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WEAKLY ALMOST PERIODIC POINT AND ERGODIC MEASURE**

ZHOU ZUOLING (周作领)*

Abstract

Let X be a compact metric space and f: X→X be continuous.

This pape introduces the notion of weakly almost periodic point, which is a generalization of the notion of almost periodic point, proves that each of f-invariant ergodic measures can be generated by a weakly almost periodic point of f and gives some equivalent conditions for that f has an invariant ergodic measure whose support is X and ones for that f has no non-atomic invariant ergodic measure, the latter is a generalization of the Blokh's work on self-maps of the interval. Also two formulae for calculating the togological entropy are obtained.

§1. Introduction

Let X be a compact metric space and $f: X\to X$ be continuous. When $X=[0, 1]$, [1] has announced the following

Theorem A. *The following* (i) *and* (ii) *are equivalent.*

(i) $R(f)=P(f)$, *that is, each recurrent point of f is periodic*,.

(ii) *f has no non-atomic invariant ergodic probability measure.*

For the general case, one may prove that in Theorem A (i) is only sufficient but not necessary for (ii). We hope to look for the necessary and sufficient condition for (ii) in Theorem A in general case. It involves the structure of ergodic measure and the levels of the set of recurrent points. [2] has introduced the notion of almost periodic point and proved that $x\in X$ is almost periodic iff $\overline{\mathrm{orb}(x)}=w(x, f)$ is a minimal set of f. It is easy to prove that the existence of a minimal set which is not a periodic orbit implies the existence of a non-atomic ergodic measure. Thus, that each almost periodic point is periodic is necessary for the non-existence of non-atomic ergodic measure. One may conjecture that there is such a subset of $R(f)$ that it coincides with $P(f)$ is a necessary and sufficient condition for (ii) in Theorem A. In this paper, we introduce the notion of weakly almost periodic point

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* Lingnan College, Zhongshan University, Guangzhou, Guangdong 510275, China.

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and give a characterization of the set mentioned above. We also obtain some other related results.

§ 2. Definitions and Lemmas

Let $f: X \to X$ be the same as in § 1. In the following we refer to [3].

Suppose that $\mathscr{B}(X)$ is the Borel σ-algebra of X. Denote by $M(X)$ the set of all probability measures on $\mathscr{B}(X)$, by $M(X, f)$ the set of all elements of $M(X)$ which are invariant for f and by $E(X, f)$ the set of all elements of $M(X, f)$ which are ergodic for f. $M(X)$ is convex compact metrizable under the weak-topology and $M(X) \supset M(X, f) \supset E(X, f) \neq \emptyset$. Each $x \in X$ determines a member δ_x of $M(X)$ defined by

$$\delta_x(A) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \bar{\in} A, \end{cases} \quad \text{for all } A \in \mathscr{B}(X).$$

$m \in M(X)$ is called atomic if there exist $x_i \in X$, $p_i \geqslant 0$, $i = 1, 2, \cdots$, with $\Sigma\, p_i = 1$ and $m = \Sigma\, p_i \delta_{x_i}$.

With respect to ergodic measure, we have

Theorem B[3]. *Let $m \in M(X, f)$. Then $m \in E(X, f)$ iff there is a $Y \in \mathscr{B}(X)$ with $m(Y) = 1$ such that*

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \to m \text{ for all } x \in Y.$$

As $m(R(f)) = 1$ for all $m \in M(X, f)$, the following corollary is clear.

Corollary. *Let $m \in E(X, f)$. Then there is $x \in R(f)$ such that*

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \to m$$

and the set of all such x has m-measure 1 and $m(w(x, f)) = 1$. In addition, when m is atomic, there is $x \in P(f)$ with the period N such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)} \to \frac{1}{N} \sum_{i=0}^{N-1} \delta_{f^i(x)} = m.$$

Recall[2] that $x \in X$ is called an almost periodic point of f if for any $\varepsilon > 0$ one may find $N > 0$ such that for any $q \geqslant 0$ there is an integer r with $q \leqslant r < N + q$ satisfying $f^r(x) \in V(x, \varepsilon)$, where $V(x, \varepsilon)$ denotes the ε-spherical neighborhood of x. Denote by $A(f)$ the set of all almost periodic points of f. It is easy to see that $P(f) \subset A(f) \subset R(f)$.

Definition 1. *$x \in X$ is called a weakly almost periodic point of f if for any $\varepsilon > 0$ one may find $N > 0$ such that $\#(\{r \mid f^r(x) \in V(x, \varepsilon),\ 0 \leqslant r < nN\}) \geqslant n$ for all $n \geqslant 0$, where $\#(\,.\,)$ denotes the cardinality.*

Denote by $W(f)$ the set of all weakly almost periodic points of f. It is easy to see that $A(f) \subset W(f) \subset R(f)$.

We shall prove that $A(.)\subsetneqq W(.)\subsetneqq R(.)$ are possible.

The proofs of the following Lemmas 1 and 2 are straightforward.

Lemma 1. $f(W(f))\subset W(f)$.

Lemma 2. *Let $x\in X$. Then $x\in W(f)$ iff for any $\varepsilon>0$,*

$$\liminf_{n\to\infty}\frac{1}{n}\#(\{r\,|\,f^r(x)\in V(x,\varepsilon),\ 0\leqslant r<n\})>0.$$

Definition 2. *Let $m\in M(X, f)$. A subset F of X is called the f-invariant minimal closed support of m if $f(F)\subset F$, $\bar{F}=F$, $m(F)=1$ and there is no any proper subset of F satisfying these conditions.*

Lemma 3. *Let $m\in M(X, f)$. Then the f-invariant minimal closed support of m exists uniquely.*

Proof Let $S_m=\{x\in X\,|\,m(V(x,\varepsilon))>0,\ \forall\,\varepsilon>0\}$. It is easy to prove that S_m is non-empty, closed and f-invariant. For each $x\in X-S_m$, there is $\varepsilon>0$ such that $m(V(x,\varepsilon))=0$ and $\alpha=\{V(x,\varepsilon)\,|\,m(V(x,\varepsilon))=0,\ \forall\,x\in X-S_m\}$ is an open cover of $X-S_m$. Since X is a Lindelof space satisfying the second countability axiom and so is $X-S_m$ (see [4]). Thus, α has a countable subcover and hence $m(X-S_m)=0$ and $m(S_m)=1$. It is easy to prove that S_m is the f-invariant minimal support of m. The uniqueness is evident.

Let $\mathscr{E}=\{F\subset X\,|\,f(F)\subset F,\ \bar{F}=F$ and $m(F)=1\}$ for $m\in M(X, f)$.

Lemmm 4. *Let $m\in M(X, f)$ and $F\in\mathscr{E}$. Then F is the f-invariant minimal closed support of m iff each non-empty open subset of F has positive m-measure.*

Proof Let F be the f-invariant minimal closed support of m and $U\subset F$ be non-empty and open. If $m(U)=0$, then $m\left(\bigcup_{n=0}^{\infty}f^{-n}(U)\right)=0$. Obviously, $F-\bigcup_{n=0}^{\infty}f^{-n}(U)$ is closed and invariant for f and $m\left(F-\bigcup_{n=0}^{\infty}f^{-n}(U)\right)=1$. It is easy to see that the f-invariant minimal closed support of m is contained in $F-\bigcup_{n=0}^{\infty}f^{-n}(U)$, a contradiction.

Now suppose that each non-empty open subset of F has positive m-measure. If $F_0\subsetneqq F$ is the f-invariant minimal closed support of m, then $F-F_0$ is non-empty and open, and so $m(F-F_0)>0$. This contradicts $m(F_0)=1$.

Lemma 5. *Let $m\in E(X, f)$ and F be the f-invariant minimal closed support of m. Then the restriction of f on F is topologically transitive, that is, there is $x\in F$ such that $\overline{\mathrm{orb}(x)}=F$.*

Proof By Corollary of Theorem B and $m(F)=1$, there is $x\in R(f)\cap F$ such that $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}\to m$. Clearly, $w(x, f)\subset F$. In the other hand, $w(x, f)$ is a f-invariant closed support of m and so $F\subset w(x, f)$. Thus, $F=w(x, f)$ and f is topologically transitive on F.

Lemma 6. *Let $m\in E(X, f)$ and $x\in R(f)$ with $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}\to m$. Then for any $\varepsilon>0$, $m(V(x, \varepsilon))>0$ iff $x\in W(f)$.*

Proof Because m is probability measure, it is easy to prove that the set $\{\varepsilon>0 \mid m(\partial V(x, \varepsilon))=0\}$ is every where dense on $(0, +\infty)$, where $\partial V(x, \varepsilon)$ denotes the boundary of $V(x, \varepsilon)$. Obviously, it suffices to prove Lemma 6 for $\varepsilon>0$ with $m(\partial V(x, \varepsilon))=0$. According to the property of weak convergence (see [3]), we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}(V(x, \varepsilon))=\frac{1}{n}\#(\{r \mid f^r(x)\in V(x, \varepsilon),\ 0\leqslant r<n\})\to m(V(x, \varepsilon)).$$

Clearly, Lemma 6 follows from Lemma 2.

§ 3. Theorems

Set

$$W_0(f)=\left\{x\in W(f) \,\middle|\, \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}\to m\in E(X, f)\right\}.$$

It is easy to check that $f(W_0(f))\subset W_0(f)$.

A subset $F\subset X$ is called an absolutely ergodic measure 1-set of f, if for each $m\in E(x, f)$ there is a subset E of F with $m(E)=1$.

Theorem 1. *$W_0(f)$ is an absolutely ergodic measure 1-set of f.*

Proof By the definition of $W_0(f)$ and Corollary of Theorem B, it is clear that $W_0(f)$ is an absolutely ergodic measure 1-set of f.

Theorem 2. $\mathrm{ent}(f)=\sup_{x\in W_0(f)}\{\mathrm{ent}(f|_{w(x,f)})\}=\mathrm{ent}(f|_{\overline{W_0(f)}})$, *where* ent (f) *denotes the topological entropy of f.*

Proof By the variational principle[3] $\mathrm{ent}(f)=\sup_{m\in E(X,f)}\{h_m(f)\}$, where $h_m(f)$ denotes the measure-theoretical entropy of f with respect to m, we have $h_m(f|_F)\leqslant \mathrm{ent}(f|_F)$, where $m\in E(X, f)$ and F is the f-invariant minimal closed support of m. By Corollary of Theorem B and Theorem 1, there is $x\in W_0(f)$ such that $F=w(x, f)$. Hence $h_m(f)\leqslant \sup_{x\in W_0(f)}\{\mathrm{ent}(f|_{w(x,f)})\}$ and so $\mathrm{ent}(f)=\sup_{m\in E(X,f)}\{h_m(f)\}\leqslant \sup_{x\in W_0(f)}\{\mathrm{ent}(f|_{w(x,f)})\}\leqslant \mathrm{ent}(f|_{\overline{W_0(f)}})\leqslant \mathrm{ent}(f)$.

Theorem 3. *Let $m\in E(X, f)$ and $x\in R(f)$ with $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(x)}\to m$. Then the following* (i)—(iii) *are equivalent.*

(i) *X is the f-invariant minimal closed support of m,*

(ii) *each non-empty open subset of X has positive m-measure,*

(iii) *$x\in W_0(f)$ with $w(x, f)=X$.*

Proof The proof of (i)⇒(ii) is similar to the one of Lemma 4.

(ii)⇒(iii) For $\varepsilon>0$, $m(V(x, \varepsilon))>0$. By Lemma 6, $x\in W(f)$ and so $x\in W_0(f)$.

$w(x, f) = X$ is clear, because if not, then $X - w(x, f)$ is a on-empty open set and so $m(X - w(x, f)) > 0$. It contradicts $m(w(x, f)) = 1$.

(iii)⇒(i) For any non-empty open subset U of X, by $w(x, f) = X$, there is r with $f^r(x) \in U$ and $V(f^r(x), \varepsilon) \subset U$ for some $\varepsilon > 0$. As $f^r(x) \in W_0(f) \subset W(f)$, so $m(U) > 0$ by Lemma 6. If $F \subsetneqq X$ is the f-invariant minimal closed support of m, then $m(X - F) > 0$, and it contradicts $m(F) = 1$.

Theorem 4. *The following* (i)—(iii) *are equivalent.*

(i) $W_0(f) = P(f)$,

(ii) $m(P(f)) = 1$ *for all* $m \in E(X, f)$,

(iii) *f has no non-atomic invariant ergodic probability measure.*

Proof (i)⇒(ii) It is clear by Theorem 1.

(ii)⇒(iii) Let $m \in E(X, f)$ and F be the f-invariant minimal closed support of m. By Theorem B, there is $x \in F \cap P(f)$ such that $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to m$. Clearly, $F = w(x, f)$ is a periodic orbit of f and m is atomic.

(iii)⇒(i) If there is $x \in W_0(f) - P(f)$, then $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to m \in E(X, f)$. By Theorem 3, $w(x, f)$ is the f-invariant minimal closed support of m. Evidently, $F \subsetneqq P(f)$ and so m can not be generated by a periodic point of f, that is, m is not atomic. It is a contradiction.

Next, let Σ_2 be the one sided symbolic space with two symbols 0 and 1 and σ: $\Sigma_2 \to \Sigma_2$ be the shift. The metric on Σ_2 is defined by

$$\rho(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}, \quad \forall\, x = (x_0 x_1 \cdots),\ y = (y_0 y_1 \cdots) \in \Sigma_2.$$

Theoem 5. $A(\sigma) \subsetneqq W(\sigma) \subsetneqq R(\sigma)$.

Proof Let m be the $\left(\frac{1}{2}, \frac{1}{2}\right)$-product measure on Σ_2. By [3], $m \in E(\Sigma_2, \sigma)$ and each non-empty open subset of Σ_2 has positive m-measure. By Theorem 3, there is $x \in W_0(\sigma)$ such that $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{\sigma^i(x)} \to m$ and $w(x, \sigma) = \Sigma_2$. As Σ_2 is not minimal, so $x \overline{\in} A(\sigma)$ (see § 1). This proves that $A(\sigma) \subsetneqq W(\sigma)$.

Let $M = (i_0 i_1 \cdots i_{n-1})$ and $N = (j_0 j_1 \cdots j_{m-1})$ be two finite sequences of $\{0, 1\}$ whose lengths are n and m, respectively. Denote $(MN) = (i_0 i_1 \cdots i_{n-1} j_0 j_1 \cdots j_{m-1})$ whose length is $n+m$. In the sequel, we form $x \in \Sigma_2$ with $x \in R(\sigma) - W(\sigma)$.

Let $P_1 = (01)$, $P_2 = (00011011)$ and inductively, for $k > 2$, P_k be a finite sequence formed by arranging all permutations of k symbols 0, 1 with repetition one after another in a line in some order, whose length is $2^k \cdot k$. Set

$$x = (P_1 Q_1 P_2 P_2 \cdots P_k Q_k \cdots) \in \Sigma_2,$$

where $Q_1 = (11)$, $Q_2 = (11 \cdots 1)$ with the length$=2$ times of the length of $(P_1 Q_1 P_2)$, and inductively, for $k>2$, $Q_k=(11\cdots1)$ with the length $=k$ times of the length of $(P_1Q_1\cdots P_k)$.

By the above construction, it is easy to see that $x\in R(\sigma)$ and $w(x, \sigma)=\Sigma_2$. Next, we prove $x\notin W(\sigma)$. By Lemma 2, it suffices to prove that

$$\liminf_{n\to\infty}\frac{1}{n}\#\left(\left\{r\,\middle|\,\sigma^r(x)\in V\left(x, \frac{1}{2}\right), 0\leqslant r<n\right\}\right)=0.$$

Note that if $y\in\Sigma_2$, then $y_0=1\Rightarrow y\overline{\in}V\left(x, \frac{1}{2}\right)$. Let $l(\,.\,)$ denote the length and $n_k=l((P_1Q_1\cdots P_kQ_k))$, $k=1, 2, \cdots$. It is easy to see that

$$\frac{1}{n_k}\#\left(\left\{r\,\middle|\,\sigma^r(x)\in V\left(x, \frac{1}{2}\right), 0\leqslant r<n_k\right\}\right)\leqslant\frac{l((P_1Q_1\cdots P_k))}{n_k}=\frac{l((P_1Q_1\cdots P_k))}{k\cdot l((P_1Q_1\cdots P_k))}$$

$$=\frac{1}{k}\to0 \text{ (as } k\to\infty).$$

We are done.

Finally, as stated in § 1, $A(f)=P(f)$ is necessary for f to have no non-atomic ergodic measure. But the author does not know whether it is sufficient also or not. Equivalently, is there any map which has a non-atomic ergodic measure but each of whose minimal sets is periodic orbit?

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