslant 0$, we can conclude when i becomes sufficient large $u(Z)\geqslant u(Z_i)$. So (21) is true.

Finally, for the given $Z\in\bar{s}_a$ we prove $\sigma(f(Z))$ falls in $f(\bar{s}_a)$. For that it is sufficient to explain that the coset $\nu(Z)=\sigma(f(Z))\setminus r(Z)$ is an empty set. In fact, we know the coset $\nu(Z)=\sigma(f(Z))\setminus r(Z)=\{\sigma(f(Z))\setminus\{0, W\}\}\setminus r(Z)$ is an open set. If $\nu(Z)$ is not empty, we can assume $t^*=\inf\{t\in[0, 1]\,|\,(1-t)f(Z)\in\nu(Z)\}$. Since $\nu(Z)$, as a subset of the segment $\sigma(f(Z))$, is open, the point $Q(t^*)=(1-t^*)\cdot f(Z)\notin\nu(Z)$, i. e., $Q(t^*)=(1-t^*)f(Z)\in f(\bar{s}_a)$. In the second step of the proceeding proof we have already shown there is an $\delta_1>0$ such that

$$(1-t')(1-t^*)f(Z)\in f(\bar{s}_a),$$

$$(1-(t^*+t'-t't^*))f(Z)\in f(\bar{s}_a) \tag{22}$$

holds for any $\delta_1>t'>0$. However, the argument is contary to the definition of infimum. Thus $\nu(Z)$ is empty. The lemma is true.

Lemma 3 *Suppose* f: $B^p\to C^n$ is a holomorphic immersion with $f(0)=0$. Let

$$\langle du\cdot f^{-1}, d\rho\rangle|_{w=f(Z)}\geqslant 0$$

hold for any $Z\in B$. *Then the biholomorphic property of* f *on* s_a *can be extended to* $\bar{s}_a$.

Proof If the statement is not true then there are two distinct points X, $Y\in\bar{s}_a$ such that $f(X)=f(Y)$. By lemma 2, for all $0\leqslant r\leqslant 1$

$$rf(X) \text{ or } rf(Y)\in f(\bar{s}_a). \tag{23}$$

Because f is holomorphic immersion, we can obtain the curve $X(r)$ with $X(1)=X$ in B such that $f(X(r))=rf(X)$ by the method of analytic continuation. That is, $X(r)=f^{-1}(rf(X))$ is a univalent component of the inverse images ofthe segment rf(X). Since

$$\frac{du(X(r))}{dr}=\frac{1}{r}\langle du\cdot f^{-1}, d\rho\rangle|_{w=rf(X)}\geqslant 0, \tag{24}$$

for $0\leqslant r\leqslant 1$

$$u(X(r))\leqslant u(X(1))=u(X)=a.$$

We have

$$X(r)\in\bar{s}_a. \tag{25}$$

Suppose $Y(r)$ is the univalent component of the inverse images of the segment $rf(X)$, but $Y(1)=Y$. A similar argument shows

$$Y(r)\in\bar{s}_a. \tag{26}$$

Let

$$\mathscr{R}=\{r\in[0, 1]\,|\,X(r)=Y(r)\}.$$

If $\mathscr{R}$ is nonempty then the supremum r^* of $\mathscr{R}$ exists. $\mathscr{R}$ is a closed set. So $X(r^*)=Y(r^*)$, $r^*<1$ because of $X(1)\neq Y(1)$, and $X(r^*+\varepsilon)\neq Y(r^*+\varepsilon)$ for $r^*<r^*+\varepsilon\leqslant 1$. But $f(X(r^*+\varepsilon))=f(Y(r^*+\varepsilon))$. This is contrary to f being biholomorphic at $X(r^*)$. If $\mathscr{R}$ is empty then $X(0)\neq Y(0)$. Since f is biholomorphic in ε_a, at least one of the two points $X(0)$ and $Y(0)$ must be a boundary point of $\bar{\varepsilon}_a$. Suppose $X(0)\in\partial\bar{\varepsilon}_a$. Let $B(X(0), \delta)$ be the open ball centred at $X(0)$ with radius δ so small that $B(X(0), \delta)\cap\mathscr{U}_0$ is empty, where $\mathscr{U}_0\subset\varepsilon_a$ is the neighborhood of the origin in B such that f is biholomrophic. Because of the open mapping theorem $f(B(X(0), \delta)\cap\varepsilon_a)$ is an open set, $f(\mathscr{U}_0)$ an open set including the origin of C^n and the origin is also a boundary point of the open set $f(B(X(0), \delta)\cap\varepsilon_a)$. So $f(B(X(0), \delta)\cap\varepsilon_a)\cap f(\mathscr{U}_0)$ is not empty. This implies for any $W\in f(B(X(0), \delta)\cap\varepsilon_a)\cap f(\mathscr{U}_0)$ it has two distinguished inverse images in ε_a. It is not possible due to the fact that f is biholomorphic in ε_a. The lemma is true.

Now we can prove Theorem 3. Let

$$\Omega=\{a\in(0, 1]\,|\quad f(Z) \text{ is biholomorphic in } \varepsilon_a\}.$$

First we prove Ω is nonempty. The hypothesis about f indicates that there is the neighbourhood of the origin in B (still call it $\mathscr{U}_0$) such that f is biholomorphic in the neighbourhood. We claim that there is a positive $\delta>0$ such that $\varepsilon_\delta\subset\mathscr{U}_0$. If it is not true, then we can choose a sequence of points $\{Z^{(i)}\}_{i=1}^{\infty}$ in B such that

$$u(Z^{(i)})<\frac{1}{i},\quad Z^{(i)}\notin\mathscr{U}_0.$$

Since $u(Z)$ is continuous with respect to Z, it follows that $\lim_{i\to\infty}u(Z^{(i)})=u(Z^*)=0$. $u(Z^*)=0$ implies Z^* is zero. Thus $\lim_{i\to\infty}Z^{(i)}=0$. This is contrary to the fact that $Z^*=0$. is an inner point of the open set $\mathscr{U}_0$.

Next we show Ω is a closed set. If $0<a_1\in\Omega$ then all $a\leqslant a_1$ fall in Ω. Therefore it is sufficient only to prove that if $a^*>a$ and all a fall in Ω then a^* is also in Ω. That is to prove f is biholomorphic in ε_{a^*}.

Assume f is not biholomorphic on ε_{a^*}, then there exists two distinct points X, Y in ε_{a^*} such that $f(X)=f(Y)$. As $u(X)<a^*$, $u(Y)<a^*$, we can find an a^{**} which satisfies $u(X)<a^{**}<a^*$, $u(Y)<a^{**}<a^*$. The formulas above imply $X, Y\in\varepsilon_{a^{**}}$. But $a^{**}\in\Omega$, $f|\varepsilon_{a^{**}}$ is biholomorphic. Thus $f(X)-f(Y)\neq0$ contradicts the assumption of $f(X)=f(Y)$. This proves Ω is a closed set.

Finally, we prove the set Ω is also an open set. For that we only to verify that if f is biholomorphic in ε_a then there is $\varepsilon>0$ such that f is also biholomorphic in $\varepsilon_{a+\varepsilon}$. If it is not so then for any $\varepsilon_n<0$ and $\lim_{u\to\infty}\varepsilon_n=0$ there are two sequences $X(n)$ and $Y(n)$ in B such that they satisfy

$$X(n) \neq Y(n), \text{ for all } n=1, 2, \cdots,$$
$$f(X(n)) = f(Y(n)), \tag{27}$$
$$\lim_{n\to\infty} u(X(n)) = \lim_{n\to\infty} u(Y(n)) = a.$$

It is easily seen that $X(n)$, $Y(n)$ are boundary sequences. So there are the convergence subsequences $X(n_k)$, $Y(n_k)$ of $X(n)$, $Y(n)$. We have

$$\lim_{k\to\infty} X(n_k) = X, \lim_{k\to\infty} Y(n_k) = Y, \tag{28}$$

and (27) gives $f(X) = f(Y)$. It is clear that X, $Y \in \partial\varepsilon_a$. If $X \neq Y$ then that is contrary to Lemma 3. If $X = Y$, (27) and (23) show f is not biholomorphic at $X \in B$. Therefore, there is a positive ε such that f is biholomorphic in $\varepsilon_{a+\varepsilon}$.

The proceeding second step proof also implies Ω is a connect set. Since Ω is open, closed and nonempty, $\Omega = [0, 1]$. That is, f is a biholomorphic mapping in B. By Iemma 1 $f(B)$ must be starlike.

Combining Theorem 2 with Theorem 3, we obtain the Suffridge-type theorem.

Theorem 4. *Suppose f: $B \to C^n$ is a holomorphic immersion mapping with $f(0)=0$. Then f is starlike if and inly if*

$$\langle du \cdot f^{-1}, d\rho \rangle |_{W=f(Z)} \geqslant 0, \quad \textit{for any } Z \in B,$$

where $\langle , \rangle$ is the inner product in C^n, $u(Z) = \sum_{i=1}^{n} |z_i|^{p_i}$ and $\rho(W)$ is the distance function from the origin in C^n.

It is reduced to the Suffridge's theorem for D_p if $p_1 = p_2 = \cdots = p_n > 1$.

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