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FINITE GROUPS WITH SCHMIDT GROUP AS AUTOMORPHISM GROUP

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Abstract

This paper continues the work of D. MacHale, D. Flannery (Proc. R. Ir. Acad. 81A, 209—215; 83A, 189—196) and the author (Proc. R. Ir. Acad. 90A, 57—62; J. Southwest China Normal University 15, No. 1, 21—28) on the topic on "Finite groups with given Automorphism group". The following result is proved:

Let G be a finite group with Aut G a Schmidt group. Then G is isomorphic to S_3 or Klain 4-group., or D such that Aut D=Inn D.

D is aSchmidt group of order $2^\alpha p$. $S_2(\in \mathrm{Syl}_2 D)$ is a normal and special group except a supersperspecial group without commutative generators.

In this paper we shall prove the following

Theorem. *Let G be a finite group with* Aut$G$ *a Schmidt group. Then G is isomorphic to S_3 or Klain 4-group or D such that Aut D=Inn D.*

Lemma 1. *Let G be a Schmidt group. Then following hold:*

1) $|G|=p^\alpha q^\beta$, *where p, q are distinct primes.*

2) *G has a normal Sylow subgroup and a cyclic Sylow subgroup, say,* $S_q \lhd G$, $S_p=\langle a\rangle$.

3) *Let N be a maximal normal subgroup of G contained in S_q. Then* $N=\phi(S_q)=S_q'$ *and* $|S_q:N|=q^b$, *where b is the order of q mod p.*

4) *Let $c\in S_q$. Then c is one generator of S_q if and only if* $[c, a]\neq 1$.

5) *If S_q is a non-Abelian group, then $N=Z(S_q)$ and N is an elementary Abelian group. If $q\neq 2$, then the exponent of S_q is q. If $q=2$, then the exponent of S_q is 4.*

6) *If S_q is an Abelian group, then S_q is an elementary Abelian group.*

7) $Z(G)=\phi(G)=\phi(S_p)\times\phi(S_q)$ (see [1]).

Lemma 2. *Sappose that Abelian group G has type*

$$(\overbrace{p^{m_1}, \cdots, p^{m_1}}^{S_1}, \overbrace{p^{m_2}, \cdots, p^{m_2}}^{S_2}, \cdots, \overbrace{p^{m_t}, \cdots, p^{m_t}}^{S_t}),\ m_1>m_2>\cdots>m_t.$$

Then $|\mathrm{Aut}G|=p^u\prod_{i=1}^{t}\prod_{k=1}^{S_i}(p^k-1)$, *where* $u=\sum_{i,j=1}^{t}S_iS_jm_{ij}-\sum_{i=1}^{t}\frac{S_i(S_i+1)}{2}$ *and* $m_{ij}=m_{\max\{i,j\}}$ (see [2]).

Lemma 3. *All Schmidt groups except S_3 and D have an outer automorphism.*

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Proof Let G be a Schmidt group. Then by Lemma 1 we have $G=S_q\rtimes S_p$ with $|S_p|=p^\alpha$ and $|S_q|=q^\beta$.

1) If $\alpha>1$, then $\phi(S_p)\neq 1$. Since $\phi(S_p)\leqslant Z(G)$, there exists $z\in Z(G)$ with $|z|=p$. Moreover there is a homomorphism f from G to $\langle z\rangle$ such that $Z(G)\leqslant \operatorname{Ker} f$. Define $\bar{\sigma}(g)=f(g)g$ for any $g\in G$. It is easy to show that $\bar{\sigma}$ is a central automorphism. Since $Z(G/Z(G))=1$, G has no inner automorphism, which is a central automorphism. Then $\bar{\sigma}$ is outher automorphism of G.

2) If $\alpha=1$ and $q\geqslant 3$, then by Lemma 1 the exponent of S_q is q. Since $G/Z(G)$ is a minimal non-nilpotent group, we have

$$G/Z(G)=\langle a,\ c_1,\ c_2,\ \cdots,\ c_b \mid a^p=c_1^q=c_2^q=\cdots=c_b^q=1,\ [c_k,\ c_l]=1, a_i^a=c_{i+1},$$
$$1\leqslant k,\ l\leqslant b,\ 1<i\leqslant b-1,\ c_b^a=c_1^{d_1}c_2^{d_2}\cdots c_b^{d_b}\rangle,$$

where $f(x)=x^b-d_bx^{b-1}-\cdots-d_2x-d_1$ is irreducible on F_q and $f(x)\mid x^p-1$, b is the order of q mod$p$. Therefore we may set

$$G=\langle a,\ c_1,\ c_2,\ \cdots,\ c_b,\ Z(G)\mid a^p=c_1^q=c_2^q=\cdots=c_b^q=1,\ [c_k,\ c_l]=z_{kl},\ c_i^a=c_{i+1},$$
$$c_b^a=c_1^{d_1}c_2^{d_2}\cdots c_b^{d_b}z_b,\ z_b,\ z_{kl}\in Z(G),\ 1\leqslant k,\ l\leqslant b,\ 1\leqslant i\leqslant b-1\rangle.$$

Put $c_b^{-d_b}c_{b-1}^{-d_{b-1}}\cdots c_1^{-d_1}=c_1^{-d_1}c_2^{-d_2}\cdots c_b^{-d_b}z'$, where $z'\in Z(G)$. Since $f(x)$ is irreducible on F_q, $f(1)=1-d_b-\cdots d_2-d_1\not\equiv 0 \pmod q$. Hence there exists an integer r such that $r(1-d_b-\cdots d_2-d_1)\equiv 1 \pmod q$. Set $\bar{z}=(z'^{-1}z_b^2)^r$. Define

$$\sigma:\ \begin{aligned} &a\mapsto a\\ &c_i\mapsto c_i^{-1}\bar{z},\ 1\leqslant i\leqslant b\\ &z\mapsto z,\ \text{for any } z\in Z(G). \end{aligned}$$

Because $z_{kl}=c_k^{-1}c_l^{-1}c_kc_l=c_kc_lc_k^{-1}c_l^{-1}=[c_k^{-1},\ c_l^{-1}]$, $(c_i^{-1}\bar{z})^a=(c_{i+1}^{-1}\bar{z})$, $1\leqslant k,\ l\leqslant b$, $1\leqslant i\leqslant b-1$, and

$$\begin{aligned}(c_b^{-1}\bar{z})^a&=(c_b^a)^{-1}\bar{z}=(c_1^{d_1}c_2^{d_2}\cdots c_b^{d_b}z_b)^{-1}\bar{z}=c_b^{-d_b}c_{b-1}^{-d_{b-1}}\cdots c_1^{d_1}z_b^{-1}\bar{z}\\ &=c_1^{-d_1}c_2^{-d_2}\cdots c_b^{-d_b}z'z_b^{-1}\bar{z}=(c_1^{-1}\bar{z})^{d_1}\cdots(c_b^{-1}\bar{z})^{d_b}\bar{z}^{-(d_1+d_2+\cdots+d_b)}\,z'z_b^{-1}\bar{z}\\ &=(c_1^{-1}\bar{z})^{d_1}\cdots(c_b^{-1}\bar{z})^{d_b}\bar{z}^{1-d_1-d_2-\cdots-d_b}\,z'z_b^{-1}\\ &=(c_1^{-1}\bar{z})^{d_1}\cdots(c_b^{-1}\bar{z})^{d_b}z_b\ (\text{this comes true by defintion of } \bar{z}),\end{aligned}$$

we see that a, $c_i^{-1}\bar{z}$, $1\leqslant i\leqslant b$, z satisfy the defining relations of G. So σ can be extended to an automorphism of G.

If $\sigma\in \operatorname{Inn}G$, then there exists $g_0\in G$ such that $\sigma(g)=g^{g_0}$ for any $g\in G$. Set $g_0=xy$, where $x\in S_p$, $y\in S_q$. We have $a^y=a^{xy}=\sigma(a)=a$, that is to say, $y\in C_{S_q}(S_p)$. By Lemma 1 $y\in C_{S_q}(S_p)\leqslant\phi(S_p)\leqslant Z(G)$. Hence $\sigma(g)=g^x$, $g\in G$. Then $c_i^x=c_i^{-1}\bar{z}$, $c_i^{x^2}=(c_i^{-1}\bar{z})^x=(c_i^x)^{-1}\bar{z}=(c_i^{-1}\bar{z})^{-1}\bar{z}=c_i$, $1\leqslant i\leqslant b$. This implies $x^2\in C_{S_p}(S_q)S_p\cap Z(G)$. Since $\alpha=1$, $S_p\cap Z(G)=1$ and $x^2=1$. Hence $p=2$. By the definition of b we know $b=1$, $|G|=2q$. Exactly $G=\langle a,\ b\mid a^2=b^q=1,\ a^{-1}ba=b^{-1}\rangle$. At this time $|\operatorname{Aut}G|=q(q-1)$. Then except $G=S_3$, G has an outer automorphism.

3) Let $\alpha=1$, $q=2$. If S_2 is an Abelian group, then

$$G=\langle a, c_1, c_2, \cdots, c_b \mid a^p=c_1^2=c_2^2=\cdots=c_b^2=1,\ c_ic_j=c_jc_i,\ 1\leqslant i,\ j\leqslant b,\ c_k^a=c_{k+1},$$
$$1\leqslant k\leqslant b-1,\ c_b^a=c_1^{d_1}c_2^{d_2}\cdots c_b^{d_b}\rangle,$$

where $f(x)=x^b-d_bx^{b-1}-\cdots-d_2x-d_1$ is irreducible on F_2 and $f(x)\mid x^p-1$ with b the order of 2 mod p. Define

$$\tau:\ \begin{array}{l} a\mapsto a^2 \\ c_i\mapsto c_j^{a^{2(i-1)}},\ 1\leqslant i\leqslant b. \end{array}$$

Then τ can be extended to an automorphism of G. If τ is inner, then there exists $g_0\in G$ such that $\tau(g)=g^{g_0}$, $g\in G$. Hence $a^{g_0}=a^2$, $[a, g_0]=a$, a contradiction to $G'=S_2$.

If S_2 is non-Abelian and superspecial group without commutative generatosr, then by [3] we have

$$G=\langle a, c_1, c_2, \cdots, c_b, z \mid a^p=c_1^4=c_2^4=\cdots=c_b^4=z^2=1,\ [c_i, c_j]=z,\ c_k^2=z,\ i\neq j,$$
$$1\leqslant i,\ j\leqslant b,\ 1\leqslant k\leqslant b,\ c_l^a=c_{l+1},\ c_b^a=c_1^{d_1}c_2^{d_2}\cdots c_b^{d_b},\ 1\leqslant l\leqslant b-1\rangle,$$

where d_i is as discribed in previous paragraph. By $\tau\in \mathrm{Aut}(G/Z(G))$ in above paragraph, we have $c_1^{a^{2b}}=c_1^{d_1}(c_1^{a^2})^{d_2}(c_1^{a^4})^{d_3}\cdots(c_1^{a^{2(b-1)}})^{d_b}z'$. Since $f(x)$ is irreducible on F_2, $f(1)\equiv 1 \pmod 2$. Define

$$\delta:\ \begin{array}{l} a\mapsto a^2 \\ z\mapsto z \\ c_i\mapsto c_1^{a^{2(i-1)}}z',\ 1\leqslant i\leqslant b. \end{array}$$

Because

$$[c_1^{a^{(b-1)2}}z']^{a^2}=c_1^{a^{2b}}z'=[c_1^{d_1}(c_1^{a^2})^{d_2}\cdots(c_1^{a^{2(b-1)}})^{d_b}z']z'$$
$$=(c_1z')^{d_1}(c_1^{a^2}z')^{d_2}\cdots(c_1^{a^{2(b-1)}}z')^{d_b}(z')^{f(1)}z'=(c_1z')^{d_1}(c^{a^2}z')_1^{d_2}\cdots(c_1^{a^{2(b-1)}}z')^{d_b},$$

the other defining relations of G are obviously satisfied by a^2, c_1z', $c_1^{a^2}z'$, $\cdots$, $c_1^{a^{2(b-1)}}z'$, z. Then δ can be extended to an automorphism of G. By the same resean as τ we know δ is an outer automorphism. The Lemma is proved.

Proof of the Theorem At first we prove that the groups satisfying the condition of the Theorem are nilpotent groups or S_3 or D with Aut $D=$ Inn D.

In fact, if $G/Z(G)\nleqslant \mathrm{Aut}G$, then $G/Z(G)$ is nilpotent. Further G is nilpotent. Suppose $G/Z(G)=\mathrm{Aut}G$. Let $G=G_1\times Z$, where $Z\leqslant Z(G)$ and G_1 has no nontrivial Abelian direct factors. Then $Z(G)=Z(G_1)\times Z$ and $G_1/Z(G_1)\cong G/Z(G)\cong \mathrm{Aut}\,G$. Since Aut $G_1\times$ Aut $Z\leqslant$ Aut$G$, Aut $G_1=$ Aut G and Aut $Z=1$, $|Z|\leqslant 2$. We assert that G_1 is a Schmidt group. Here we may assume $|G_1|=p^\alpha q^\beta$.

Let H be a proper subgroup of G_1. Then $HZ(G_1)/Z(G_1)\leqslant G_1/Z(G_1)$. If $HZ(G_1)/Z(G_1)\nleqslant G_1/Z(G_1)$, then H is nilpotent. Otherwise $G_1=HZ(G_1)$. Hence H is a normal subgroup. Further G_1 has a maximal normal subgroup M such that there is an element z' which is not in M but in $Z(G_1)$. Suppose $|G_1/M|=r$, where $r=p$ or q. Then $(z')^r\in M$. Suppose there is $z\in M\cap Z(G_1)$ such that $|z|=r$. Define $\sigma(g)=f(g)g$, $g\in G_1$, where f is a homomorphism from G_1 to $\langle z\rangle$ such that $M\leqslant \mathrm{Ker}\,f$. Then $\sigma\in \mathrm{Aut}\,G_1$. Since $\sigma(z')=f(z')z'\neq z'$, σ is an outer automorphism. This contradicts $\mathrm{Aut}\,G_1=\mathrm{Aut}\,G=G_1/Z(G_1)$. Therefore $r\nmid|Z(G_1)\cap M|$ and we have $G_1=M\times\langle z'\rangle$, a contradiction to supposition of G_1. Then H is nilpotent, G_1 is a Schmidt group. By Lemma 3 G_1 is S_3. If $G=S_3\times Z_2$, $D\times Z_2$ then by [3] Lemma 4, G has an automorphism, a contradiction. Therefore $G=S_3$ or D with $\mathrm{Aut}\,D=\mathrm{Inn}\,D$.

Secondly, if G is nilpotent, set $G=S_{p_1}\times S_{p_2}\times\cdots\times S_{p_t}$. Then $\mathrm{Aut}\,G=\mathrm{Aut}\,S_{p_1}\times\mathrm{Aut}S_{p_2}\times\cdots\times\mathrm{Aut}\,S_{p_t}$. If there exist two $\mathrm{Aut}\,S_{p_i}$, $|\mathrm{Aut}\,S_{p_i}|\neq1$, then as Aut G is a Schmidt group we know that all $|\mathrm{Aut}\,S_{p_i}|$ are nilpotent, and so is Aut G, a contradiction. Then we may set $\mathrm{Aut}\,S_{p_1}=\mathrm{Aut}\,G$. Since $S_{p_1}/Z(S_{p_1})\leqslant\mathrm{Aut}\,S_{p_1}=\mathrm{Aut}\,G$ and the nilpotent class of the Sylow subgroup of Aut G is at most 2, the nilpotent class of S_p is at most 3.

When S_{p_1} is commutative, suppose that S_{p_1} has type

$$(\overbrace{p^{m_1},\cdots,p^{m_1}}^{S_1},\overbrace{p^{m_2},\cdots,p^{m_2}}^{S_2},\cdots,\overbrace{p^{m_t},\cdots,p^{m_t}}^{S_t}),\ m_1>m_2>\cdots>m_t.$$

If $s_i=1$, $1\leqslant i\leqslant t$, then by Lemma 2, Aut G is a p-group, a contradiction. This implies that there exists an s_i, say s_1, larger than 1. So S_{p_1} has a direct factor N of type (p^{m_1},p^{m_1}). Therefore $GL_2(p)\leqslant GL_2(p^{m_1})=\mathrm{Aut}\,N\leqslant\mathrm{Aut}\,G$. Since $GL_2(p)$ is not nilpotent and Aut G is a Schmidt group, $\mathrm{Aut}\,G=GL_2(p)$. If $p>3$, then $GL_2(p)$ is unsolvable, a contradiction. If $p=3$, we have $SL_2(3)\nleqslant GL_2(3)$, but $SL_2(3)$ is not nilpotent, a contradiction to the fact that all subgroups of Aut G are nilpotent. Then $\mathrm{Aut}\,G=GL_2(2)=S_3$, this implies that G is the Klain 4-group by [4].

When the class of nilpotency of S_{p_1} is 3, we see that $S_{p_1}/Z(S_{p_1})$ is a non-Abelian normal subgroup of $\mathrm{Aut}\,S_{p_1}=\mathrm{Aut}\,G$. By Lemma 1 2), 3), $S_{p_1}/Z(S_{p_1})$ is a Sylow subgroup of Aut S_{p_1}. Then $p_1\nmid|\mathrm{Aut}\,S_{p_1}/\mathrm{Inn}\,S_{p_1}|$. But any p_1-group with order larger than p_1 has an outer automorphism of order p_1, a contradiction.

When the class of nilpotency of S_{p_1} is 2, if $S_{p_1}/Z(S_{p_1})$ is a Sylow subgroup of Aut S_{p_1}, we can obtain a contradiction by using the method in above paragraph. Otherwise by Lemma 1 2), 3), 7), we know $S_{p_1}/Z(S_{p_1})\leqslant Z(\mathrm{Aut}\,S_{p_1})$. Therefore for any $\tau\in\mathrm{Aut}\,S_{p_1}$ and $I_{g_0}\in\mathrm{Inn}\,S_{p_1}$, $\tau I_{g_0}=I_{g_0}\tau$ holds. We have $I_{g_0}\tau(g)=\tau I_{g_0}(g)$, $g\in G$,

$$g_0^{-1}\tau(g)g_0=\tau(g_0)^{-1}\tau(g)\tau(g_0),$$
$$\tau(g)=g_0\tau(g_0)^{-1}\tau(g)\tau(g_0)g^{-1},$$

which implies $g_0\tau(g_0)^{-1}\in Z(S_{p_1})$ for any $g_0\in G$. Then τ is a central automorphism, that is to say, Aut S_{p_1} consists of central automorphisms. Since $|\mathrm{Aut}\,S_{p_1}|$ has two distinct prime factors, by the formula of the order of the group of the central automorphisms of finite p-group displayed in [4] (p. 280) we know that S_{p_1} has a direct factor E of type (p_1^k,p_1^k). Put $S_{p_1}=E\times D$. Then $\mathrm{Aut}\,E\times\mathrm{Aut}\,D\leqslant\mathrm{Aut}\,S_{p_1}$. Since $\mathrm{Aut}\,E=GL_2(p_1^k)$ is non-nilpotent and $\mathrm{Aut}\,S_{p_1}=\mathrm{Aut}\,G$ is a Schmidt group, we have $\mathrm{Aut}\,E=\mathrm{Aut}\,S_{p_1}$, $\mathrm{Aut}\,D=1$. Hence D is Abelian, and so is S_{p_1}, a contradiction to S_{p_1} having nilpotent class 2. This concludes the proof of the theorem.

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