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UNIQUENESS OF SOLUTIONS FOR HIGHER DIMENSIONAL QUASILINEAR DEGENERATE PARABOLIC EQUATION

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Abstract

This paper studies the uniqueness of generalized solutions for the first boundary value problem of the form

$$u_t = \Delta A(u, x, t) = \frac{\partial b^i(u, x, t)}{\partial x_i} + c(u, x, t),$$

where $A_u \geqslant 0$.

It is proved that if $u \in BV_x(Q_T)$ and $A(u, x, t)$ is strictly increasing with respect to u, the solution is unique.

§ 1. Introduction

In this paper, we cosider the uniqueness of generalized solutions for the first boundary value problem of the form

$$u_t = \Delta A(u, x, t) + \frac{\partial b^i(u, x, t)}{\partial x_i} + c(u, x, t) = 0 \text{ in } Q_T, \tag{1.1}$$

$$u(x, t) = \psi(x, t) \text{ on } \Sigma \times (0, T), \tag{1.2}$$

$$u(x, 0) = u_0, \text{ on } \Omega, \tag{1.3}$$

where $Q_T = \Omega \times (0, T)$, $\Omega \subset R^m$ is a smooth bounded region with boundary Σ, and $A_u(u, x, t) \geqslant 0$.

Equation (1.1) is a degenerate parabolic equation. It has been suggested as a mathematical model for a variety of physical problem, we shall not recall them here, but refer to [1], where the very extensive literature is summarized. The uniqueness of solution for $m=1$ is well established in [2, 3] The paper [4] proved the uniqueness of nonnegative solution under the following conditions

$$A'(0) = 0, \quad A'(s) > 0 \quad \text{if } s > 0.$$

It is well known that for $A'(s) \geqslant 0$, $m > 1$ the uniqueness of generalized solution of (1.1)—(1.3) is an openproblem[5]. In this paper, we shall give a sufficient condition such that the generalized solutions of (1.1)—(1.3) are unique.

We assume that

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H1. $A(u, x, t)\in C(R\times\bar{Q}_T)$, $A_u\in L^\infty(R\times Q_T)$ and, for every $(x, t)\in Q_T$, $A(u, x, t)$ is strictly increasing with respect to u;

H2. $b^i(u, x, t)\in C^1(R\times\bar{Q}_T)$, $c(u, x, t)$, $c_u(u, x, t)\in L^\infty(R\times Q_T)$, ψ, $u_0\in L^\infty$.

As in [5, 6], by $BV(Q_T)$, we mean the class of all functions with bounded variation on Q_T. In other words $u\in BV(Q_T)$ if and only if $u\in L^1(Q_T)$, and $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i}$, $i=1, 2, \cdots, m$, are regular measure with bounded variations on Q_T.

An equivalent definition of $BV(Q_T)$ is $u\in BV(Q_T)$ if and only if $u\in L^1(Q_T)$ and

$$\iint_{Q_T}|u(x_1+h_1, \cdots, x_m+h_m, t+h_{m+1})-u(x, t)|\leqslant K|h|$$

for some constant $K>0$ and any vector $h=(h_1, h_2, \cdots h_m, h_{m+1})$, where we set $u(x, t)=0$ if $(x, t)\overline{\in}\bar{Q}_T$ (see [6]).

As in [3], we say that a function $v(x, t)\in BV_x(Q_T)$ if $v(x, t)$ is integrable on Q_T and satisfies

$$\iint_{Q_T}|v(x+h, t)-v(x, t)|\,dx\,dt\leqslant K|h|$$

for some constant $K>0$, and any vector $h=(h_1, \cdots, h_m)$. Here we set $v(x, t)=0$ if $(x, t)\overline{\in}\bar{Q}_T$.

Clearly $BV(Q_T)\subset BV_x(Q_T)$ and the generalized derivatives of every function in $BV_x(Q_T)$ with respect to $x_i(i=1, 2, \cdots, m)$ are regular measures with bounded variations on Q_T (see [6]), but in general, the generalized derivative with respect to t is not.

By Fubini's Theorem and

$$\int_0^T\int_\Omega\frac{|u(x+h, t)-u(x, t)|}{|h|}\,dx\,dt\leqslant K,$$

we deduce that for almost all $t\in(0, T)$

$$\int_\Omega|u(x+h, t)-u(x, t)|\,dx\leqslant C(t)|h|.$$

Thus $u\in BV_x(Q_T)$ implies, for almost all $t\in(0, T)$, $u(\cdot, t)\in BV(\Omega)$.

Definition. *A function u is called a generalized solution of the problem* (1.1)—(1.3), *if*

1) $u(x, t)\in BV_x(Q_T)\cap L^\infty(Q_T)$, $A(u, x, t)\in W_2^1(Q_T)$;

2) *for any* $\phi\in C^2(\bar{Q}_T)$, $\phi=0$ *on* $\Sigma\times(0, T)$, $\phi(x, T)=0$, *u satisfies*

$$\iint_{Q_T}[u\phi_t+A(u, x, t)\Delta\phi-b^i(u, x, t)\phi_{x_i}-c(u, x, t)\phi]\,dx\,dt$$
$$=\int_0^T\int_\Sigma A(\psi, x, t)\phi_{x_i}n_i\,d\sigma\,dt-\int_\Omega u_0(x)\phi(x, 0)\,dx,$$

where $n=(n_1, n_2, \cdots, n_m)$ denote the outward pointing normal on Σ.

The existence of the generalized solution of (1.1)—(1.3) is well established in [7]. The uniqueness obtained here is as follows.

Theorem. *Suppose that the hypotheses* H1, H2 *hold. Then the solution of* (1.1)—(1.3) *is unique.*

Clearly the hypothesis **H1** permits, for every $(x, t)\in Q_T$, the set of zero points of $A_u(u, x, t)$ to have measure zero.

§ 2. Proof of Theorem

For fixed $t>0$, let Γ_u^t be the set of jump points of $u(\cdot, t)\in BV(\Omega)$; $\nu(x, t)$ be the normal of Γ_u^t at (x, t); $u^+(x, t)$, $u^-(x, t)$ be the approximate limits of $u(\cdot, t)$ at $(x, t)\in\Gamma_u^t$ with respect to $(\nu, y-x)>0$, $(\nu, y-x)<0$ respectively.

Set

$$\bar{u}=\frac{1}{2}(u^+ + u^+),\quad F^\delta(u, x, t)=\max\{F(u, x, t)-\delta, 0\},$$

$$\hat{f}(u)=\int_0^1 f(su^+ + (1-s)u^-)\,ds,$$

where $F(u, x, t)\in C(R\times\bar{Q}_T$, $f(u)\in C^1(R)$.

In order to prove Theorem, we need the following lemmas.

Lemma 2.1. *Let* $u,\ v\in BV_x(Q_T)\cap L^\infty(Q_T)$. *Then* $\bar{v}$ *is integrable on* Q_T *with respect to measures* $\frac{\partial u}{\partial x_i}$, $i=1, 2, \cdots, m$. *Moreover* $\int_\Omega \bar{v}\frac{\partial u}{\partial x_i}dx$ *are integrable on* $(0, T)$, *and*

$$\iint_{Q_T}\bar{v}\,\frac{\partial u}{\partial x_i}\,dx\,dt=\int_0^T\Big(\int_\Omega\bar{v}\,\frac{\partial u}{\partial x_i}\,dx\Big)dt.$$

The proof of Lemma 2.1 can be found in [3, Theorem 2.1 and Remark 2.1].

By Lemma 2.1 and measure theory, if $u,\ v\in BV_x(Q_T)$, the set $Q_T\cap(\bar{v}(x, t)>\delta\}$ is measurable with rerpect to $\frac{\partial u}{\partial x_i}$, $i=1, 2, \cdots, m$.

Lemma 2.2. *Suppote that* H_1 *holds and* u *is a solution of* (1.1)—(1.3). *Then for every* $\phi\in C^1(\bar{Q}_T)$, $\phi=0$ *on* $\Sigma\times(0, T)$, $\delta>0$ *and* $F(s, x, t)\in C^1(R\times\bar{Q}_T)$

$$\iint_{(|F|>\delta)\cap Q_T}\phi\,\frac{\partial F}{\partial x_i}\,dx\,dt=-\iint_{(|F|>\delta)\cap Q_T}(F-\delta\,\mathrm{sgn}\,F)\phi_{x_i}\,dx\,dt,\ i=1, 2, \cdots, m. \tag{2.1}$$

Proof Since $\frac{\partial A(u, x, t)}{\partial x_i}\in L^2(Q_T)$, there exists a set $S\subset(0, T)$ with mes $S=0$, such that

$$\frac{\partial A(u, x, t)}{\partial x_i}\in L^2(\Omega),\ t\in(0, T)\setminus S,\quad i=1, 2, \cdots, m.$$

Hence by [6], for all measurable set $E\subset\Gamma_u^t$, $t\in(0, T)\setminus S$ we have

$$\int_E\frac{\partial A(u, x, t)}{\partial x_i}\,dx=\int_E(A(u^+, x, t)-A(u^-, x, t))\nu_{x_i}dH_{m-1}=0, \tag{2.2}$$

where H_{m-1} denotes the $m-1$ dimensional Hausdorff measure. From (2.2) we can obtain for almost all $x\in\Gamma_u^t$

$$\nu_{x_i}(A(u^+, x, t)-A(u^-, x, t))=0. \tag{2.3}$$

Hence by $(\nu_{x_1}, \nu_{x_2}, \cdots, \nu_{x_m})\neq 0$ and hypothesis H1 we get from (2.3)

$$u^+=u^-$$

and

$$F^+(u, x, t)=F^-(u, x, t) \quad \text{for almost all } x\in\Gamma_u^t.$$

Thus for any $f(s)\in C^1(R)$ and almost all $t\in(0, T)$,

$$\frac{\partial}{\partial x_i} f(u(\cdot, t))=\hat{f}_u(u(\cdot, t))\frac{\partial u}{\partial x_i}=f_u(u(\cdot, t))\frac{\partial u}{\partial x_i}.$$

Since $F^\delta(u, x, t)\in BV_x(Q_T)$ and $F^\delta(u^+, x, t)=F^\delta(u^-, x, t)$, H_{m-1} a. e. on Ω, we have for almost all $t\in(0, T)$

$$\frac{\partial}{\partial x_i}\frac{F^\delta(u)}{F^\delta(u)+\epsilon}=\frac{\epsilon F^\delta(u)}{(F^\delta(u)+\epsilon)^2}\frac{\partial F^\delta}{\partial x_i}.$$

Hence by the theory of BV space and dominated convergence theorem, we have

$$\begin{aligned}\iint_{Q_T\cap(F>\delta)}\phi\frac{\partial F}{\partial x_i}\,dx\,dt&=\lim_{\epsilon\to0^+}\iint_{Q_T}\phi\frac{F^\delta}{F^\delta+\epsilon}\frac{\partial F}{\partial x_i}\,dx\,dt \\ &=-\lim_{\epsilon\to0^+}\iint_{Q_T}F^\delta\frac{F^\delta}{F^\delta+\epsilon}\phi_{x_i}\,dx\,dt \\ &\quad-\lim_{\epsilon\to0^+}\iint_{Q_T}\phi\frac{\epsilon F^\delta}{(F^\delta+\epsilon)^2}\frac{\partial F}{\partial x_i}\,dx\,dt \\ &=-\iint_{Q_T\cap(F>\delta)}(F-\delta)\phi_{x_i}\,dx\,dt.\end{aligned}\tag{2.4}$$

In analogous fashion we can prove

$$\iint_{Q_T\cap(F<-\delta)}\phi\frac{\partial F}{\partial x_i}\,dx=-\iint_{Q_T\cap(F<-\delta)}(F+\delta)\phi_{x_i}\,dx\,dt. \tag{2.5}$$

From (2.4) (2.5), we obtain (2.1).

Lemma 2. 3. *Suppose that the hypotheses of Lemma 2.2 hold. Then for every* $\phi\in C^1(\bar{Q}_T)$, $\phi(x, t)=0$ *on* $\Sigma\times(0, T)$, $F(u, x, t)\in C^1(R\times\bar{Q}_T)$

$$\lim_{\delta\to0_+}\iint_{Q_T\supset(|F|<\delta)}\phi\frac{\partial F}{\partial x_i}\,dx\,dt=0, \quad i=1, 2, \cdots, m.$$

Proof Because

$$\lim_{\delta\to0^+}\iint_{Q_T\subset(|F|<\delta)}\phi\frac{\partial F}{\partial x_i}\,dx\,dt=\iint_{Q_T\cap(F=0)}\phi\frac{\partial F}{\partial x_i}\,dx\,dt,$$

it is enough to prove

$$\iint_{Q_T\cap(F=0)}\phi\frac{\partial F}{\partial x_i}\,dx\,dt=0.$$

Clearly

$$\iint\limits_{Q_T\cap(F=0)} \phi \frac{\partial F}{\partial x_i} dx\,dx = \iint\limits_{Q_T} \phi \frac{\partial F}{\partial x_i} dx\,dt - \lim_{\eta\to 0^+} \iint\limits_{Q_T\cap(F>\eta)} \phi \frac{\partial F}{\partial x_i} dx$$

$$-\lim_{\eta\to 0^+} \iint\limits_{Q_T\cap(F<-\eta)} \phi \frac{\partial F}{\partial x_i} dx\,dt = I_1 - I_2 - I_3.$$

By Lemma 2.2

$$I_2 = -\lim_{\eta\to 0^+} \iint\limits_{Q_T\cap(F>\eta)} \phi_{x_i}(F-\eta)dx\,dt = -\iint\limits_{Q_T\cap(F>0)} \phi_{x_i} F\,dx\,dt;$$

$$I_3 = -\lim_{\eta\to 0^+} \iint\limits_{Q_T\cap(F<-\eta)} \phi_{x_i}(F+\eta)dx\,dt = -\iint\limits_{Q_T\cap(F<0)} \phi_{x_i} F\,dx\,dt.$$

Thus

$$I_2 + I_3 - = \iint\limits_{Q_T} \phi_{x_i} F\,dx\,dt = \iint\limits_{Q_T} \phi \frac{\partial F}{\partial x_i} dx\,dt = I_1$$

and Lemma 2.2 is proved.

Lemma 2. 4. *Let $F(u, x, t)\in C(R\times\bar{Q}_T)$ be strictly monotonous with respect to u. Then for every $\delta>0$, $M>0$ there exists a constant $K(\delta, M)>0$ such that if $|u_1|$, $|u_2|\le M$, $|u_1-u_2|>\delta$, then*

$$|F(u_1, x, t) - F(u_2, x, t)| > K(\delta, M),\ (x, t)\in Q_T.$$

The proof of Lmma 2.4 is simple, we omit it.

Proof of Theorem Suppose that the solution of (1.1)—(1.3) is not unique. Then there exist two solutions u_1, u_2 of (1.1)—(1.3) and $\chi\in C_0^\infty(Q_T)$ such that

$$\iint\limits_{Q_T} (u_1-u_2)\chi\,dx\,dt \ne 0. \tag{2.6}$$

By Definition of generalized solutions, we have

$$\iint\limits_{Q_T} (u_1-u_2)\left(\phi_t + \frac{A(u_1, x, t)-A(u_2, x, t)}{u_1-u_2}\Delta\phi + \frac{c(u_1, x, t)-c(u_2, x, t)}{u_1-u_2}\phi\right)dx\,dt$$

$$-\iint\limits_{Q_T} (b^i(u_1, x, t) - b^i(u_2, x, t))\phi_{x_i}\,dx\,dt = 0. \tag{2.7}$$

Denote

$$U^i(x, t) = b^i(u_1, x, t) - b^i(u_2, x, t),\quad i=1, 2, \cdots, m.$$

Then $U^i\in BV(Q_T)$ and $|U^i|>\delta$ implies that

$$|u_1-u_2| > C(\delta) > 0. \tag{2.8}$$

By Lemma 2.2

$$\iint\limits_{Q_T} (b^i(u_1, x, t) - b^i(u_2, x, t))\phi_{x_i}\,dx\,dt$$

$$= \iint\limits_{Q_T} U^i\phi_{x_i}\,dx\,dt = -\iint\limits_{Q_T} \phi \frac{\partial U^i}{\partial x_i} dx\,dt$$

$$= \iint\limits_{Q_T\cap(|U^i|>\delta)} (U^i - \delta\,\mathrm{sgn}\,U^i)\phi_{x_i}\,dx\,dt - \iint\limits_{Q_T\cap(|U^i|<\delta)} \phi \frac{\partial U^i}{\partial x_i} dx.$$

We take χ_δ^i such that

$$\chi_\delta^i=1 \text{ if } |U^i|>\delta; \quad \chi_\delta^i=0 \text{ if } |U^i|\leqslant\delta.$$

Let

$$B_\delta^i=\frac{\chi_\delta^i(U^i-\delta\operatorname{sgn}U^i)}{u_1-u_2}.$$

Then by (2.8) $|B_\delta^i|\leqslant K(\delta)$, $i=1, 2, \cdots, m$, and (2.7) can be rewritten as

$$\iint_{Q_T}(u_1-u_2)(\phi_t+\widetilde{A}\Delta\phi-B_\delta^i\phi_{x_i}+\widetilde{C}\phi)dx\,dt+\iint_{Q_T\cap(|U^i|\leqslant\delta)}\phi\frac{\partial U^i}{\partial x_i}dx\,dt=0, \tag{2.9}$$

where

$$\widetilde{A}=\frac{A(u_1, x, t)-A(u_2, x, t)}{u_1-u_2} \quad \widetilde{C}=\frac{C(u_1, x, t)-C(u_2, x, t)}{u_1-u_2}.$$

Set

$$D_\delta^i(x, t)=(\eta+\widetilde{A}_\epsilon)^{-1/2}B_\delta^i,$$

where $\eta>0$ and $\widetilde{A}_\epsilon$ is a positive C^∞ approximation of $\widetilde{A}$ such that

$$\iint_{Q_T}(\widetilde{A}_\epsilon-\widetilde{A})^2dx\,dt\leqslant\eta^3. \tag{2.10}$$

By the strictly monotonicity of $A(u, x, t)$ with respect to u,

$$|D_\delta^i|\leqslant D(\delta, M). \tag{2.11}$$

In fact, if $|U^i|\leqslant\delta$, (2.11) is obvious; if $|U^i|>\delta$, then by (2.8) and Lemma 2.4

$$\widetilde{A}=\frac{A(u_1, x, t)-A(u_2, x, t)}{u_1-u_2}\geqslant\frac{K(\delta, M)}{2M}.$$

We choose $D_{\delta h}^i$, $\widetilde{C}_h\in C^3(\overline{Q}_T)$ such that

$$|\widetilde{C}_h|, \{D_{\delta h}^i|\leqslant K(\delta), \lim_{h\to0^+}D_{\delta h}^i=D_\delta^i, \lim_{h\to0^+}\widetilde{C}_h=\widetilde{C} \text{ a. e. on } Q_T.$$

We consider the first boundary value problem

$$\phi_t(\eta+\widetilde{A}_\epsilon)\Delta\phi-D_{i\delta h}^i(\eta+\widetilde{A}_\epsilon)^{1/2}\phi_{x_i}+\widetilde{C}_h\phi=\chi \text{ in } Q_T, \tag{2.12}$$

$$\phi(x, t)=0 \text{ on } \Sigma\times(0, T), \tag{2.13}$$

$$\phi(x, T)=0. \tag{2.14}$$

By classical theory, (2.12)—(2.14) has a unique classical solution

$$|\phi|\leqslant N,$$

where N is a constant independent of η, ϵ, δ, h.

Below for simplicity of notation, we will denote the constants depending only on M δ by $K(\delta)$, and the constants independent of ϵ η h δ by C, although they may change from line to line in the proof.

We now prove that the solution of (2.12)—(2.14) satisfies

$$\iint_{Q_T}(\eta+\widetilde{A}_\epsilon)(\Delta\phi)^2dx\,dt\leqslant K(\delta)\eta^{-1}, \tag{2.15}$$

$$\iint_{Q_T}|\nabla\phi|^2dx\,dt\leqslant K(\delta)\eta^{-1}. \tag{2.16}$$

Multiplying (2.12) by $\Delta\phi$, integrating over Q_T we have

$$\iint_{Q_T}\{\phi_t\Delta\phi+(\widetilde{A}_\epsilon+\eta)(\Delta\phi)^2-D^i_{\delta h}(\eta+\widetilde{A}_\epsilon)^{1/2}\phi_{x_i}\Delta\phi+\widetilde{C}_h\phi\Delta\phi-\chi\Delta\phi\}dx\,dt=0. \quad (2.17)$$

Clearly

$$|D^i_{\delta h}(\eta+\widetilde{A}_\epsilon)^{1/2}\phi_{x_i}\Delta\phi|\leqslant\frac{1}{4}(\eta+\widetilde{A}_\epsilon)(\Delta\phi)^2+\sum_{i=1}^{m}(D^i_{\delta h})^2\phi^2_{x_i}, \quad (2.18)$$

$$\iint_{Q_T}\phi_t\Delta\phi\,dx\,dt=\frac{1}{2}\int_\Omega|\nabla\phi(x,0)|^2\,dx>0, \quad (2.19)$$

$$\left|\iint_{Q_T}\widetilde{C}_h\phi\,\Delta\phi\,dx\,dt\right|\leqslant\frac{1}{4}\iint_{Q_T}(\eta+\widetilde{A}_\epsilon)(\Delta\phi)^2\,dx\,dt+\frac{C}{\eta}, \quad (2.20)$$

$$\left|\iint_{Q_T}\chi\,\Delta\phi\,dx\,dt\right|=\left|\iint_{Q_T}\Delta\chi\,\phi\,dx\right|\leqslant C, \quad (2.21)$$

$$\iint_{Q_T}|\nabla\phi|^2\,dx\,dt=-\iint_{Q_T}\phi\,\Delta\phi\,dx\,dt\leqslant\frac{C}{\alpha\eta}+\frac{\alpha}{4}\iint_{Q_T}(\eta+\widetilde{A}_\epsilon)(\Delta\phi)^2\,dx\,dt. \quad (2.22)$$

Substituting (2.18)—(2.22) into (2.17), we obtain (2.15). From (2.15) and (2.22) we obtain (2.16).

Substituting the solution of (2.12)—(2.14) into (2.9), we have

$$\begin{aligned}&\left|\iint_{Q_T}(u_1-u_2)\chi\,dx\,dt\right|\\&=\left|\iint_{Q_T}(u_1-u_2)\{\phi_t+(\eta+\widetilde{A}_\epsilon)\Delta\phi-D^i_{\delta h}(\eta+\widetilde{A}_\epsilon)^{1/2}\varphi_{x_i}+\widetilde{C}_h\phi\}dx\,dt\right|\\&\leqslant\left|\iint_{Q_T}(u_1-u_2)\eta\,\Delta\phi\,dx\,dt\right|+\left|\iint_{Q_T}(u_1-u_2)(\widetilde{A}_\epsilon-\widetilde{A})\Delta\phi\,dx\,dt\right|\\&\quad+\left|\iint_{Q_T}(u_1-u_2)(\widetilde{C}_h-\widetilde{C})dx\,dt\right|+\left|\iint_{Q_T}(u_1-u_2)\{D^i_{\delta h}(\eta+\widetilde{A}_\epsilon)^{1/2}-B^i_\delta\}dx\,dt\right|\\&\quad+\left|\iint_{Q_T\cap(|U^i|<\delta)}\phi\,\frac{\partial U^i}{\partial x_i}\,dx\,dt\right|.\end{aligned} \quad (2.23)$$

First by Lemma 2.4 and (2.15)

$$\iint_{Q_T\cap(|u_1-u_2|>\gamma)}(\Delta\phi)^2\,dx\,dt\leqslant K(\gamma)K(\delta)\eta^{-1},\ \gamma>0.$$

Thus

$$\begin{aligned}\left|\iint_{Q_T}\eta(u_1-u_2)\Delta\phi\,dx\,dt\right|&\leqslant C\left\{\iint_{Q_T}\eta^2(u_1-u_2)^2(\Delta\phi)^2\,dx\,dt\right\}^{1/2}\\&\leqslant C\left\{\iint_{Q_T\cap(|u_1-u_2|>\gamma)}(u_1-u_2)^2\eta^2(\Delta\phi)^2\,dx\,dt\right\}^{1/2}\\&\quad+C\left\{\iint_{Q_T\cap(|u_1-u_2|<\gamma)}(u_1-u_2)^2\eta^2(\Delta\phi)^2\,dx\,dt\right\}^{1/2}\\&\leqslant CK(\gamma)K(\delta)\eta^{1/2}+C\gamma K(\delta).\end{aligned} \quad (2.24)$$

Secondly by (2.10)' and (2.15)

$$\left|\iint_{Q_T}(\tilde{A}_\varepsilon-\tilde{A})\Delta\phi\,dx\,dt\right|\leqslant C\left(\iint_{Q_T}(\tilde{A}_\varepsilon-\tilde{A})^2\,dx\,dt\right)^{1/2}\left(\iint_{Q_T}(\Delta\phi)^2\,dx\,dt\right)^{1/2}\leqslant CK(\delta)\eta^{1/2}. \tag{2.25}$$

Moreover, we can choose h small enough such that

$$\left|\iint_{Q_T}(u_1-u_2)\{D^i_{\delta h}(\eta+\tilde{A}_\varepsilon)^{1/2}-B^i_\delta\}\phi_{x_i}\,dx\,dt\right|\leqslant\eta K(\delta), \tag{2.26}$$

$$\left|\iint_{Q_T}(u_1-u_2)(\tilde{C}_h-\tilde{C})dx\,dt\leqslant\eta K(\delta)\right.. \tag{2.27}$$

Last, by Lemma 2.3

$$\iint_{Q_T\cap(|U^i|<\delta)}\phi\frac{\partial U^i}{\partial x_i}\,dx\,dt\leqslant w(\delta), \tag{2.28}$$

where $w(\delta)$ is a nondecreasing function with $\lim_{\delta\to0^+}w(\delta)=0$.

Substituting (2.24)—(2.28) into (2.23), we obtain

$$\left|\iint_{Q_T}(u_1-u_2)\chi\,dx\,dt\right|\leqslant C(K(\gamma)K(\delta)\eta^{1/3}+\gamma K(\delta)+K(\delta)\eta^{1/2}+w(\delta)).$$

Hence we choose $\gamma=\delta(K(\delta))^{-1}$ and let $\eta\to0^+$ to obtain

$$\left|\iint_{Q_T}(u_1-u_2)\chi\,dx\,dt\right|\leqslant C(\delta+w(\delta)).$$

Let $\delta\to0^+$ to obtain

$$\iint_{Q_T}(u_1-u_2)\chi\,dx\,dt=0.$$

This contradicts (2.6) and Theorem is proved.

References

[1] Wu Zhuoqun, Degenerate quasilinear parabolic equations, *Advances in Mathematics*, 16: 2(1987), 121—158.

[2] Zhao Junning, Uniqueness of solutions of quasilinear degenerate parabolic equation, *Northeastern Math. J.*, 1: 2(1985), 153—165.

[3] Wu Zhuoqun & Yin Jingxue, Some properties of functions in BV_x and their applications to the uniqueness of solutions for degenerate quasilinear parabolic equations, *Northeastern Math. J.*, 5: 4(1989), 395—422.

[4] Wu Dequan & Jiang Lishang, On the boundary value problem of a multidimensional quasilinear equation, *Northeastern Math. J.*, 1: 1(1985), 54—67.

[5] Volpert, A. I. & Hudjaev, S. I., Cauchy problem of second order quasilinear degenerate parabolic equations, *Math. CB.*, 78: 120(1969), 374—396.

[6] Vol'pert, A. I., *BV* spaces and quasilinear equations, *Mat. Sb.*, 73: 115(1967), 255—302.

[7] Wu Zhuoqun & Zhao Junning, Some general result on the first boundary value problem for quasilinear degenerate paraboliceqution, *Chinese Annals of Math.*, 4B: 3(1983), 319—328.