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EXISTENCE OF RADIAL LIMITS OF HARMONIC FUNCTIONS IN BANACH SPACES

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Abstract

The following result is established: let X be a Banach space without the Radon-Nikodym property, there exists a uniformly bounded harmonic function f defined on the open unit disk of $\mathbf{C}$ with values in X, such that for almost all $\theta\in[0, 2\pi]$, $\lim\limits_{r\to 1} f(re^{i\theta})$ does not exist.

In the last twenty years, several remarkable results have been established in the theory of infinite dimensional Banach spaces. Most of these results emphasize the interplay between the topological, geometrical and measure theoretical structures of a Banach space. Here is a well known prototype of such interrelations. It is due to the combined efferts of several authors and we refer the reader to the books [1], [6] and the paper [4] for a detailed account of its history and for the notions involved in its statement.

Theorem (**A**). *Let X be a Banach space. The following properties are equivalent.*

1. *For every closed convex bounded sulset C of X, all C-valued martingales norm converge almost surely.*

2. *For every non empty bounded subset C of X, C has linear slices of arbitrily small diameter.*

3. *Every uniformly bounded harmonic function defined on the open unit disk of the complex plane with values in X has radial limits almost everywhere on the torus.*

A Banach space verifying the conditions of Theorem (A) is said to have the Radon-Nikodym property. In contrast with the Radon-Nikodym property, Bukhvalov and Danilevich have introduced the analytic Radon-Nikodym property in complex Banach spaces[4]. Recall that a complex Banach space X is said to have the analytic Radon-Nikodym property if every analytic function f: $\mathbf{D}\mapsto X$, defined on the open unit disk of $\mathbf{C}$ with values in X, has radial limits almost everywhere on the torus, this means that for almost all $\theta\in[0, 2\pi]$, $\lim\limits_{r\uparrow 1} f(re^{i\theta})$ exists. As every

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analytic function is harmonic, a complex Banach space with the Radon–Nikodym property has also the analytic Radon–Nikodym property. Several remarkable results have been established for the analytic Radon–Nikodym property, for instance, we have an analogous theorem of Theorem (A) in the "analytic" setting, where "martingale" is replaced by "plurisubharmonic martingale" in the first statement[3,7] while "linear slices" is replaced by "plurisubharmenic slices" in the second [7]. Among other results about the analytic Radon–Nikodym property, we recall the following result[2].

Theorem (B). *Let X be a complex Banach space without the analytic Radon–Nikodym property. There exists a uniformly bounded analytic function f: $\mathbf{D}\to X$, defined on the open unit disk of $\mathbf{C}$ with values in X, such that for almost all $\theta\in[0, 2\pi]$, $\lim_{r\uparrow 1} f(re^{i\theta})$ does not exist in X; more precisely, we have*

$$\lim_{r,s\uparrow 1}\sup\|f(re^{i\theta})-f(se^{i\theta})\|\geqslant 1.$$

The purpose of this paper is to establish an analogue of Theorem (B) in the "Radon–Nikodym property" setting. We shall show the following result:

Theorem 1. *Let X be a Banach space without the Radon–Nikodym property. There exists a uniformly bounded harmonic function f: $\mathbf{D}\to X$, defined on the open unit disk of $\mathbf{C}$ with values in X, such that for al most all $\theta\in[0, 2\pi]$, $\lim_{r\uparrow 1} f(re^{i\theta})$ does not exist.*

Let us recall some notions and notations. Throughout this paper, $\mathbf{T}$ will de the torus $\{e^{i\theta}: \theta\in[0, 2\pi]\}$ with normalized Lebesgue measure $d\theta/2\pi$, μ will be the Lebesgue measure on $[0, 2\pi]$ and $\mathbf{D}$ denotes the open unit disk in the complex plane. Let X be a Banach space, $1\leqslant p\leqslant\infty$, we shall denote by $h^p(\mathbf{D}, X)$ the space of all harmonic functions f: $\mathbf{D}\to X$ satisfying

$$\|f\|_{h^p}=\sup_{0<r<1}\left(\int_0^{2\pi}\|f(re^{i\theta})\|^p d\theta/2\pi\right)^{1/p}<\infty,$$

for $1\leqslant p<\infty$, and for $p=\infty$

$$\|f\|_{h^\infty}=\sup_{z\in\mathbf{C}}\|f(z)\|.$$

It is easy to verify that for $1\leqslant p\leqslant q\leqslant\infty$, $h^q(\mathbf{D}, X)\subset h^p(\mathbf{D}, X)$ and with the norm $\|\cdot\|_{h^p}$, $h^p(\mathbf{D}, X)$ becomes a Banach space.

Let $\theta\in[0, 2\pi]$, $0\leqslant\alpha<\pi/2$ and $S_\alpha(\theta)=\{z\in\mathbf{D}: \operatorname{Arg}\{(z-e^{i\theta})/e^{i\theta}\}<\alpha\}$, where for every $z\in\mathbf{D}$, $\operatorname{Arg}(z)$ denotes the unique point $\theta\in[0, 2\pi]$ such that $z=re^{i\theta}$. Recall that if $f\in h^1(\mathbf{D}, X)$, f is said to have nontangential limit on $e^{i\theta}$, if for every $0\leqslant\alpha<\pi/2$, $(f(z_n))_{n\geqslant 1}$ converges in X whenever $(z_n)_{n\geqslant 1}$ is a sequence of $\mathbf{D}$ such that z_n belongs to $S_\alpha(\theta)$ when n is big enough and that $z_n\to e^{i\theta}$. It will be useful to notice that every harmonic function $f\in h^1(\mathbf{D}, X)$ has nontangential limits almost everywhere on $\mathbf{T}$.

If $z\in\mathbf{D}$, $z=re^{i\theta}$, the Poisson kernel at z on $[0, 2\pi]$ is defined by

$$P_z(\theta)=\frac{1-r^2}{1-2r\cos(\theta-\alpha)+r^2}.$$

Let f be an integrable function with values in some Banach space X, $f\in L^1([0, 2\pi], X)$, its harmonic extension by the Poisson kernel in $\mathbf{D}$ is defined by

$$f(z)=\int_0^{2\pi} f(\theta)P_z(\theta)d\theta/2\pi.$$

It is known that such a harmonic function has radial limits $f(\theta)$ almost everywhere on $\mathbf{T}$ in X[4]. Inversely, if $h\in h^1(\mathbf{D}, X)$ has radial limits and if $g(\theta)=\lim\limits_{r\uparrow 1} h(re^{i\theta})$, then the harmonic extension by the Poisson kernel of g in $\mathbf{D}$ coincides with h[4].

Firstly, we discuss the relation between the existence of radial limits and the existence of nontangential limits of harmonic functions in Banach spaces.

Theorem 2. *Let X be a Banach space and let $f\in h^\infty(\mathbf{D}, X)$. If f has radial limits almost everywhere on the torus in X, then f has also nontangential limits almost everywhere on the torus in X.*

Proof If h is any element of $h^1(\mathbf{D}, X)$ having radial limits almost everywhere on in X and $g(e^{i\theta})=\lim\limits_{r\uparrow 1} h(re^{i\theta})$, then the harmonic extension by the Poisson kernel in $\mathbf{D}$ of g coincides with h and $g\in L^1(\mathbf{T}, X)$. If $g\in L^1(\mathbf{T}, X)$, its harmonic extension by the Poisson kernel in $\mathbf{D}$ has radial limits almost everywhere on $\mathbf{T}$ in X with $g(e^{i\theta})=\lim\limits_{r\to 1} g(re^{i\theta})$. If h is any element of $h^1(\mathbf{D}, X)$ having radial limits almost everywhere on $\mathbf{T}$ in X, we shall denote by $h(e^{i\theta})$ its radial limit on $e^{i\theta}$. If g is any functson in $L^1(\mathbf{T}, X)$, its harmonic extension by the Poisson kernel in $\mathbf{D}$ will be denoted by the same letter.

Let $n\in\mathbf{N}$. There exists an X-valued simple function h_n in $L^1(\mathbf{T}, X)$, such that $\|h_n-f\|_1\leqslant 2^{-2n}$. Put $f_n=h_n-f$ and $g_n=\|f_n\|$, so $f_n\in L^1(\mathbf{T}, X)$ and $g_n\in L^1(\mathbf{T}, \mathbf{R})$.

It is easy to verify that for each X-valued simple function in $L^1(\mathbf{T}, X)$, its harmonic extension by the Poisson kernel in $\mathbf{D}$ has nontangential limits almost everywhere on $\mathbf{T}$ in X, so $h_n\in h^1(\mathbf{D}, X)$ has nontangential limits almost everywhhere on $\mathbf{T}$ in X. Let $B_n=\{\theta\in[0, 2\pi]: \|f_n(e^{i\theta})\|\leqslant 2^{-n}\}$. It is easy to see that $\mu(B_n)\geqslant 2\pi-2^{-n}$ and $A=\bigcap\limits_{n\geqslant 1}(\bigcup\limits_{k\geqslant n} B_k)$ is of measure 2π. If we denote $C_n=\{\theta\in[0, 2\pi]: \lim\limits_{r\uparrow 1} g_n(re^{i\theta})$ exists$\}$, we have $\mu(C_n)=2\pi$ and $\mu(A\cap(\bigcap\limits_{n\geqslant 1} C_n)=2\pi$.

Let $\theta\in A\cap)\bigcap\limits_{n\geqslant 1} C_n)$, $0\leqslant\alpha<\pi/2$ and $(Z_n)_{n\geqslant 1}$ is any sequence of $\mathbf{D}$ inside the region $S_\alpha(\theta)$ such that $Z_n\to e^{i\theta}$. For every $n\in\mathbf{N}$, $\theta\in\bigcup\limits_{k\geqslant n} B_k$, there exists then $m\in\mathbf{N}, m\geqslant n$, such that $\theta\in B_m$ and

$$\begin{aligned}
&\limsup_{p,q\to\infty} \|f(Z_p)-f(Z_q)\| \\
&\leqslant \limsup_{p,q\to\infty} \|h_m(Z_p)-h_m(Z_q)\| + \limsup_{p,q\to\infty} \|f_m(Z_p)-f_m(Z_q)\| \\
&\leqslant \limsup_{p\to\infty} \|f_m(Z_p)\| + \limsup_{q\to\infty} \|f_m(Z_q)\| \\
&\leqslant \limsup_{p\to\infty} g_m(Z_p) + \limsup_{q\to\infty} g_m(Z_r) \\
&= 2g_m(e^{i\theta}) \leqslant 2^{-n},
\end{aligned}$$

where, in the last line, we have used the well known fact that each function in $h^1(\mathbf{D}, \mathbf{R})$ has nontangential limits almost everywhere on $\mathbf{T}$. Since $n\in\mathbf{N}$ is arbitrary, we get

$$\limsup_{p,q\to\infty} \|f(Z_p)-f(Z_q)\| = 0,$$

which shows that f has nontangential limit on $e^{i\theta}$ and finishes the proof of Theorem 2.

The following "localization" of Theorem 2 is also true.

Theorem 3. *Let X be a Banach space, $A\subset\mathbf{T}$ be a Lebesgue measurable subset with positive measure and let $f\in h^\infty(\mathbf{D}, X)$, such that for every $e^{i\theta}\in A$, $\lim_{r\uparrow 1} f(re^{i\theta})$ exists. Then for almost all $e^{i\theta}\in A$, f has nontangential limit on $e^{i\theta}$.*

Proof Let $g\in L^\infty(\mathbf{T}, X)$ be the function defined on $\mathbf{T}$ by

$$g(e^{i\theta}) = \begin{cases} \lim_{r\uparrow 1} f(re^{i\theta}), & \text{if } e^{i\theta}\in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then the harmonic extension of g by the Poisson kernel has nontangential limits almost everywhere on $\mathbf{T}$ in X by Theorem 2, and for almost all $e^{i\theta}\in A$, $\lim_{r\uparrow 1}\{f(re^{i\theta})-g(re^{i\theta})\}=0$. So, without loss of generality, we can suppose that for every $e^{i\theta}\in A$, $\lim_{r\uparrow 1} f(re^{i\theta})=0$.

Let $h\in L^\infty(\mathbf{T})$ be the function defined by

$$h(e^{i\theta}) = \begin{cases} M, & \text{if } e^{i\theta}\in A^c, \\ 0, & \text{if } e^{i\theta}\in A, \end{cases}$$

where $M=\|h\|_{h^\infty}$. Let $B=\{e^{i\theta}\in A: \lim_{r\uparrow 1} h(re^{i\theta})=0\}$. If $e^{i\theta}\in B$, $0\leqslant\alpha<\pi/2$ and $(Z_n)_{n\geqslant 1}$ is any sequence of $\mathbf{D}$ inside the region $S(\theta)$ such that $z_n\to e^{i\theta}$, we have

$$\|f(sz_n)\| = \left\| \int_0^{2\pi} f(se^{i\alpha}) P_{z_n}(\alpha)\, d\alpha/2\pi \right\|$$

for each $0\leqslant s<1$. So for every $n\in\mathbf{N}$

$$\begin{aligned}
\|f(z_n)\| &= \lim_{s\uparrow 1} \|f(sz_n)\| \\
&= \lim_{s\uparrow 1} \left\| \int_0^{2\pi} f(se^{i\alpha}) P_{z_n}(\alpha)\, d\alpha/2\pi \right\| \\
&\leqslant \int_0^{2\pi} \lim_{s\uparrow 1} \|f(se^{i\alpha})\| P_{z_n}(\alpha)\, d\alpha/2\pi
\end{aligned}$$

$$\leqslant \int_0^{2\pi} h(e^{i\alpha}) P_{z_n}(e^{i\alpha}) d\alpha/2\pi = h(z_n).$$

We get $\lim_{n\uparrow\infty} \|f(z_n)\| \leqslant \lim_{n\to\infty} h(z_n) = 0$, which completes the proof since $B \subset A$ and the measure of B equals to that of A.

Theorem 1 is an easy consequence of the theorem above and the subsequent result.

Theorem 4. *Let X be a Banach space without the Radon-Nikodym property. There exists $f \in h^\infty(\mathbf{D}, X)$ such that for almost all $\theta \in [0, 2\pi]$, f has not nontangential limit on $e^{i\theta}$; this means that there exists $0 \leqslant \alpha < \pi/2$, and a sequence $(Z_n)_{n\geq 1}$ in $\mathbf{D}$ inside the region $S_\alpha(\theta)$, such that $Z_n \to e^{i\theta}$ and $\lim_{n\to\infty} f(Z_n)$ does not exist.*

The proof of Theorem 4 will use the subsequent two lemmas which may be, we hope, useful in other situations.

Lemma 1. *Let $a \in \mathbf{D}$ and let h_a be the conformal mapping from D onto $\mathbf{D}$ defined by $h_a(z) = (z-a)/(1-\bar{a}z)$. There exists $0 \leqslant \beta < \pi/2$ such that for every $\theta \in [0, 2\pi]$, $h_a(re^{i\theta})$ belongs to the region $S_\beta(h_a(e^{i\theta}))$ whenever $1-r$ is small enough.*

Lemma 2. *Let A be a Lebesgue measurable subset of $[0, 2\pi]$ with positive measure and let $\varepsilon > 0$. There exists $a \in \mathbf{D}$, such that if h_a is the conformal mapping from $\mathbf{D}$ onto $\mathbf{D}$ defined by $h_a(z) = (z-a)/(1-\bar{a}z)$, $\mu(\{\theta \in [0, 2\pi]: e^{i\theta} = h_a(e^{i\alpha})$ for some $\alpha \in A\})$ is more than $2\pi - \varepsilon$.*

Before proving these two lemmas, we will give the proof of Theorem 4.

Proof of Theorem 4 Let X be a Banach space without the Radon-Nikodym prperty. There exists $g \in h^\infty(\mathbf{D}, X)$ and a Lebesgue measurable subset $A \subset [0, 2\pi]$ with positive measure, such that for every $\theta \in A$, $\lim_{r\uparrow 1} g(re^{i\theta})$ does not exist.

By Lemma 2, for every $n \in \mathbf{N}$ there exists $a_n \in \mathbf{D}$, such that if h_n is the conformal mapping from $\mathbf{D}$ onto $\mathbf{D}$ defined by $h_n(z) = (z-a_n)/(1-\bar{a}_n z)$, $\mu(\{\theta \in [0, 2\pi/: e^{i\theta} = h_n(e^{i\alpha})$ for some $\alpha \in A\}) > 2\pi - 1/n$.

For $f \in h^\infty(\mathbf{D}, X)$, $n \in \mathbf{N}$ and $\theta \in [0, 2\pi]$, let us consider

$$E(f)(n, \theta) = \limsup_{r,s\uparrow 1} \|f \circ h_n(re^{i\alpha}) - f \circ h_n(se^{i\alpha})\|,$$

where $e^{i\alpha} \in \mathbf{T}$ is such that $h_n(e^{i\alpha}) = e^{i\theta}$. It is easy to verify that the application $\theta \to E(f)(n, \theta)$ is measurable for each $n \in \mathbf{N}$. Put

$$N_f = \{\theta \in [0, 2\pi]: \sup_{n\in\mathbf{N}} E(f)(n, \theta) > 0\}$$

and

$$G_n = \{f \in h^\infty(\mathbf{D}, X): \mu(N_f) > 2\pi - 1/\mathrm{n}\}.$$

We claim that for each $n \in \mathbf{N}$, G_n is not empty. Indeed. $\mu(\{\theta \in [0, 2\pi]: e^{i\theta} = h_n(e^{i\alpha})$ for some $\alpha \in A\}) > 2\pi - 1/n$, if $e^{i\theta} = h_n(e^{i\alpha})$ for some $\alpha \in A$.

$$E(g \circ h_n^{-1})(n,\theta) = \limsup_{r,s\uparrow 1} \|g \circ h_n^{-1}(h_n(re^{i\alpha})) - g \circ h_n^{-1}(h_n(se^{i\alpha}))\|$$

$$= \limsup_{r,s\uparrow 1} \|g(re^{i\alpha}) - g(se^{i\alpha})\| > 0.$$

We get $\sup_{m\in\mathbf{N}} E(g\circ h_n^{-1})(m, \theta)>0$ and $\{\theta\in[0, 2\pi]: e^{i\theta}=h_n(e^{i\alpha})$ for some $\alpha\in A\}\subset N_{g\circ h_n^{-1}}$. This shows that $g\circ h_n^{-1}\in G_n$ and so G_n is not empty.

G_n is an open subset of $h^\infty(\mathbf{D}, X)$. Indeed, let $f\in G_n$, then $N_f=\bigcup_{k\geq 1} D_k$. where

$$D_k=\{\theta\in[0, 2\pi]: \sup_{n\in\mathbf{N}} E(f)(n, \theta)>1/k\}.$$

There exists then $k\in\mathbf{N}$, such that $\mu(D_k)>2\pi-1/n$. Hence for any $h\in h^\infty(\mathbf{D}, X)$ with $\|h\|_{h^\infty}\leqslant 1/4k$, $f+h$ belongs to G_n.

We shall show that for each $n\in\mathbf{N}$, G_n is dense in $h^\infty(\mathbf{D}, X)$. Let $f\in h^\infty(\mathbf{D}, X)$ and $h\in G_{2n}$ with $\|h\|_{h^\infty}\leqslant 1$. Then $\xi+\alpha h$ belongs to G_n if $\alpha>0$ is small enough; this proves that G_n is dense in $h^\infty(\mathbf{D}, X)$. In deed, let $\delta>0$ be sufficiently small so that

$$\mu(\{\theta\in N_f: \sup_{n\in\mathbf{N}} E(f)(n, \theta)\leqslant\delta\}<1/2n.$$

If we take $\alpha=\delta/4$, we get

$$B=([0, 2\pi]\setminus N_f)\cap N_h\subset N_{f+\alpha h}$$

and

$$C=\{\theta\in[0, 2\pi]: \sup_{n\in\mathbf{N}} E(f)(n, \theta)>\delta\}\subset N_{f+\alpha h}.$$

We have

$$\begin{aligned}\mu(N_{f+\alpha h})&\leqslant\mu(B\cup C)=\mu(B)+\mu(C)\\&\geqslant(2\pi-\mu(N_f)-1/2n)+(\mu(N_f)-1/2n)\\&=2\pi-1/n.\end{aligned}$$

This proves that $f+\alpha h\in G_n$.

Now as $h^\infty(\mathbf{D}, X)$ is a Banach space, by Baire category theorem, $\bigcap_{n\geq 1} G_n$ is not empty. Let $f\in\bigcap_{n\geq 1} G_n$. We have $\mu(N_f)=2\pi$; this means that for almost all $\theta\in[0, 2\pi]$, there exists $n\in N$, such that

$$E(f)(n, \theta)=\limsup_{r,s\uparrow 1}\|h_n(ke^{i\alpha})-fh_n(se^{i\alpha})\|>0,$$

where $e^{i\alpha}\in\mathbf{T}$ is such that $h_n(e^{i\alpha})=e^{i\theta}$. Since h_n is continuous from $\mathbf{D}$ onto $\mathbf{D}$, we get that $h_n(re^{i\alpha})$ converges to $e^{i\theta}$ when r tends to 1, and by Lemma 1, there exists $0\leqslant\beta<\pi/2$, such that $h_n(re^{i\alpha})\in S_\beta(\theta)$ if $1-r$ is small enough. f is then a uniformly bounded harmonic function defined on $\mathbf{D}$ with values in X satisfying Theorem 4. This finishes the proof of Theorem 4.

The argument used in the proof of Theorem 4 is similar to that given in [5], where the author has shown that if X is a Banach space without the Radon-Nikodym property, there exists a Lipschitz map f: $[0, 1]\to X$, such that the set of points of non differentiability of f is of measure 1.

By the proof of Theorem 4, the set of functions in $h^\infty(\mathbf{D}, X)$ satisfying Theorem 4 is a G_δ-dense subset of $h^\infty(\mathbf{D}, X)$ and so the set of harmonic functions in $h^\infty(\mathbf{D}, X)$ satifying Theorem 1 is also a G_δ-dense subset of $h^\infty(\mathbf{D}, X)$ by Theorem 2.

We end this paper by giving the proofs of Lemma 1 and Lemma 2.

Proof of Lemma 1 Without loss of generality, we can suppose that $a\in[0, 1[$. Let $e^{i\alpha}\in\mathbf{T}$, $r_n\uparrow 1$, and $e^{i\theta}\in\mathbf{T}$ is such that $h_a(e^{i\theta})=e^{i\alpha}$. Then

$$e^{i\theta}=(e^{i\alpha}+a)/(1+ae^{i\alpha}),$$

and

$$h_a(r_ne^{i\theta})=\frac{(r_n-1)a+(r_n-a^2)e^{i\alpha}}{1-a^2r_n+a(1-r_n)e^{i\alpha}}.$$

Letting $z_n=(h_a(r_ne^{i\alpha})-e^{i\alpha})/e^{i\alpha}$, we get

$$z_n=\frac{(r_n-1)(a+(a^2+1)e^{i\alpha}-ae^{2i\alpha})}{(1-a^2r_n+a(1-r_n)e^{i\alpha})e^{i\alpha}}.$$

To show that $h_a(r_ne^{i\theta})$ belongs to the region $S_\beta(\alpha)$ for some $0\leqslant\beta<\pi/2$ when n is big enough, it suffices to show that

$$\left|\operatorname{tg}\operatorname{Arg}\frac{a+(a^2+1)e^{i\alpha}-ae^{2i\alpha}}{(1-a^2)e^{i\alpha}}\right|\leqslant M$$

for some constant $M>0$ depending only on $a\in\mathbf{D}$. A simple calculation shows that we can take $M=2/(1+a^2)$ which completes the proof of the lemma.

In the proof of Lemma 2 we shall use the following notation: let α, $\beta\in[0, 2\pi[$, then $[\alpha, \beta]=\{\theta\in[0, 2\pi]: \alpha\leqslant\theta\leqslant\beta\}$ if $\alpha<\beta$ and $[\alpha, \beta]=\{\theta\in[0, 2\pi]: 2\pi>\theta\geqslant\alpha$ or $0\leqslant\theta\leqslant\beta\}$ if $\alpha>\beta$. Let s, $t\in\mathbf{R}$, $e^{is}=e^{i\alpha}$, $e^{it}=e^{i\beta}$, $0\leqslant\alpha$, $\beta<2\pi$, $[s, t]$ will denote the interval $[\alpha, \beta]$ of $[0, 2\pi[$.

Proof of Lemma 2 It is not hard to verify that the conformal mapping h_a is one to one continuous mapping from $\mathbf{T}$ onto $\mathbf{T}$, and h_a maps intervals of $\mathbf{T}$ to intervals of $\mathbf{T}$. If we denote by g_a the application from $[0, 2\pi[$ defined by $g_a(\theta)=\operatorname{Arg}(h_a(e^{i\theta}))$, then for every α, $\beta\in[0, 2\pi[$, $g_a([\alpha, \beta])=[g_a(\alpha), g_a(\beta)]$, to show the lemma, it is sufficient to show that for every Lebesgue measurable subset $A\subset[0, 2\pi]$ with positive measure and for every $\varepsilon>0$ there exists $a\in\mathbf{D}$, such that $g_a(A)$ is of measure more than $2\pi-\varepsilon$. Fix then A and ε as above.

A simple calculation shows that

$$\operatorname{tg}(g_a(\theta))=\operatorname{tg}\left(\operatorname{Arg}(\alpha)+\operatorname{Act}\operatorname{tg}\frac{(1-r^2)\sin(\theta-\alpha)}{(1+r^2)\cos(\theta-\alpha)+2r}\right),$$

if $a=re^{i\alpha}$. Let G be an open subset of $[0, 2\pi]$ containing α. We claim that $\mu(g_{re}(G))$ converges to 2π when r tends to 1. Indeed, if $\eta>0$ is such thbt $[\alpha-\eta, \alpha+\eta]\subset G$, and $\delta>0$. there exists $0\leqslant r<1$, such that $\operatorname{Arccos}\{2r/(1+r^2)\}\in[0, \eta]$ and

$$\frac{(1-r^2)\sin\eta}{(1+r^2)\cos\eta-2r}\leqslant\operatorname{tg}(\pi-\delta/2).$$

For such $0\leqslant r<1$, $\alpha+\pi/2\in g^{i\alpha}_{re}([\alpha-\eta, \alpha+\eta])$, $\alpha\in g^{i\alpha}_{re}([\alpha-\eta, \alpha+\eta])$ and therefore $[\alpha, \alpha+\pi-\delta/2]\subset g^{i\alpha}_{re}([\alpha-\eta, \alpha+\eta])$ since $g^{i\alpha}_{re}$ maps any interval $[s, t]$ to $[g^{i\alpha}_{re}(s), g^{i\alpha}_{re}(t)]$. An analogous argument shows that for such $0\leqslant r<1$, $[\alpha+\pi+\delta/2, \alpha]\subset g^{\alpha i}_{er}([\alpha-\eta, \alpha-\eta])$. We get $[\alpha+\pi+\delta/2, \alpha+\pi-\delta/2]\subset g^{i\alpha}_{er}([\alpha-\eta, \alpha-\eta])$ and $\mu(g^{i\alpha}_{ie}(G))\geqslant\mu(g^{i\alpha}_{er}([\alpha-\eta, \alpha-\eta]))\geqslant2\pi-\delta$.

It is easy to verify from the special expression of g_a that there exists integrable functions $f_a \in L^1([0, 2\pi])$, such that for almost all $(\alpha, \theta) \in [0, 2\pi]^2$, $f_{re}^{i\alpha}(\theta) = f_{re}^{i\alpha}(\alpha)$ and for any measurable sudset B of $[0, 2\pi]$.

$$\mu(g_{re}^{i\alpha}(B)) = \int_B f_{re}^{i\alpha}(\theta) d\theta.$$

We leave the verification of this fact to the interested reader.

Now if $(G_n)_{n\geq 1}$ is a sequence of open subsets of $[0, 2\pi]$, such that $G_{n+1} \subset G_n$, $A \subset G_n$ for every $n \in \mathbf{N}$, and $\mu(\bigcap_{n\geq 1} G_n)\ \mu(A)$, we have

$$\begin{aligned}
&\lim_{r\uparrow 1} \int_A \mu(g_{re}^{i\alpha}(A)) d\alpha \\
&= \lim_{r\to 1} \int_A \int_A f_{re}^{i\alpha}(\theta) d\theta\, d\alpha \\
&= \int_A \lim_{r\uparrow 1} f_{re}^{i\alpha}(\theta) d\theta\, d\alpha \\
&= \lim_{n\to\infty} \int G_n \lim_{r\uparrow 2} \int_A f_{re}^{i\alpha}(\theta) d\theta\, d\alpha \\
&= \lim_{n\to\infty} \lim_{r\uparrow 1} \int_A G_n f_{re}^{i\alpha}(\theta) d\theta\, d\alpha \\
&= \lim_{n\to\infty} \lim_{r\uparrow 1} \int_A \int G_n f_{re}^{i\alpha}(\theta) d\alpha\, d\theta \\
&= \lim_{n\to\infty} \int_A \lim_{r\uparrow 1} \int G_n f_{re}^{i\alpha}(\alpha) d\alpha\, d\theta \\
&= \lim_{n\to\infty} \int_A 2\pi d\theta = 2\pi\mu(A),
\end{aligned}$$

where, in the last line, we have used the fact that for $\theta \in A \subset G_n$, $\lim_{r\uparrow 1} \int_{G_n} f_{re}^{i\alpha}(\alpha) d\alpha = \lim_{r\uparrow 1} \mu(g_{re}^{i\alpha}(G_n)) = 2\pi$. There exists then $0 \leqslant r < 1$ and $\theta \in [0, 2\pi]$, such that $\mu(g_{re}^{i\alpha}(A)) \geqslant 2\pi - \varepsilon$. This finishes the proof of Lemma 2.

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