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ON THE DUALITY OF GENERALIZED GEOMETRIC PROGRAMMING

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Abstract

It is found that generalized geometric programming (GGP) is in fact a special case of generalized convex programming. By selecting a suitable bifunction and calculating its adjoint function, the dual form of the standard GGP problem is derived. Some duality theorems are also obtained with this point of view. The method used is simpler and more general than what appeared in the literature.

§1. Introduction

Generalized geometric program is a class of extremum problems defined as follows.

Let $g_k(x^k)$ be convex functions for $k\in\{0\}\cup I\cup J$, where I and J are two non-intersecting (possibly empty) positive integer sets with finite cardinality $|I|$ and $|J|$, respectively; $x^k\in R^{m_k}$. Denote the conjugate function of g_k by h_k, i. e.,

$$h_k(y^k)=g_k^*(y^k)\equiv\sup_{x^k\in R^{m_k}}\{\langle y^k,\ x^k\rangle-g_k(x^k)\}. \tag{1.1}$$

Denote by C_k and D_k the effetive domains of g_k and h_k, respectively:

$$C_k\equiv\text{dom } g_k,\quad D_k\equiv\text{dom } h_k. \tag{1.2}$$

For each $j\in J$, define

$$C_j^\dagger\equiv\{(x^j,\ K_j)\,|\,K_j=0 \text{ and } \sup_{d^j\in D_j}\langle x^j,\ d^j\rangle<\infty, \text{ or } K_j>0 \text{ and } x^j\in K_jC_j\}. \tag{1.3}$$

Let

$$x=(x^0,\ x^I,\ x^J)\in R^{\overline{m}},$$

where $\overline{m}=m_0+\sum_{i\in I}m_i+\sum_{j\in J}m_j$,

$$K=(K_1,\ \cdots,\ K_{|J|})\in R^{|J|},$$

and

$$C\equiv\{(x,\ K)\,|\,x^k\in C_k,\ k\in\{0\}\cup I \text{ and } (x^j,\ K_j)\in C_j^\dagger,\ j\in J\}.$$

Letting X be a closed convex cone in $R^{\overline{m}}$, we define

$$S\equiv\{(x,\ K)\,|\,(x,\ K)\in C,\ x\in X, \text{ and } g_i(x^i)\leqslant 0,\ i\in I\}.$$

Furthermore, define functions

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$$g_j^\dagger(x^j, K_j) \equiv \begin{cases} \sup\limits_{d^j \in D_j} \langle x^j, d^j \rangle \text{ if } K_j = 0 \text{ and } (x^j, K_j) \in C_j^\dagger, \\ K_j g_j(x^j/K_j) \text{ if } K_j > 0 \text{ and } (x^j, K_j) \in C_j^\dagger, \\ +\infty \quad \text{otherwise} \end{cases} \tag{1.4}$$

and

$$G(x, K) \equiv g_0(x^0) + \sum_{j \in J} g_j^\dagger(x^j, K_j). \tag{1.5}$$

The extremum problem

(P) $$\phi = \inf_{(x,K) \in S} G(x, K)$$

is called a generalized geometric programming problem, and is expressed as GGP for short in this paper.

GGP is a class of important optimization problems, which includes many typical extremun problems as its special cases, such as the minimization of signomials, quadratic programming, best l_p-norm approximation, optimal location, multicommodity transportation networks, minimum discrimination information, discrete optimal control with linear dynamics, dynamic programming with linear transition equations and so on (see [1] and [2] for detail). Peterson has made a systematic study on GGP (see [3—6]). One of the very important properties is that the dual of GGP has nice symmetry——it is still a GGP problem. As shown by Peterson, the dual of problem (P) is

$$\text{(D)} \quad \psi = \inf_{(y,\lambda) \in T} H(y, \lambda),$$

where

$$D_i^+ \equiv \{(y^i, \lambda_i) \mid \lambda_i = 0 \text{ and } \sup_{c^i \in C_i} \langle y^i, c^i \rangle < +\infty, \text{ or } \lambda_i > 0 \text{ and } y^i \in \lambda_i D_i\}, \tag{1.6}$$

$$D \equiv \{(y, \lambda) \mid y^k \in D_k,\ k \in \{0\} \cup J, \text{ and } (y^i, \lambda_i) \in D_i^+,\ i \in I\},$$

$$Y \equiv \{y \mid \langle x, y \rangle \geqslant 0,\ \forall x \in X\},$$

$$T \equiv \{(y, \lambda) \mid (y, \lambda) \in D,\ y \in Y \text{ and } h_j(y^j) \leqslant 0,\ j \in J\},$$

$$h_i^+(y^i, \lambda_i) \equiv \begin{cases} \sup\limits_{c^i \in C_i} \langle y^i, c^i \rangle \quad \text{if } \lambda_i = 0 \text{ and } (y^i, \lambda_i) \in D_i^+, \\ \lambda_i h_i(y_i/\lambda_i) \quad \text{if } \lambda_i > 0 \text{ and } (y^i, \lambda_i) \in D_i^+ \\ +\infty \text{ otherwise}, \end{cases} \tag{1.7}$$

$$H(y, \lambda) \equiv h_0(y^0) + \sum_{i \in I} h_i^+(y^i, \lambda_i). \tag{1.8}$$

We found that in fact the duality property of GGP can be derived as a special case of the general generalized convex programming theory. More exactly, if we select a suitable bifunction, express problem (P) as a generalized convex program associated with this bifunction, and find its adjoint bifunction, then the generalized convex program associated with the adjoint bifunction is just problem (D). This method is much simpler and more general, and also can be used to derive dual for some problems which are more complicated than standard GGP. Furthermore, with this point of view, the proof of the duality theorem of GGP will be clearer and simpler.

In section 2 we construct a bifunction F corresponding to problem (P) and derive dual problem (D) based on F. In section 3 we prove a strong duality theorem between problems (P) and (D). The notation for bifunction and adjoint function in this paper is similar to that used in [7] of Rockafellar.

§ 2. Dual Problem Derivation

We introduce bifunction F from R^t to $R^{\overline{m}}$:

$$(FW)(x, K) \equiv g_0(x^0 - z^0) + \sum_J g_j^\dagger(x^j - z^j, K_j - \eta_j) + \delta((x, K) \mid g_i(x^i - z^i) \leqslant \mu_i,\ i \in I) + \delta((x, K) \mid x + \xi \in X), \tag{2.1}$$

where $\delta(.\mid C)$ is the indicator function of set C, i. e.,

$$\delta(x \mid C) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \overline{\in} C, \end{cases} \tag{2.2}$$

and

$$\begin{aligned} W &= (z, \xi, \mu, \eta) \in R^t, \\ z &= (z^0, z^I, z^J) \in R^{\overline{m}}, \\ \xi &= (\xi^0, \xi^I, \xi^J) \in R^{\overline{m}}, \\ \mu &= (\mu_1, \cdots, \mu_{|I|}) \in R^{|I|}, \\ \eta &= (\eta_1, \cdots, \eta_{|J|}) \in R^{|J|}, \\ t &= 2\overline{m} + |I| + |J|. \end{aligned}$$

It is easy to see that now problem (P) can be expressed as

(P′) $\inf\ (F0)(x, K)$.

By Section 30 of [7], the adjoint bifunction of F is defined as

$$\begin{aligned} (F^*(x^*, K^*))(W^*) = \inf_{W,x,K} \{ &(FW)(x, K) - \langle x^0, x^{*0}\rangle - \sum_J \langle x^j, x^{*j}\rangle - \sum_I \langle x^i, x^{*i}\rangle \\ &- \sum_J K_j K_j^* + \langle z^0, z^{*0}\rangle + \sum_J \langle z^j, z^{*j}\rangle + \sum_I \langle z^i, z^{*i}\rangle \\ &+ \langle \xi, \xi^*\rangle + \sum_J \eta_j \eta_j^* + \sum_I \mu_i \mu_i^* \}, \end{aligned} \tag{2.3}$$

where

$$W^* = (z^*, \xi^*, \mu^*, \eta^*).$$

If there is a $\mu_i^* < 0\ (i \in I)$, then we can let $\mu_i \to +\infty$ which keeps the value of $(FW)(x, K)$ finite, so that $\mu_i \mu_i^* \to -\infty$, indicating that

$$(F^*(x^*, K^*))(W^*) = -\infty. \tag{2.4}$$

Because the dual problem will be $\sup\ (F^*(x^*, K^*))(W^*)$, obviously we can ignore this case, i. e., in what follows, we only need to consider the case

$$\mu_i^* \geqslant 0,\ \forall\ i \in I. \tag{2.5}$$

On the other hand, if $\xi^* \overline{\in} Y$, then there exists $v \in X$ such that

$$\langle \xi^*, v\rangle < 0.$$

Taking a feasible solution of problem (P): $(x, K)\in S$ and taking $z=0$, $\eta=0$, $\mu_i\geqslant g_i(x^i)$ for $i\in I$, and

$$\xi_\lambda=\lambda v-x \quad (\lambda>0)$$

so that $x+\xi_\lambda\in X$, we see that for this pair of (x, K) and $W=(z, \xi_\lambda, \mu, \eta)$, $(FW)(x, K)$ is finite.

But

$$\langle\xi_\lambda, \xi^*\rangle=\lambda\langle v, \xi^*\rangle-\langle x, \xi^*\rangle\to-\infty \quad (\lambda\to+\infty).$$

The above facts and (2.3) indicate that when $\xi^*\in Y$,

$$(F^*(x^*, K^*))(W^*)=-\infty. \tag{2.6}$$

Therefore, we only need to consider the case

$$\xi^*\in Y. \tag{2.7}$$

In fact when $\xi^*\in Y$, we have $\langle v, \xi^*\rangle\geqslant 0$ for any v in X. Thus

$$\min_{\xi\in X-x}\langle\xi, \xi^*\rangle=\min_{v\in V}\langle v-x, \xi^*\rangle=\min_{v\in X}\langle v, \xi\rangle-\langle x, \xi^*\rangle=-\langle x, \xi^*\rangle \tag{2.8}$$

and the minimum is attained at $\xi=-x(v=0)$.

By (2.1), (2.3) and (2.8),

$$\begin{aligned}(F^*(x^*, K^*))(W^*)=\inf_{W,x,K}\{&g_0(x^0-z^0)-\langle x^0, x^{*0}\rangle+\langle z^0, z^{*0}\rangle-\langle x^0, \xi^{*0}\rangle\\&+\sum_J[g_j^+(x^j-z^j, K_j-\eta_j)-\langle x^j, x^{*j}\rangle-K_jK_j^*\\&+\langle z^j, z^{*j}\rangle-\langle x^j, \xi^{*j}\rangle+\eta_j\eta_j^*]\\&+\sum_I[-\langle x^i, x^{*i}\rangle+\langle z^i, z^{*i}\rangle-\langle x^i, \xi^{*i}\rangle+\mu_i^*g_i(x^i-z^i)]\}.\end{aligned} \tag{2.9}$$

Let

$$\begin{aligned}b_0&\equiv\inf_{x^0,z^0}\{g_0(x^0-z^0)-\langle x^0, x^{*0}\rangle+\langle z^0, z^{*0}\rangle-\langle x^0, \xi^{*0}\rangle\}\\&=\inf_{x^0,z^0}\{g_0(x^0-z^0)-\langle x^0-z^0, z^{*0}\rangle+\langle x^0, z^{*0}-x^{*0}-\xi^{*0}\rangle\}\\&=-\sup_{x^0-z^0}\{\langle x^0-z^0, z^{*0}\rangle-g_0(x^0-z^0)\}+\inf_{x^0}\langle x^0, z^{*0}-x^{*0}-\xi^*\rangle\\&=\begin{cases}-h_0(z^{*0}) & \text{if } z^{*0}-x^{*0}-\xi^{*0}=0,\ z^{*0}\in D_0,\\-\infty & \text{otherwise,}\end{cases}\end{aligned} \tag{2.10}$$

$$\begin{aligned}b_i&\equiv\inf_{x^i,z^i}\{\mu_i^*g_i(x^i-z^i)-\langle x^i, x^{*i}\rangle+\langle z^i, z^{*i}\rangle-\langle x^i, \xi^{*i}\rangle\}\\&=\inf_{x^i-z^i}\{\mu_i^*g_i(x^i-z^i)-\langle x^i-z^i, z^{*i}\rangle\}+\inf_{x^i}\langle x^i, z^{*i}-x^{*i}-\xi^{*i}\rangle\}.\end{aligned} \tag{2.11}$$

Obviously,

$$\inf_{x^i}\langle x^i, z^{*i}-x^{*i}-\xi^{*i}\rangle=\begin{cases}0 & \text{if } z^{*i}-x^{*i}-\xi^{*i}=0,\\-\infty & \text{otherwise.}\end{cases} \tag{2.12}$$

If $\mu_i^*>0$, then

$$\begin{aligned}\inf_{x^i-z^i}\{\mu_i^*g(x^i-z^i)-\langle x^i-z^i, z^{*i}\rangle\}&=-\sup_{x^i-z^i}\{\langle x^i-z^i, z^{*i}\rangle-(\mu_i^*g_i)(x^i-z^i)\}\\&=-(\mu_i^*g^i)^*(z^{*i})=-(h_i\mu_i^*)(z^{*i})\\&=\begin{cases}-\mu_i^*h_i(z^{*i}/\mu_i^*) & \text{if } z^{*i}\in\mu_i^*D_i,\\-\infty & \text{otherwise.}\end{cases}\end{aligned} \tag{2.13}$$

If $\mu_i^*=0$, then

$$\inf_{x^i-z^i}\{\mu_i^* g_i(x^i-z^i)-\langle x^i-z^i,\ z^{*i}\rangle\}=-\sup_{x^i-z^i}\langle x^i-z^i,\ z^{*i}\rangle.$$

When deriving (2.9) from (2.3), we take $\mu_i=g_i(x_i-z_i)$, which means $x^i-z^i\in C_i$ and hence

$$\sup_{x^i-z^i}\{\langle x^i-z^i,\ z^{*i}\rangle\}=\sup_{c^i\in C_i}\{\langle c^i,\ z^{*i}\rangle\}. \tag{2.14}$$

Combining (2.13) and (2.14), we see that

$$-\inf_{x^i-z^i}\{\mu_i^* g_i(x^i-z^i)-\langle x^i-z^i,\ z^{*i}\rangle\}$$
$$=\begin{cases}\mu_i^* h_i(z^{*i}/\mu^{*i}) & \text{if } \mu^{*i}>0 \text{ and } z^{*i}\in D_i,\\ \sup\limits_{c^i\in C_i}\langle z^{*i},\ c^i\rangle & \text{if } \mu^{*i}=0 \text{ and } \sup\limits_{c_i\in C_i}\langle c^i,\ z^{*i}\rangle<+\infty,\\ +\infty & \text{otherwise.}\end{cases}$$

By defining

$$D_i^+\equiv\{(z^{*i},\ \mu_i^*)\mid \mu_i^*=0 \text{ and } \sup_{c^i\in C_i}\langle z^{*i},\ x^i\rangle<+\infty;\ \text{or } \mu_i^*>0 \text{ and } z^{*i}\in\mu_i^* D_i\} \tag{2.15}$$

and

$$h_i^+(z^{*i},\ \mu_i^*)\equiv\begin{cases}\sup\limits_{c^i\in C_i}\langle z^{*i},\ c^i\rangle & \text{if } \mu_i^*=0 \text{ and } (z^{*i},\ \mu_i^*)\in D_i^+,\\ \mu_i^* h_i(z^{*i}/\mu_i^*) & \text{if } \mu_i^*>0 \text{ and } (z^{*i},\ \mu_i^*)\in D_i^+\end{cases} \tag{2.16}$$

for each $(z^{*i},\ \mu_i^*)\in D_i^+$, we obtain

$$-\inf_{x^i-z^i}\{\mu_i^* g_i(x^i-z^i)-\langle x^i-z^i,\ z^{*i}\rangle\}=\begin{cases}h_i^+(z^{*i},\ \mu_i^*) & \text{if } (z^{*i},\ \mu_i^*)\in D_i^+\\ +\infty & \text{otherwise}\end{cases} \tag{2.17}$$

and therefore, for each $i\in I$,

$$-b_i=\begin{cases}h_i^+(z^{*i},\ \mu_i^*) & \text{if } z^{*i}-\xi^{*i}=x^{*i} \text{ and } (z^{*i},\ \mu_i^*)\in D_i^+,\\ +\infty & \text{otherwise.}\end{cases} \tag{2.18}$$

We now condsider

$$b_j\equiv\inf_{x^j,z^j,K_j,\eta_j}\{g_j^+(x^j-z^j,\ K_j-\eta_j)-\langle x^j,\ x^{*j}\rangle-K_jK_j^*-\langle z^j,\ z^{*}\rangle-\langle x^j,\ \xi^{*}\rangle+\eta_j\eta_j^*\}. \tag{2.19}$$

As in (2.19) an infimum is wanted, we only need to observe the case $K_j-\eta_j\geqslant 0$ (otherwise the first term in the braces will be $+\infty$). For notational simplicity, let

$$t^j=x^j-z^j,\quad \tau_j=K_j-\eta_j,\quad \forall j\in J.$$

Then we have

$$b_j=\inf_{t^j,\tau_j,z^j,\eta_j}\{g_j^+(t^j,\ \tau_j)-\langle t^j+z^j,\ x^{*j}+\xi^{*j}\rangle-(\tau_j+\eta_j)K_j^*+\langle z^j,\ z^{*}\rangle+\eta_j\eta_j^*\}$$
$$=\inf_{t^j,\tau_j}\{g_j^+(t^j,\ \tau_j)-\langle t^j,\ x^{*j}+\xi^{*j}\rangle-\tau_jK_j^*\}$$
$$+\inf_{z^j}\langle z^j,\ z^{*j}-x^{*j}-\xi^{*j}\rangle+\inf_{\eta_j}\eta_j(\eta_j^*-K_j^*). \tag{2.20}$$

Clearly,

$$\inf_{z^j}\langle z^j,\ z^{*j}-x^{*j}-\xi^{*j}\rangle=\begin{cases}0 & \text{if } z^{*j}-\xi^{*j}-x^{*j}=0\\ -\infty & \text{otherwise,}\end{cases} \tag{2.21}$$

$$\inf_{\eta_j} \eta_j(\eta_j^* - K_j^*) = \begin{cases} 0 & \text{if } \eta_j^* - K^{*j} = 0, \\ -\infty & \text{otherwise,} \end{cases} \tag{2.22}$$

$$\begin{aligned} &\inf_{t^j, \tau_j} \{g_j^\dagger(t^j, \tau_j) - \langle t^j, x^{*j} + \xi^{*j} \rangle - \tau_j K_j^*\} \\ &\quad = -\sup_{\tau_j \geq 0} \{\tau_j K_j^* + \sup_{t^j} \{\langle t^j, x^{*j} + \xi^{*j} \rangle - g_j^\dagger(t^j, \tau_j)\}\}, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} &\sup_{t^j} \{\langle t^j, x^{*j} + \xi^{*j} \rangle - g_j^\dagger(t^j, \tau_j)\} \\ &= \begin{cases} \sup_{t^j} \{\langle t^j, x^{*j} + \xi^{*j} \rangle - \sup_{d^j \in D_j} \langle t^j, d^j \rangle\} & \text{if } \tau_j = 0, \\ \sup_{t^j} \{\langle t^j, x^{*j} + \xi^{*j} \rangle - (g_j \tau_j)(t^j)\} & \text{if } \tau_j > 0, \end{cases} \\ &= \begin{cases} 0 & \text{if } x^{*j} + \xi^{*j} \in D_j,\ \tau_j = 0, \\ +\infty & \text{if } x^{*j} + \xi^{*j} \in D_j,\ \tau_j = 0, \\ \tau_j h_j(x^{*j} + \xi^{*j}) & \text{if } x^{*j} + \xi^{*j} \in D_j,\ \tau_j > 0, \\ +\infty & \text{if } x^{*j} + \xi^{*j} \in D_j,\ \tau_j > 0. \end{cases} \end{aligned} \tag{2.24}$$

The first conclusion of (2.24) is obvious as we can take $t^j = 0$; the second one is derived by separation property of convex sets. Notice that as h_j is a closed convex function, its effective domain D_j is a closed convex set. By [7], for $\tau_j > 0$, $(g_j \tau_j)(t^j) \equiv \tau_j g_j(t^j/\tau_j)$ and $(g_j \tau_j)^*(.) = \tau_j g_j^*(.)$, thus the third and the fourth conclusions of (2.24) are obtained according as $x^{*j} + \xi^{*j}$ belongs to the effective domain of h_j or not.

(2.24) tells us that to guarantee that this supremum value is finite, we must have

$$x^{*j} + \xi^{*j} \in D_j \tag{2.25}$$

and when (2.25) holds,

$$\sup_{t^j} \{\langle t^j, x^{*j} + \xi^{*j} \rangle - g_j^\dagger(t^j, \tau_j)\} = \tau_j h_j(x^{*j} + \xi^{*j}),$$

from which we see that

$$\sup_{t^j, \tau_j} \{\langle t^j, x^{*j} + \xi^{*j} \rangle + \tau_j K_j^* - g_j^\dagger(t^j, \tau_j)\} = \sup_{\tau_j \geq 0} \{\tau_j (K_j^* + h_j(x^{*j} + \xi^{*j}))\}.$$

Substituting this result into (2.23), we obtain

$$\begin{aligned} \inf_{t^j, \tau_j} \{g_j^\dagger(t^j, \tau_j) - \langle t^j, x^{*j} + \xi^{*j} \rangle - \tau_j K_j^*\} &= -\sup_{\tau_j \geq 0} \{\tau_j (K_j^* + h_j(x^{*j} + \xi^{*j}))\} \\ &= \begin{cases} 0 & \text{if } K_j^* + h_j(x^{*j} + \xi^{*j}) \leq 0 \\ -\infty & \text{otherwise.} \end{cases} \end{aligned} \tag{2.26}$$

Combining (2.20), (2.21), (2.22) and (2.26), and noticing (2.25), we see that

$$-b_j = \begin{cases} 0 & \text{if } z^{*j} - \xi^{*j} = x^{*j},\ \eta_j^* = K_j^*,\ x^{*j} + \xi^{*j} \in D_j, \\ & \quad \text{and } K_j^* + h(x^{*j} + \xi^{*j}) \leq 0, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.27}$$

By (2.9), (2.10), (2.18), (2.27), (2.5) and (2.7), we obtain the adjoint bifunction of F:

$$(F^*(x^*, K^*))(W^*) = b_0 + \sum_I b_i + \sum_J b_j$$

$$= \begin{cases} -h_0(z^{*0}) - \sum_I h_i^+(z^{*i}, \mu_i^*) & \text{if } z^{*0} \in D_0;\ \mu_i^* \geqslant 0,\ (z^{*i}, \mu_i^*) \in D_i^+, \\ & i \in I;\ x^{*j} + \xi^{*j} \in D_j,\ \eta_j^* = K_j^* \text{ and} \\ & K_j^* + h_j(x^{*j} + \xi^{*j}) \leqslant 0,\ j \in J \\ & z^* - \xi^* = x^* \text{ and } \xi^* \in Y, \\ -\infty & \text{otherwise.} \end{cases} \tag{2.28}$$

By section 30 of [7], we know that the dual of problem (P′) is

$$\sup_{W^*} (F^*(0, 0))(W^*)$$

or equivalently,

$$\inf_{W^*} -(F^*(0, 0))(W^*). \tag{2.29}$$

Setting $x^* = 0$, $K^* = 0$ in (2.28) and hence $z^* = \xi^*$, we can write problem (2.29) as

$$\inf \{h_0(z^{*0}) + \sum_I h_i^+(z^{*i}, \mu_i^*) \mid z^{*0} \in D_0,\ (z^{*i}, \mu_i^*) \in D_i^+,\ \mu_i^* \geqslant 0,\ i \in I,$$
$$z^{*j} \in D_j,\ h_j(z^{*j}) \leqslant 0,\ j \in J,\ z^* \in Y\}. \tag{2.30}$$

It is easy to find that the above extremum problem is exactly the problem (D) in Section 1 if we replace z^* by y, and μ_i^* by λ_i.

The method used in the above derivation is more direct and more general than what used in [5] and [6]. This method can also be employed to derive dual problems of some more complicated convex programs.

§ 3. Duality Theorems

Peterson gave a duality theorem for constrained GGP, the proof of which is divided into two parts, i. e., [5] and [6]. Here we give some different results based on generalized convex programming theory.

Theorem 3.1 *If there exists* $(\bar{x}, \bar{K})$ *such that*

$$\bar{x}^0 \in \text{ri } C_0, \tag{3.1a}$$

$$(\bar{x}^j, \bar{K}_j) \in \text{ri } C_j^+,\ j \in J, \tag{3.1b}$$

$$\bar{x}^i \in \text{ri } C_i \text{ and } g_i(\bar{x}^i) < 0,\ i \in I, \tag{3.1c}$$

$$\bar{x} \in \text{ri } X, \tag{3.1d}$$

then GGP problem (P) *and its dual* (D) *have no gap, i. e.,* $\phi + \psi = 0$.

Also, if problem (D) *is feasible, it must have an optimal solution.*

Proof We only need to prove that for the bifunction F given by (2.1),

$$0 \in \text{ri dom } F, \tag{3.2}$$

because (3.2) means that problem (P) is strongly consistent (see Section 29 of [7]), and therefore by Theorem 30.4 of [7], program (P) is normal, which means that the two conclusions of this theorem holds.

In order to prove (3.2), by Theorem 6.4 of [7], it suffices to prove that for any $W=(z, \xi, \mu, \eta)\in$ dom F, there exists $\rho>1$ such that

$$(1-\rho)W+\rho\, 0=(1-\rho)W\in \text{dom } F. \tag{3.3}$$

In fact $W=(z, \xi, \mu, \eta)\in$ dom F implies $\text{dom}(FW)\neq\phi$, i. e., there is (x, K) satisfying

$$x^0-z^0\in C_0, \tag{3.4a}$$

$$(x^j-z^j, K_j-\eta_j)\in C_j^\dagger,\ j\in J, \tag{3.4b}$$

$$g_i(x^i-z^i)\leqslant\mu_i,\ i\in I, \tag{3.4c}$$

$$x+\xi\in X. \tag{3.4d}$$

Because of (3.1a) and (3.4a), according to Theorem 6.4 of [7], there exists $\rho_0>1$ such that if $1<\rho\leqslant\rho_0$, then

$$(1-\rho)(x^0-z^0)+\rho\bar{x}^0\in C_0$$

or equivalently,

$$[(1-\rho)x^0+\rho\bar{x}^0]-(1-\rho)z^0\in C_0. \tag{3.5a}$$

Similarly, by (3.1b) and (3.4b), there are $\rho_j>1$, $j\in J$, such that when $1<\rho\leqslant\rho_j$,

$$(1-\rho)(x^j-z^j, K_j-\eta_j)+\rho(\bar{x}^j, \overline{K}_j)\in C_j^\dagger$$

or

$$[(1-\rho)(x^j, K_j)+\rho(\bar{x}^j, \overline{K}_j)]-(1-\rho)(z^j, \eta_j)\in C_j^\dagger. \tag{3.5b}$$

By (3.1d) and (3.4d), there is $\bar{\rho}>1$ such that

$$(1-\rho)(x+\xi)+\rho\bar{x}\in X$$

or equivalently

$$[(1-\rho)x+\rho\bar{x}]+(1-\rho)\xi\in X \tag{3.5d}$$

for $1<\rho\leqslant\bar{\rho}$.

As $g_i(\bar{x}^i)<0$, there exists $\rho_i'>1$ for each $i\in I$ such that when $1<\rho\leqslant\rho_i'$,

$$(1-\rho)\mu_i>g_i(\bar{x}^i)/2.$$

On the other hand, since convex function g_i is continuous at relative interior point $\bar{x}^i$, there is $\rho_i''>1$ such that when $1<\rho\leqslant\rho_i''$,

$$|g_i((1-\rho)(x^i-z^i)+\rho\bar{x}^i)-g_i(\bar{x}^i)|<-g_i(\bar{x}^i)/2.$$

From the above two inequalities we know that if $1<\rho\leqslant\rho_i\equiv\min\{\rho_i', \rho_i''\}$, then

$$g_i((1-\rho)(x^i-z^i)+\rho\bar{x}^i)<g_i(\bar{x}^i)/2<(1-\rho)\mu_i$$

or equivalently,

$$g_i([(1-\rho)x^i+\rho\bar{x}^i]-(1-\rho)z^i)<(1-\rho)\mu_i. \tag{3.5c}$$

Combining (3.5a)—(3.5d), we see that when $1<\rho\leqslant\min_{k\in(0)\cup I\cup J}\{\bar{\rho}, \rho_k\}$, for the perturbation vector

$$(1-\rho)W=(1-\rho)(z, \xi, \mu, \eta)$$

there exists

$$(x^*, K^*) \equiv (1-\rho)(x, K) + \rho(\bar{x}, \bar{K})$$

such that

$$(F(1-\rho)W)(x^*, K^*) < -\infty.$$

Hence $(1-\rho)W \in \text{dom } F$, which proves (3.3) and (3.2).

Symmetrically, we have another duality theorem, the conditions of which are stated in terms of dual problem (D).

Theorem 3.2. *If there exists* $(\bar{y}, \bar{\lambda})$ *such that*

$$\bar{y}^0 \in \text{ri } D_0, \tag{3.6a}$$

$$(\bar{y}^i, \bar{\lambda}_i) \in \text{ri } D_i^+, \ i \in I, \tag{3.6b}$$

$$\bar{y}^j \in \text{ri } D_j \text{ and } h_j(\bar{y}^j) < 0, \ j \in J, \tag{3.6c}$$

$$\bar{y} \in \text{ri } Y, \tag{3.6d}$$

then GGP problem (P) *and its dual* (D) *have no gap, i. e.,* $\phi + \psi = 0$.

Also, if problem (P) *is feasible, it must have an optimal solution.*

Remark The conditions (3.6a)—(3.6d) of Theorem 3.2 are somehow different from what Petenson imposed in [6] for his strong duality theorem. His conditions are:

(1) there exists (y', λ') such that

$$h_j(y'^j) < 0, \quad j \in J;$$

(2) Problem (D) has a finite infimum ψ;

(3) there exists $(\bar{y}, \bar{\lambda})$ satisfying

$$\bar{y} \in \text{ri } Y,$$

$$\bar{y}^k \in \text{ri } D_k, \quad k \in \{0\} \cup J,$$

$$(\bar{y}^i, \bar{\lambda}_i) \in \text{ri } D_i, \quad i \in I.$$

Based on the generalized convex programming theory, we can obtain other duality theorems. For instance, by Theorem 30.4 of [7], we have

Theorem 3.3. *If*

(a) *the optimal solutions to* (P) *form a non-empty bounded set; or*

(b) *the optimal solutions to* (D) *form a non-empty bounded set,*

then $\phi + \psi = 0$.

Throughout this paper, we know that GGP problem is in fact a special case ofgeneralized convex programming. By using the tool of bifunction and its adjoint function, the study of the properties of GGP can be considerably eased.

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